# Chapter 1
# A Nonconforming Mixed Method for the Time-Harmonic Maxwell Equations

Jim Douglas, Jr.*  Juan E. Santos†  Dongwoo Sheen‡

**Abstract**

We present a nonconforming mixed finite element scheme for the time-harmonic Maxwell equations in a three-dimensional, bounded domain with absorbing boundary conditions on artificial boundaries. The numerical procedure is employed to solve a direct problem in magnetotellurics.

## 1 Introduction

The magnetotelluric method, which is of interest in petroleum exploration and detection of groundwater reservoirs and mineral deposits [1], is used to infer the distribution of the earth's electric conductivity from measurements of natural electric and magnetic fields on the earth's surface and is based on a form of Maxwell's equations. The object of this paper is to present a numerical procedure to determine the scattered electromagnetic fields induced inside the earth when a plane electromagnetic wave arrives normally to the earth's surface; the earth is modelled as a horizontally-layered medium containing arbitrarily shaped conductivity anomalies.

We present a nonconforming mixed finite element scheme for solving the time-harmonic Maxwell equations in a bounded domain with absorbing boundary conditions on artificial exterior boundaries; the method employs a nonconforming element discussed in [4] for solving second-order elliptic problems. The convergence of the numerical solution to that of the differential problem is demonstrated in [3], where in addition the method is hybridized and a parallelizable domain decomposition iterative procedure is described and analyzed. Other numerical methods to solve the direct problem in magnetotellurics have been proposed previously by several authors; see the references in [3].

The organization of the paper is as follows. In §2 we describe the physical problem and the differential equations and boundary conditions employed for its mathematical description and develop a mixed weak formulation of the problem. In §3 the nonconforming mixed finite element spaces used for the spatial discretization are defined, along with our nonconforming mixed finite element method. In §4 we state an *a priori* error estimate for the finite element method.

*Department of Mathematics, Purdue University, West Lafayette, IN 47907-1395

†Department of Mathematics, Purdue University, West Lafayette, IN 47907-1395 and CONICET, Observatorio Astronómico, Universidad Nacional de La Plata, La Plata (1900), Argentina

‡Department of Mathematics, Seoul National University, Seoul 151-742, Seoul, Korea

## 2 The Differential Model

If $E$ and $H$ denote the electric and magnetic fields for a given angular frequency $\omega$, the time-harmonic Maxwell equations in a region free of sources are given by

$$(\sigma + i\omega\varepsilon)E - \nabla \times H = 0, \tag{1a}$$

$$i\omega\mu H + \nabla \times E = 0, \tag{1b}$$

where $\sigma$, $\varepsilon$, and $\mu$ denote the conductivity, electric permittivity, and magnetic permeability, respectively. The terms $\sigma E$ and $i\omega\varepsilon E$ in (1a) represent conductive and displacement currents, respectively.

In magnetotelluric modelling the medium parameters $\sigma$, $\varepsilon$, and $\mu$ fall within a limited range for which

$$\omega\varepsilon << \sigma,$$

and displacement currents can be neglected. Thus, Maxwell's equations reduce to

$$\sigma E - \nabla \times H = 0, \tag{2a}$$

$$i\omega\mu H + \nabla \times E = 0. \tag{2b}$$

Our differential model is formulated in terms of scattered fields (see also [2]). Consider the primary model, without a scatterer, identified with $\mathbf{R}^3_+ = \{(x, y, z) \in \mathbf{R}^3 : z > 0\}$ where the medium parameters $\sigma$ and $\mu$ are assumed to have known values $\sigma_p$ and $\mu_p$, respectively. Suppose that a bounded scatterer $\Omega_s$ is embedded in $\mathbf{R}^3_+$; the primary medium parameters $\sigma_p$ and $\mu_p$ in $\Omega_s$ are then changed into the parameters $\sigma = \sigma_p + \sigma_s$ and $\mu = \mu_p + \mu_s$ with $\mathrm{supp}(\sigma_s) \cup \mathrm{supp}(\mu_s) \subset \Omega_s$. Let $E_p$ and $H_p$ be physically meaningful solutions of Maxwell's equations (2) in $\mathbf{R}^3_+$ for the primary model. Then, let $E_t = E_p + E_s$ and $H_t = H_p + H_s$ denote the total electromagnetic fields in $\mathbf{R}^3_+$ with $\sigma$ and $\mu$ induced by a plane, monochromatic electromagnetic wave of frequency $\omega$ incident upon the boundary $z = 0$ of $\mathbf{R}^3_+$. Finally, let $E_s$ and $H_s$ be the scattered electromagnetic fields due to the presence of the anomalies of $\Omega_s$; they satisfy the equations

$$\begin{aligned} \sigma E_s - \nabla \times H_s &= -\sigma_s E_p \equiv F \quad \text{in } \mathbf{R}^3_+, \\ i\omega\mu H_s + \nabla \times E_s &= -i\omega\mu_s H_p \equiv G \quad \text{in } \mathbf{R}^3_+. \end{aligned}$$

