One dimensional electroseismic modeling using the finite element method.
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Abstract

This work presents a finite element procedure for the numerical approximation of the electrosesismic problem at seismic frequencies. The electrosesismic equations used are the ones derived by S. Pride, consisting in the coupled Biot’s equations of motion and Maxwell equations where the seismoelectric feedback is being neglected. The modeled domain comprises two half-spaces, one being air, and the other one a horizontally layered medium. The chosen electromagnetic source is an infinite plane of time dependent electric current located above the surface of the Earth. Under these two assumptions, the electric and magnetic fields and the solid and fluid displacements depend only on one coordinate, namely the one chosen to describe the vertical variations of the Earth; therefore the problem can be considered to be one dimensional.

The existence of a unique solution for both the continuous and discrete weak problems is analyzed, considering a finite computational domain by recursing to absorbing boundary conditions.

The finite element procedure is carried out by using $C^0$ linear functions for the electric field and the solid displacementes, and piecewise constant functions for the magnetic field and fluid displacements, respectively.

Examples showing the capabilities of the numerical method are presented.
Variables and parameters used in this document

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1 Introduction

There is empiric evidence demonstrating that seismic waves propagating through near-surface layers of the Earth induce electromagnetic disturbances that can be measured at the surface (seismoelectric effect)\cite{1, 2, 3}; and also recent tests suggest that the reciprocal process, that is surface measurable acoustic disturbances induced by electromagnetic fields (electroseismic effect), is also possible \cite{4, 5, 6}.

In order to explain these phenomena, Thompson and Gist \cite{7} and Pride \cite{8} suggested that they are generated by an electrokinetic coupling mechanism which can be shortly explained as follows \cite{9, 10}: Within a fluid saturated porous medium there exists a nanometer-scale separation of electric charge in which a bound charge existing on the surface of the solid matrix (normally of negative sign) is balanced by adsorbed positive ions of the surrounding fluid, setting an immobile layer. Further from the surface there exists a distribution of mobile counter ions, forming the so called diffuse layer. The effective thickness of this double layer is of about 10 nm. When an electric field is applied to this system, the ions in the diffuse layer move, dragging the pore fluid along with it because of the viscous traction. This is known as electro-osmosis and is responsible for the electroseismic phenomena. On the other hand, the reciprocal situation arises when an applied pressure gradient creates fluid flow and hence, an ionic convection current, which in turn produces an electric field. This is known as electrofiltration and is responsible for the so-called seismoelectric phenomena.

Pride \cite{8} derived a set of equations controlling both electroseismic and seismoelectric effects in electrolyte-saturated porous media. In these equations the coupling mechanism acts through the (generally frequency dependent) electrokinetic coupling coefficient $L(\omega)$. When this coefficient is set to zero, Pride’s set of equations turns to the uncoupled Maxwell’s and Biot’s equations, describing the latter mechanical wave propagation in a fluid saturated porous medium \cite{11, 12}.

There exist already some works implementing different numerical methods to solve the set of equations modeling both mentioned processes. Han \cite{13}, Pain et. al \cite{14}, Heines and Pride \cite{10} and White \cite{15, 16} have proposed several different approaches to numerically study these phenomena.

The authors have already worked in the numerical solution of each of the two sets of equations into which Pride’s equations turn to be when $L$ is set to zero. Maxwell’s equa-
tions were solved within the frame of magnetotelluric modeling [19, 20, 21, 22, 23], while numerical modeling of Biot’s equations can be found in [24, 25, 26].

Based upon this previous experience, in this work a finite element method to numerically approximate the solution of the electroseismic set of equations is presented. In this approach the electromagnetic source considered is an infinite plane of current, while the Earth is assumed to be horizontally layered. Under these hypotheses the equations can be solved considering only one component of the involved fields, and they depend only on one coordinate direction, namely the one associated with depth.

The finite elements can be chosen as \( C^0 \)-linear functions for the electric field and solid displacements, and piecewise constant functions for the magnetic field and fluid displacements respectively.

The structure of this report is as follows: In Section a review of the hypothesis leading to Pride’s equations is performed, and the latter are stated in the space-frequency domain. Section is devoted to a description of the Earth model and electromagnetic source chosen, followed by a section explaining the absorbing boundary conditions used to uniquely define the studied problem. Later, the weak form of Pride’s equation is stated, and the existence of an unique solution is demonstrated. Then the finite element method is presented, followed by a set of examples showing the implementation of the algorithm.

