The speed of propagation of fronts for the reaction–diffusion equation

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The problem of determining the speed of propagation of fronts for the reaction–diffusion equation dates back to the classical work of Fischer and of Kolmogorov, Petrovskii and Piscounov (KPP) in the 1930's. This equation is used in physics (to model, for example, the propgation of flames) and in population dynamics among other fields. In one space dimension the reaction diffusion equation reads as follows,

$$u_t = u_{xx} + f(u),$$

for $x \in \mathbb{R}$ and $t > 0$. If one has a sufficiently localized (decaying faster than an exponential) initial condition $u(x, 0)$ the solution $u(x, t)$ converges to travelling fronts of the form $q(x - ct)$ or $q(x + ct)$, travelling either to the right or to the left with speed c. The speed of propagation of the fronts, say c, only depends of the reaction profile $f(u)$. In the original formulation of Fischer (or KPP), the reaction term is given by $f(u) = u(1 - u)$, and they showed that for this profile $c = 2$. In general one can consider profiles that satisfy: $f(u)$ is continuous in $[0, 1]$, positive in $(0, 1)$ and $f(0) = f(1) = 0$. If $f(u)$ satisfies the so called *KPP condition*, i.e., $f(u) \leq f'(0)u$, all $u \in [0, 1]$, one can prove that $c = 2\sqrt{f'(0)}$. For all other profiles, including most of the problems involving propagation of flames, the problem of determining c is a hard one. About a decade ago, in collaboration with M. Cristina Depassier we proved a variational characterization of the speed of propagation of fronts for rather general reaction profiles. In this minicourse I will review the extensive literature on the subject. I will present the variational characterization and use it for different applications. In particular, I will show how to use it to determine the speed of propagation of fronts when the reaction term has a cutoff at low values of u, a problem that is of recent interest (after the classical work of Brunet and Derrida).