Truncate the problem to a compact domain, so that a practical computational procedure can be defined. Let $\Omega \subset \mathbf{R}^3_+$ be a cube containing $\Omega_s$ and big enough so that $\Gamma \equiv \partial\Omega$ is far away from $\Omega_s$. Without loss of generality, the problem can be scaled so that $\Omega$ is the unit cube whose bottom face is included in the boundary $z = 0$ of $\mathbf{R}^3_+$. Now, consider the scattering problem to find $(E, H) \equiv (E_s, H_s)$:

$$\sigma E - \nabla \times H = F \quad \text{in } \Omega, \tag{3a}$$

$$i\omega\mu H + \nabla \times E = G \quad \text{in } \Omega, \tag{3b}$$

for given $F$ and $G$. To minimize the effect of reflections from the artificial boundary $\Gamma$, impose the absorbing boundary condition

$$(1 - i)P_\tau aE + \nu \times H = 0 \quad \text{on } \Gamma, \qquad a = [\sigma/(2\omega\mu)]^{1/2}, \tag{4}$$

where $\nu$ denotes the unit outer normal to $\Gamma$ and $P_\tau\varphi = \varphi - \nu(\nu \cdot \varphi) = -\nu \times (\nu \times \varphi)$ is the projection of the trace of $\varphi$ on $\Gamma$.Assume that $0 < \sigma_{\min} \le \sigma \le \sigma_{\max}$, $0 < \mu_{\min} \le \mu \le \mu_{\max}$, and that $a$ is a real-valued, Lipschitz-continuous function on $\Gamma$ such that $0 < a_{\min} \le a(x)$ for $x \in \Gamma$. The following existence and uniqueness results for (3)–(4) is proved in [5].

THEOREM 2.1. *Let $F, G \in [L^2(\Omega)]^3$ and $\omega \neq 0$. Then, there exists a unique electromagnetic field $(E, H) \in [H(curl;\Omega)]^2$ satisfying (3)-(4). If, in addition, $F$ and $G$ belong to $H(div;\Omega)$ and $\sigma$ and $\mu$ are Lipschitz-continuous on $\overline{\Omega}$, then $E$ and $H$ belong to $[H^{1/2}(\Omega)]^3$; more precisely, $\{E, H\} \in [H(curl;\Omega) \cap H(div;\Omega)]$ with boundary values in $[L^2(\Gamma)]^6$.*

Here, $(H^s(\Omega), \|\cdot\|_s)$ and $(H^s(\Gamma), |\cdot|_s)$ indicate standard, complex Sobolev spaces for any real number $s$, with $H^0(\Omega) = L^2(\Omega)$; $\|\cdot\|_{0,\Omega} = \|\cdot\|$ denotes the usual $L^2$-norm with the usual complex inner product$(\varphi, \psi)$. Also, for a face $f$ of $\Omega$, let$\langle \varphi, \psi \rangle_f$ denote the inner product on $L^2(f)$, with associated norm $|\cdot|_{0,f}$. Let the Hilbert space

$$H(curl;\Omega) = \left\{ \varphi \in [L^2(\Omega)]^3 : \nabla \times \varphi \in [L^2(\Omega)]^3 \right\}$$

be equipped with the natural norm and inner product

$$\|\varphi\|_{H(curl;\Omega)} = (\|\varphi\|_0^2 + \|\nabla \times \varphi\|_0^2)^{\frac{1}{2}}, \qquad (\varphi, \psi)_{H(curl;\Omega)} = (\varphi, \psi) + (\nabla \times \varphi, \nabla \times \psi).$$

Finally, denote by $Lip(\Gamma)$ the space of all Lipschitz-continuous functions on $\Gamma$ and by $Lip(\Gamma)'$ the dual space of $Lip(\Gamma)$. It is shown in [6] that $(\nu \times \varphi) \cdot P_\tau \psi \in Lip(\Gamma)'$ for all $\varphi, \psi \in H(curl;\Omega)$.

The following generalized Green's formula on $H(curl;\Omega)$ [6, 7] will be useful:

$$(5) \qquad (\nabla \times \varphi, \psi) - (\varphi, \nabla \times \psi) = \langle \nu \times \varphi, \psi \rangle_\Gamma = \langle \nu \times \varphi, P_\tau \psi \rangle_\Gamma, \quad \forall \varphi, \psi \in H(curl;\Omega),$$

where the boundary integral term $\langle \nu \times \varphi, P_\tau \psi \rangle_\Gamma$ is understood as $\langle (\nu \times \varphi) \cdot \overline{P_\tau \psi}, 1 \rangle$, the duality pairing between $\nu \times \varphi \cdot \overline{P_\tau \psi} \in Lip(\Gamma)'$ and $1 \in Lip(\Gamma)$. Note that $\nu \times \varphi$ and $P_\tau \psi$ belong only to $[H^{-1/2}(\Gamma)]^3$ for $\varphi, \psi \in H(curl;\Omega)$.