2 The differential model for electroseismic

Assuming an \( e^{+i\omega t} \) time dependence, the fully coupled system of equations describing both electroseismic and seismoelectric phenomena, expressed in the space-frequency domain is the following:

\[
\begin{align*}
\nabla \times E &= -i\omega \mu H, \\
\nabla \times H &= J + J^{\text{ext}}, \\
J &= \sigma E + i\varepsilon \omega E + L(\omega) \left( -\nabla p_f + \omega^2 \rho_f \mathbf{u}^s \right) \quad (2.1) \\
-\omega^2 \rho_b \mathbf{u}^s - \omega^2 \rho_f \mathbf{u}^f &= \nabla \cdot \tau(u) + F^{(s)}, \quad (2.2) \\
i\omega \mathbf{u}^f &= L(\omega)E + \frac{k(\omega)}{\eta} \left( -\nabla p_f + \omega^2 \rho_f \mathbf{u}^s \right) + F^{(f)}, \quad (2.3) \\
\tau_{ij}(u) &= 2G \varepsilon_{ij}(\mathbf{u}^s) + \delta_{ij}(\lambda_c \nabla \cdot \mathbf{u}^s - D \xi), \quad (2.4) \\
p_f(u) &= -D \nabla \cdot \mathbf{u}^s + K_{av} \xi. \quad (2.5)
\end{align*}
\]
In this set of equations, a porous solid saturated by a single phase, compressible viscous fluid is considered, and it is also assumed that the whole aggregate is isotropic. The symbols \( u^s = (u^s_i) \) and \( \tilde{u}^f = (\tilde{u}^f_i) \), \( i = 1, \ldots, d \) denote the averaged displacement vectors of the solid and fluid phases, respectively, where \( d \) stands the Euclidean dimension, i.e., \( d = 1, 2, 3 \). Also

\[
    u^f = \phi(\tilde{u}^f - u^s) \tag{2.8}
\]

is the average relative fluid displacement per unit volume of bulk material, where \( \phi \) denotes the fraction of connected pore space per unit bulk volume, sometime referred to as effective porosity; and

\[
    \xi = -\nabla \cdot u^f, \tag{2.9}
\]

represents the change in fluid content.

In (2.1)-(2.7) the notation \( u = (u^s, u^f) \) has been used. In (2.2) and (2.5) \( L(\omega) \) is the electrokinetic mobility and \( J^{ext} \) is the electromagnetic (EM) current source density, the primary source of the EM fields. Also, if \( \rho_s \) and \( \rho_f \) denote the mass densities of the solid grains composing the solid matrix, \( \rho_b \) is the mass density of the bulk material, given by

\[
    \rho_b = \phi \rho_f + (1 - \phi) \rho_s. \tag{2.10}
\]

The dynamic permeability \( k(\omega) \) will be defined using the theory of dynamic permeability presented by Johnson et al. [37] and Pride and Garambois [45] as follows:

\[
    k(\omega) = k_0 \left[ \left( 1 + \frac{i \omega}{\omega_c} \right)^2 + i \frac{\omega}{\omega_c} \right]^{-1}, \tag{2.11}
\]

where \( k_0 \) is the effective permeability and \( \omega_c \) is a critical frequency at which the viscous-boundary layers begin to form in the pores. The value of \( \omega_c \) can be estimated from the relation

\[
    \omega_c = \frac{\eta \phi}{\rho_f F k_0}, \tag{2.12}
\]

where \( \eta \) is the fluid viscosity and \( F \) is the electrical formation factor. Here \( F \) is computed from the equation [37]

\[
    F = \frac{T}{\phi}, \tag{2.13}
\]
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with $T$ denoting the formation tortuosity, assumed in this work to depend on the porosity by the relation

$$ T = \frac{1}{2} \left( 1 + \frac{1}{\phi} \right). $$

(2.14)

It is remarked that (2.11) corresponds to the choice $n = 4$ in [45]. Note that

$$ k(\omega) \to k_0, \quad \text{as} \quad \omega \to \infty. $$

(2.15)

Let the coefficients $H_1(\omega)$ and $g(\omega)$ be defined by the equations

$$ H_1(\omega) + iH_2(\omega) = \frac{\eta}{k(\omega)}, \quad g(\omega) = \frac{H_2(\omega)}{\omega}. $$

(2.16)

Then (2.5) can be rewritten in the form

$$ -\omega^2 \rho_f u^s - \omega^2 g(\omega) u^f + i\omega H_1(\omega) u^f = -\nabla p_f + \frac{\eta}{k(\omega)} L(\omega) E(\omega),. $$

(2.17)

The mass coupling coefficient $g(\omega)$ in (2.17) represent the inertial effects associated with dynamic interactions between the solid and fluid phases, while the coefficient $H_1(\omega)$ include the viscous coupling effects between such phases.