Test (3a) and (3b) against $\varphi \in H(curl;\Omega)$ and $\psi \in [L^2(\Omega)]^3$ and apply (5) to obtain the mixed, weak problem of finding $(E, H) \in H^*(curl;\Omega) \times [L^2(\Omega)]^3$ such that

$$(6a) \quad (\sigma E, \varphi) - (H, \nabla \times \varphi) + (1-i)\langle P_\tau a E, P_\tau \varphi \rangle_\Gamma = (F, \varphi), \quad \varphi \in H(curl;\Omega),$$

$$(6b) \qquad i\omega(\mu H, \psi) + (\nabla \times E, \psi) = (G, \psi), \quad \psi \in [L^2(\Omega)]^3,$$

where $H^*(curl;\Omega) = \{\varphi \in H(curl;\Omega) : P_\tau a \varphi = \nu \times \chi \text{ for some } \chi \in H(curl;\Omega)\}$. The boundary term in (6a) makes sense since $\langle P_\tau a E, P_\tau \varphi \rangle_\Gamma = \langle \nu \times \chi, P_\tau \varphi \rangle_\Gamma = \langle \nu \times \chi \cdot \overline{P_\tau \varphi}, 1 \rangle$, the last term being understood as a duality between $Lip(\Gamma)'$ and $Lip(\Gamma)$.

**begin of Dongwoo's remark** Since we have the absorbing boundary condition as in (4), we are considering $H \in H(curl;\Omega)$ and this implies that $P_\tau a E$ really belongs to $H^*(curl;\Omega)$ in practice. In the existence paper, we have $E \in H^*(curl;\Omega)$ implicitly since we seek solutions $E, H$ in $H(curl;\Omega)$ satisfying the absorbing boundary condition, and we do not use much about the weak formulation except that we take integration by parts only; I should have stated it clearly in that paper.

Also alternatively, in the weak problem (6) we can say that the solution and the test functions belong to the same space $H^*(curl;\Omega)$.

**end of Dongwoo's remark**

## 3 A Nonconforming Mixed Finite Element Procedure

For $0 < h < 1$, let $\mathcal{T}^h$ be a quasiregular partition of $\Omega$ into three-dimensional rectangles $\Omega_j$, $j = 1, \cdots, J$, with diameters bounded by $h$. Let $\widehat{K}$ be the cube $[-1,1]^3$ and let

$\widehat{Q} = \widehat{Q}_x \times \widehat{Q}_y \times \widehat{Q}_z$, where (see [4])

$$\widehat{Q}_x = \operatorname{Span}\left\{1, y, z, \left(y^2 - \frac{5}{3}y^4\right) - \left(z^2 - \frac{5}{3}z^4\right)\right\},$$
$$\widehat{Q}_y = \operatorname{Span}\left\{1, z, x, \left(z^2 - \frac{5}{3}z^4\right) - \left(x^2 - \frac{5}{3}x^4\right)\right\},$$
$$\widehat{Q}_z = \operatorname{Span}\left\{1, x, y, \left(x^2 - \frac{5}{3}x^4\right) - \left(y^2 - \frac{5}{3}y^4\right)\right\}.$$

Let $\xi_i$, $i = 1, \dots, 6$, be the centroid of the $i^{th}$ face of $\widehat{K}$. The local degrees of freedom for $\varphi \in \widehat{Q}(\widehat{K})$ can be taken to be $\{(P_\tau\varphi)(\xi_i),\ i = 1, \cdots, 6\}$. Define a local interpolant $\widehat{\pi} : [H^2(\widehat{K})]^3 \to \widehat{Q}(\widehat{K})$ by requiring that $P_\tau(\widehat{\pi}\varphi - \varphi)(\xi_i) = 0, \quad i = 1, \cdots, 6$. Then, let$\widehat{S} = \widehat{S}_x \times \widehat{S}_y \times \widehat{S}_z$, where

$$\widehat{S}_x = \operatorname{Span}\left\{1, y - \frac{10}{3}y^3, z - \frac{10}{3}z^3\right\},$$
$$\widehat{S}_y = \operatorname{Span}\left\{1, z - \frac{10}{3}z^3, x - \frac{10}{3}x^3\right\},$$
$$\widehat{S}_z = \operatorname{Span}\left\{1, x - \frac{10}{3}x^3, y - \frac{10}{3}y^3\right\},$$

and define a local interpolant $\widehat{P} : [L^2(\widehat{K})]^3 \to \widehat{S}(\widehat{K})$ by requiring that