It can be immediately seen from (2.16) that

$$ H_1(\omega) \to \frac{\eta}{k_0}, \quad \text{as} \quad \omega \to 0, $$

$$ g(\omega) \to g_0 = \frac{3}{2} \rho_f T \phi, \quad \text{as} \quad \omega \to 0, $$

(2.18)

so that the low-frequency Biot’s equations of motion in [11] are recovered if the electrokinetic coupling coefficient is set to zero.

Concerning the coefficients in the constitutive equations (2.6) and (2.7), the coefficient $G$ is equal to the elastic shear modulus of the dry matrix. Also,

$$ \lambda_c = K_c - 2/3G, $$

(2.19)

with $K_c$ being the bulk modulus of the saturated material. Following [46], [35] the coefficients in (2.6)-(2.7) can be obtained from the relations

$$ \alpha = 1 - \frac{K_m}{K_s}, \quad K_{av} = \left[ \frac{\alpha - \phi}{K_s} + \frac{\phi}{K_f} \right]^{-1}, $$

$$ K_c = K_m + \alpha^2 K_{av}, \quad D = \alpha K_{av}, $$

(2.20)

where $K_s, K_m$ and $K_f$ denote the bulk modulus of the solid grains composing the solid matrix, the dry matrix and the saturant fluid, respectively. The coefficient $\alpha$ is known as the effective stress coefficient of the bulk material.
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2.1 Assumptions on the model

In the present work some assumptions will be made that allow for a simplification of the original set of equations (2.1)-(2.7):

(a) in the low-frequency range (up to several tens of Hz) displacement currents \( i \omega \varepsilon E \) can be ignored because they are about four orders of magnitude smaller than the conduction currents \( \sigma E \) for the seismic frequency range and for all materials of interest.

(b) The second approximation is to disregard the electrofiltration feedback, i.e., the generation of an electric current due to the induced pressure gradient, which is represented by the term \( L(\omega)(-\nabla p_f + \omega^2 \rho_f u^s) \) in equation (2.3). In order to understand why this assumption is reasonable, consider an electric field propagating through an homogeneous material. In this case, the fluid filtration given by \( L(\omega)E \) in (2.5) is compensated by the Darcy fluid filtration \( \frac{k(\omega)}{\eta}(-\nabla p_f + \omega^2 \rho_f u^s) \) so that the outcome is no fluid flow, i.e.,

\[
L(\omega)E + \frac{k(\omega)}{\eta}(-\nabla p_f + \omega^2 \rho_f u^s) = 0 \tag{2.21}
\]

wherefrom

\[
(-\nabla p_f + \omega^2 \rho_f u^s) = -\frac{L(\omega)E\eta}{k(\omega)}. \tag{2.22}
\]

Using the equality above, (2.3) can be rewritten as

\[
J = \sigma E \left( 1 - \frac{\eta L^2}{k(\omega)\sigma} \right). \tag{2.23}
\]

Haines and Pride [10] affirm that the correction term satisfies \( \frac{\eta L^2}{k(\omega)\sigma} < 10^{-5} \) for all materials of interest; therefore it can be neglected relative to one. As such, the flow driven electric current \( L(-\nabla p_f + \omega^2 \rho_f u^s) \) is very small compared to the conduction current \( \sigma E \), and can be neglected in Maxwell equations.

(c) It is also considered that in the low frequency range, \( L(\omega) \approx L_0 \) as defined in [13]; further, the low frequency limits in (2.15) and (2.18) are used.

(d) It must be taken into account that no seismic source will be here considered, because the interest of this work is towards electroseismics; therefore \( F^{(s)} = F^{(f)} = 0 \) in equations (2.5) and (2.6) respectively.

(e) Turning now to the choice of the Earth model and the electromagnetic source, the latter is assumed to be a time dependent infinite plane of electric current flowing in the direction of, say, the \( x \)-coordinate axis. The former is assumed to be a layered three-dimensional half space with the horizontal layers in the \( z \)-direction; in Fig. 1 a scheme is depicted, together with the one dimensional computational domain.
Figure 1: Scheme for the Earth model and the electromagnetic source. On the right the 1D computational domain $\Omega$ is shown; the portion corresponding to the Biot medium is referred to as $\Omega^B$. The symbols $\Gamma^{i,j}$, $i=b,t$, $j=B,M$ denote the bottom and top boundaries, for Biot’s and Maxwell’s equations.

Taking into account all the above enumerated considerations leads to a one dimensional problem, where the electromagnetic field has a diffusive behaviour, and has no contribution -through the coupling coefficient- from the mechanical displacements. The latter can exist only in the $SH$-direction [13], i.e., the solid and fluid displacements will only occur in the $x$-direction. Moreover, both the electromagnetic fields and mechanical displacements have only a depth dependence, i.e., using the coordinates axes displayed in Fig. 1, they
are only function of \( z \). Therefore the system (2.1)-(2.7) can be rewritten as follows:

\[
\begin{align*}
\frac{\partial E_x}{\partial z}(z, \omega) + i \omega \mu H_y(z, \omega) &= 0 \quad \text{in } \Omega, \\
\sigma E_x(z, \omega) + \frac{\partial H_y}{\partial z}(z, \omega) &= -J_{\text{ext}}, \quad \text{in } \Omega, \\
-\omega^2 \rho_u u_x^s(z, \omega) - \omega^2 \rho_f u_x^f(z, \omega) - \frac{\partial}{\partial z} \left( G \frac{\partial u_x^s(z, \omega)}{\partial z} \right) &= 0, \quad \text{in } \Omega_B, \\
-\omega^2 \rho_f u_x^s(z, \omega) - \omega^2 g_0 u_x^f(z, \omega) + i \omega \frac{\eta}{k_0} u_x^f(z, \omega) &= \frac{\eta}{k_0} L_0 E_x(z, \omega) \quad \text{in } \Omega_B.
\end{align*}
\]

The symbol \( \Omega \) stands for the whole computational domain, while \( \Omega_B \) is the portion of the domain where Biot equations are to be solved, i.e., the subsurface. The top and bottom boundaries are called \( \Gamma_{t,M} \) and \( \Gamma_{t,B} \) and \( \Gamma_{b,M} \) and \( \Gamma_{b,B} \) for Maxwell’s and Biot’s equations respectively.

This system of equations must be equipped with appropriate boundary conditions on the computational borders \( \Gamma_{i,j} \), \( i=t,b, j=M,B \) in order to be solved. The ones chosen for this work are absorbing boundary conditions [19, 26]. How they are obtained from their three dimensional counterparts is detailed below.

### 2.2 Absorbing boundary conditions

First we analyze the absorbing boundary conditions associated with (2.26)-(2.27).

Notice that, as has been done in all previous sections, no notation change is done when dealing with three dimensional vectors or scalars; where each is used is clear from the context. It is remarked, however, that one dimensional boundary conditions for Biot and Maxwell equations, suitable for the above described electroseismic system are deduced from their three dimensional versions. Here the three-dimensional domain \( \Omega_B \) is assumed to be a rectangular prism with boundary \( \Gamma_B \); the coordinate system is the same as in Fig. 1.

Following [26], define the operators

\[
\begin{align*}
\mathcal{G}_{\Gamma}(u) &= \left( \tau(u) \nu \cdot \nu, \tau(u) \nu \cdot \chi^1, \tau(u) \nu \cdot \chi^2, p_f(u) \right), \\
\mathcal{S}_{\Gamma}(u) &= \left( u^s \cdot \nu, u^s \cdot \chi^1, u^s \cdot \chi^2, u^f \cdot \nu \right)^t,
\end{align*}
\]

where \( \nu \) is an outer unit normal vector, and \( \chi^1 \) and \( \chi^2 \) are two unit tangents on \( \Gamma \) so that \( \{\nu, \chi^1, \chi^2\} \) is an orthonormal system on \( \Gamma_B \). Here \( \tau \) and \( p_f \) are given by (2.6) and (2.7).
Consider next the restriction to an interval in the positive \( z \)-direction of the following 3D absorbing boundary condition [26]:

\[
-\mathcal{G}_T(u(x, y, z, \omega)) = i\omega \mathcal{B} \mathcal{S}_T(u(x, y, z, \omega)), \quad (x, y, z, \omega) \in \Gamma^B \times (0, \omega^*). \tag{2.30}
\]