$$\int_{\widehat{K}} (\widehat{P}\psi_\ell - \psi_\ell)dx\,dy\,dz = 0, \quad \int_{\widehat{K}} curl(\widehat{P}\psi_\ell - \psi_\ell)dx\,dy\,dz = 0, \quad \ell = x, y, z,$$

for $\psi = (\psi_x, \psi_y, \psi_z)$, where the two-dimensional $curl$ is defined as usual:

$$curl\,\psi_x = \left(\frac{\partial\psi_x}{\partial z}, -\frac{\partial\psi_x}{\partial y}\right), \quad curl\,\psi_y = \left(\frac{\partial\psi_y}{\partial x}, -\frac{\partial\psi_y}{\partial z}\right), \quad curl\,\psi_z = \left(\frac{\partial\psi_z}{\partial y}, -\frac{\partial\psi_z}{\partial x}\right).$$

Note that $\nabla \times \widehat{Q} = \widehat{S}$.

If $P_\tau\widehat{Q}$ or $P_\tau\widehat{S}$ vanishes at the center of a face of $\widehat{K}$, it is orthogonal to constants on that face. This fundamental property of $\widehat{Q}$ and $\widehat{S}$ is important in obtaining effective nonconforming methods [4].

Define $Q(\Omega_j)$ and $S(\Omega_j)$ by scaling and translating from $\widehat{Q}$ and $\widehat{S}$.

Let$\Gamma_j = \partial\Omega_j \cap \Gamma$ and $\Gamma_{jk} = \partial\Omega_j \cap \partial\Omega_k = \Gamma_{kj}$, and set

$$\widetilde{\Lambda}^h = \left\{\widetilde{\lambda}^h : \widetilde{\lambda}^h|_{\Gamma_{jk}} = \widetilde{\lambda}_{jk} \in P_0 \times P_0 \text{ for each face } \Gamma_{jk} \text{ of } \Omega_j;\ \widetilde{\lambda}_{jk} + \widetilde{\lambda}_{kj} = 0\right\}.$$

Denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\Gamma_{jk}}$ the approximation to $\langle \cdot, \cdot \rangle_{\Gamma_{jk}}$ obtained by using the midpoint rule on $\Gamma_{jk}$; i.e., if $\xi_{jk}$ is the centroid of $\Gamma_{jk}$, then

$$\langle\!\langle u, v \rangle\!\rangle_{\Gamma_{jk}} = |\Gamma_{jk}|(u\overline{v})(\xi_{jk}), \qquad |\Gamma_{jk}| = meas(\Gamma_{jk}).$$

Define the nonconforming mixed finite element space $V^h \times W^h$ as follows:

$$V^h = \left\{\varphi \in [L^2(\Omega)]^3 : \varphi|_{\Omega_j} \in Q(\Omega_j) \text{ and } \sum_{jk} \langle\!\langle \theta, P_\tau\varphi \rangle\!\rangle_{\Gamma_{jk}} = 0, \quad \forall\theta \in \widetilde{\Lambda}^h\right\},$$
$$W^h = \left\{\psi \in [L^2(\Omega)]^3 : \psi|_{\Omega_j} \in S(\Omega_j)\right\},$$

and set

$$V_j^h = V^h|_{\Omega_j} \quad \text{and} \quad W_j^h = W^h|_{\Omega_j}.$$

Then, our nonconforming mixed finite element procedure is to find $(E^h, H^h) \in V^h \times W^h$ such that

$$(\sigma E^h, \varphi) - \sum_j (H^h, \nabla \times \varphi)_j + (1-i)\langle\langle P_\tau a E^h, P_\tau \varphi \rangle\rangle_\Gamma = (F, \varphi), \quad \varphi \in V^h, \tag{7a}$$

$$i\omega(\mu H^h, \psi) + \sum_j (\nabla \times E^h, \psi)_j = (G, \psi), \quad \psi \in W^h. \tag{7b}$$

## 4 Convergence of the Nonconforming Mixed Finite Element Procedure

Since the regularity theorem stated earlier is not so strong as those for more standard elliptic problems, it is not clear that optimal order convergence in comparison to approximability, which is optimal in terms of broken norms, should result. What has been proved [3] is the following. If $(E, H)$ and $(E^h, H^h)$, $0 < h < 1$, are the solutions to (6) and (7), respectively, then,

$$\begin{aligned} &\|E - E^h\|_0 + \|H - H^h\|_0 + \|\nabla \times (E - E^h)\|_{0,h} \\ &\quad \le C h^{1/2} \left( \|E\|_2 + h^{1/2} \|H\|_1 \right), \end{aligned}$$

where

$$\|u\|_{m,h}^2 = \sum_j \|u\|_{m,\Omega_j}^2.$$

## References


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