The matrix \( \mathcal{B} \) in (2.30) is positive definite and is given by \( \mathcal{B} = \mathcal{M}^{\frac{1}{2}} \mathcal{S}^{\frac{1}{2}} \mathcal{M}^{\frac{1}{2}} \), where

\[
\tilde{\alpha} = \rho_b - \rho_f^2/g_0,
\]

\[
\mathcal{M} = \begin{bmatrix}
\rho_b & 0 & 0 & \rho_f \\
0 & \tilde{\alpha} & 0 & 0 \\
0 & 0 & \tilde{\alpha} & 0 \\
\rho_f & 0 & 0 & g
\end{bmatrix},
\]

\[
\mathcal{E} = \begin{bmatrix}
\lambda_c + 2G & 0 & 0 & D \\
0 & G & 0 & 0 \\
0 & 0 & G & 0 \\
D & 0 & 0 & K_{av}
\end{bmatrix},
\]

and

\[
\mathcal{S} = \mathcal{M}^{\frac{1}{2}} \mathcal{E} \mathcal{M}^{\frac{1}{2}}.
\]

Let us compute the operators \( \mathcal{G}_T(u), \mathcal{S}_T(u) \) on each face of \( \Omega^B \), assuming that \( \Gamma^{j,B}, j=b,t \) are contained in planes parallel to the \( xy \)-plane, the one containing \( \Gamma^{t,B} \) located over the one containing \( \Gamma^{b,B} \). Take on this faces as unit normal and unit tangent vectors

\[
\nu = (0, 0, -1), \quad \chi^1 = (0, 1, 0), \quad \chi^2 = (1, 0, 0) \quad \text{on} \quad \Gamma^{t,B},
\]

\[
\nu = (0, 0, 1), \quad \chi^1 = (0, 1, 0), \quad \chi^2 = (-1, 0, 0) \quad \text{on} \quad \Gamma^{b,B}.
\]

Then,

\[
\mathcal{S}_T(u) = (0, 0, u_x^s, 0), \quad \text{on} \quad \Gamma^{t,B}, \quad \mathcal{S}_T(u) = (0, 0, -u_x^s, 0), \quad \text{on} \quad \Gamma^{b,B}. \tag{2.31}
\]

Next, on \( \Gamma^{t,B} \),

\[
\tau_{xj} \nu_j = -\tau_{xz}, \quad \tau_{yj} \nu_j = -\tau_{yz} = 0, \quad \tau_{zj} \nu_j = -\tau_{zz} = 0,
\]
so that

\[ G_{\Gamma_t,B}(u) = (0, 0, -\tau_{xz}, 0, p_f). \] (2.32)

Similarly,

\[ G_{\Gamma_b,B}(u) = (0, 0, -\tau_{xz}, 0, p_f). \] (2.33)

Consequently (2.30) on \( \Gamma_t,B \) reduces to

\[-(0, 0, -\tau_{xz}, p_f) = i\omega(B_{13}, B_{23}, B_{33}, B_{43})(u_x^s), \] (2.34)

so that

\[ \tau_{xz} = G \frac{\partial u_x^s}{\partial z} = i\omega B_{33} u_x^s, \quad \text{on} \quad \Gamma_t,B. \] (2.35)

Proceeding similarly, (2.30) on \( \Gamma_b,B \) reduces to

\[-(0, 0, -\tau_{xz}, p_f) = i\omega(B_{13}, B_{23}, B_{33}, B_{43})(-u_x^s), \] (2.36)

and

\[ \tau_{xz} = G \frac{\partial u_x^s}{\partial z} = -i\omega B_{33} u_x^s, \quad \text{on} \quad \Gamma_b,B. \] (2.37)

Now in any one dimensional interval in the z-direction, the faces \( \Gamma_t,B \) and \( \Gamma_b,B \), have unit outward normal \( \nu = -1 \) and \( \nu = 1 \), respectively. Thus (2.35) and (2.37) can be written together as

\[-G \frac{\partial u_x^s}{\partial z} \cdot \nu = i\omega B_{33} u_x^s, \quad \text{on} \quad \Gamma^B \equiv \Gamma_t,B \cup \Gamma_b,B. \] (2.38)

This last expression is the first one of the two one dimensional boundary conditions looked for; however, the coefficient \( B_{33} \) in (2.38) still has to be calculated explicitly.

For that purpose, it is convenient to rearrange the entries in the matrices \( \mathcal{M} \) and \( \mathcal{E} \) as follows. let \( I_2 \) denote the \( 2 \times 2 \) identity matrix and define the matrices

\[
\tilde{\mathcal{M}} = \begin{bmatrix} \rho_b & \rho_f & 0 & 0 \\ \rho_f & g_0 & 0 & 0 \\ 0 & 0 & \tilde{\alpha} & 0 \\ 0 & 0 & 0 & \tilde{\alpha} \end{bmatrix} \equiv \begin{bmatrix} P & 0I_2 \\ 0I_2 & \tilde{\alpha}I_2 \end{bmatrix},
\]
\[\bar{E} = \begin{bmatrix}
\lambda_c + 2G & D & 0 & 0 \\
D & K_{av} & 0 & 0 \\
0 & 0 & G & 0 \\
0 & 0 & 0 & G
\end{bmatrix} \equiv \begin{bmatrix}
Q & 0I_2 \\
0I_2 & GI_2
\end{bmatrix}.\]

Then,
\[\bar{S} = \bar{M}^{-\frac{1}{2}}\bar{E}\bar{M}^{-\frac{1}{2}} = \begin{bmatrix}
\bar{S} & 0I_2 \\
0I_2 & G/\bar{\alpha}I_2
\end{bmatrix},\]
and
\[\bar{B} = \bar{M}^{\frac{1}{2}}\bar{S}^{\frac{1}{2}}\bar{M}^{\frac{1}{2}} = \begin{bmatrix}
R & 0I_2 \\
0I_2 & \bar{\alpha}((G/\bar{\alpha}))^{1/2}I_2
\end{bmatrix},\]
where \(\bar{S} = P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\) and \(R = P^{\frac{1}{2}}\bar{S}^{\frac{1}{2}}P^{\frac{1}{2}}\).

Hence,
\[\bar{B}_{33} = B_{33} = \bar{\alpha}((G/\bar{\alpha}))^{1/2}\]
(2.39)

Thus if the boundary operators are redefined as
\[\mathcal{G}_\Gamma(u) = \left(\tau(u)\nu \cdot \nu, p_f(u)\tau(u)\nu \cdot \chi, \tau(u)\nu \cdot \chi^2\right),\]
(2.40)
\[S_\Gamma(u) = \left(u^s \cdot \nu, u^f \cdot \nu, u^s \cdot \chi, u^s \cdot \chi^2, u^s \cdot \chi^2\right)^t,\]
(2.41)
a repetition of the above argument leading to (2.38) yields
\[-G\frac{\partial}{\partial z} u^s_x \cdot \nu = i\omega\bar{\alpha}((G/\bar{\alpha}))^{1/2}u^s_x, \quad \text{on} \quad \Gamma^B.\]
(2.42)

Next we proceed to obtain the absorbing boundary conditions for Maxwell equations (2.24)-(2.25).

For a 3D bounded domain \(\Omega\) with boundary \(\Gamma\), the 3D absorbing boundary condition for Maxwell equations is [19]
\[aP_T E + \nu \times H = 0, \quad \text{on} \quad \Gamma\]
(2.43)
where \(a = (1 - i)\sqrt{\frac{\sigma}{2\mu}\omega}\) and \(P_T E = -\nu \times \nu \times E\) and as before \(\nu\) is the unit outer normal to the boundary \(\Gamma\).

In our 1D-model, \(E^s = (E^s_x, 0, 0)\) and \(H^s = (0, H^s_y, 0)\) and a straightforward calculations show that (2.43) reduces to the two scalar conditions
\[aE^s_x + H^s_y = 0 \quad \text{on} \quad \Gamma^{t,M},\]
\[aE^s_x - H^s_y = 0 \quad \text{on} \quad \Gamma^{b,M}.\]
(2.44)
These two equations can be written, using again that \( \nu = \mp 1 \) on \( \Gamma^{t,M} \) and \( \Gamma^{b,M} \) respectively as

\[
aE_x^s - \nu \cdot H_y^s = 0 \quad \text{on} \quad \Gamma = \Gamma^{t,M} \cup \Gamma^{b,M}.
\]

(2.45)

Therefore, the complete set of equations needed to guarantee a unique solution of our 1D electroseismic model have been formulated; the equations are (2.24)-(2.27), (2.42) and (2.45). However, instead of solving Maxwell's equations as are presented in (2.24)-(2.25), a modified version of these equations in terms of primary and secondary fields will be considered. In the next subsection it is explained how it is obtained.

### 2.3 Treatment of Maxwell equations

When numerically solving Maxwell equations -actually this is done with other equations also- it is customary to think the fields \( E \) and \( H \) as a combination of the so-called primary and secondary fields, denoted \( E^p, H^p \) and \( E^s, H^s \) respectively. Usually this is done when the primary fields can be analytically calculated; in this case the secondary ones are thought as perturbations to the primary fields. This approach is the one used here.

In order to see how it works, consider the whole space with conductivity \( \sigma^p \), and an (parallel to the \( xy \)-plane as in Fig. 1) infinite sheet of current density \( J^{ext} \) located in \( z = z_S \geq 0 \). Further, consider that the current density is of the form

\[
J^{ext} = I(\omega)\delta(z - z_S)e_x, \quad (x,y,z) \in \mathbb{R}^3,
\]

(2.46)

where \( e_x = (1,0,0) \).

Under the present hypotheses, Maxwell’s equations for the primary fields can be written -following [43]- as

\[
\nabla \times E^p = -i\omega \mu H^p
\]

(2.47)

\[
\nabla \times H^p = \sigma^p E^p + J^{ext}.
\]

(2.48)

Taking divergence in (2.47),

\[
\nabla \cdot H^p = 0,
\]

(2.49)

and consequently

\[
H^p = \nabla \times A,
\]

(2.50)
2. THE DIFFERENTIAL MODEL FOR ELECTROSEISMIC

Now using (2.50) in (2.47), it follows that
\[ E^p + i\omega A = -\nabla V, \tag{2.51} \]

Now use (2.51) in (2.48) and use (2.50) to get
\[ \nabla \times \nabla \times A - \sigma^p i\omega \mu A = -\sigma^p \nabla V + J^{ext}. \tag{2.52} \]

Impose the Lorentz condition
\[ \nabla \cdot A = -\sigma^p V, \tag{2.53} \]

to see that (2.52) becomes
\[ \Delta A + k^2 A = -I(\omega)\delta(z - z_S)\hat{e}_x, \tag{2.54} \]

where
\[ k^2 = \sigma^p i\omega \mu. \]

Let \( G_1(z) \) be the solution of the following scalar Helmholtz equation in an unbounded three dimensional space
\[ \Delta G_1 + k^2 G_1 = -\delta(z). \tag{2.55} \]

By analogy with the 2D harmonic line source in [43], it turns out that
\[ G_1(z) = \frac{1}{2ik}e^{-ik|z-z_S|} \tag{2.56} \]

so that
\[ A(z) = \int_{-\infty}^{\infty} I(\omega)G_1(z - z')\delta(z')dz'\hat{e}_x = I(\omega)G_1(z)\hat{e}_x = (A_x(z), 0, 0). \tag{2.57} \]

Next, since \( \nabla \cdot A = 0 \), from (1.122)-(1.123) in [43]
\[ E^p(z) = -i\omega \mu A(z) = -i\omega \mu I(\omega)G_1(z)\hat{e}_x = (E_x(z), 0, 0), \tag{2.58} \]
\[ H^p(z) = \nabla \times A = (0, \frac{\partial A_x(z)}{\partial z}, 0) = (0, H_y(z), 0). \tag{2.59} \]

Therefore, as it can be expected from the fact that an infinite plane source source is being considered, the electromagnetic field does not depend on \( x \) and \( y \), but only on the \( z \) coordinate.
Set now within the considered whole space at \( z = z_{sf}, \) \( 0 \leq z_{s} \leq z_{sf} \) a half-space of horizontal layers of different electrical conductivities such that

\[
\sigma_{c}(z) = \sigma^{p} + \sigma^{s}(z),
\]

where, as already stated, \( \sigma^{p} \) is the constant background conductivity and \( \sigma^{s}(z) \) is the conductivity anomaly. Define the scattered electrical and magnetic fields by

\[
E^{s} = E - E^{p}, \quad H^{s} = H - H^{p},
\]

where \( (E, H) \) are the solution of

\[
\nabla \times E = -i\omega \mu H, \quad \nabla \times H = \sigma_{c}(z)E + I(\omega)\delta(z - z_{s})\hat{e}_{x},\]

Then subtract (2.47) from (2.62) and (2.48) from (2.63) to obtain

\[
\nabla \times E^{s} = -i\omega \mu H^{s}, \quad \nabla \times H^{s} = \sigma_{c}E^{s} + \sigma^{s}E^{p}.
\]

Then, the source for the secondary fields is a conduction current where the electrical conductivity involve just the conductivity anomalies, and the electric field is the primary one; the latter can be, as was shown above, analytically calculated. The one dimensional version of equations (2.64)- (2.65) is the one to be numerically approximated. Therefore, the full set of equations to be solved by means of the finite element method is

\[
\frac{\partial E_{x}^{s}}{\partial z}(z, \omega) + i\omega \mu H_{y}^{s}(z, \omega) = 0 \quad \text{in } \Omega, \quad \text{(2.66)}
\]

\[
\sigma_{c}E_{x}^{s}(z, \omega) + \frac{\partial H_{y}^{s}}{\partial z}(z, \omega) = -\sigma^{s}E^{p}(z), \quad \text{in } \Omega, \quad \text{(2.67)}
\]

\[
-\omega^2 \rho_{b}u_{x}^{s}(z, \omega) - \omega^2 \rho_{f}u_{x}^{f}(z, \omega) - \frac{\partial}{\partial z} \left( G \frac{\partial u_{x}^{s}(z, \omega)}{\partial z} \right) = 0, \quad \text{in } \Omega^{B}, \quad \text{(2.68)}
\]

\[
-\omega^2 \rho_{f}u_{x}^{s}(z, \omega) - \omega^2 g_{0}u_{x}^{f}(z, \omega) + i\omega \frac{\eta}{k_{0}} u_{x}^{f}(z, \omega) = \frac{\eta}{k_{0}} L_{0} E_{x}(z, \omega) \quad \text{in } \Omega^{B}, \quad \text{(2.69)}
\]

\[
aE_{x}^{s} - \nu \cdot H_{y}^{s} = 0 \quad \text{on } \Gamma, \quad \text{(2.70)}
\]

\[
-G \frac{\partial}{\partial z} u_{x}^{s} \cdot \nu = i\omega \tilde{\alpha}(\frac{G}{\tilde{\alpha}})^{1/2} u_{x}^{s}, \quad \text{on } \Gamma^{B}. \quad \text{(2.71)}
\]

Notice that because it is assumed that there is no seismoelectric feedback, Maxwell equations equations (2.66), (2.67) and (2.70) can be independently solved. Once \( E^{s} \) is obtained, \( E = E^{p} + E^{s} \) is calculated and the solid and fluid displacements \( u^{s}, u^{f} \) are obtained by solving equations (2.68), (2.69) and (2.71).
2.4 Modification of the elastic coefficient G to introduce viscoelasticity

It is possible to take into account the lossy nature of the subsurface by considering that the Earth presents viscoelastic behaviour. In order to introduce viscoelasticity the correspondence principle stated by M. Biot [17, 18] can be used, i.e., the (real) relaxed elastic shear modulus coefficient $G$ is replaced by a complex frequency dependent viscoelastic modulus $\hat{G} = \hat{G}(\omega)$. In this work the linear viscoelastic model presented in [39] is used through the following formula:

$$\hat{G}(\omega) = \frac{G}{R(\omega) - iT(\omega)} = G_r(\omega) + iG_i(\omega).$$ (2.72)

Here the frequency dependent functions $R$ and $T$, associated with a continuous spectrum of relaxation times, characterize the viscoelastic behaviour and are given by [39]

$$R(\omega) = 1 - \frac{1}{\pi \hat{Q}} \ln \frac{1 + \omega^2 T_1^2}{1 + \omega^2 T_2^2}, \quad T(\omega) = \frac{2}{\pi \hat{Q}} \tan^{-1} \frac{\omega (T_1 - T_2)}{1 + \omega^2 T_1 T_2}.$$

The model parameters $\hat{Q}$, $T_1$ and $T_2$ are taken such that the quality factor

$$Q(\omega) = \frac{T(\omega)}{R(\omega)} = \frac{G_r(\omega)}{G_i(\omega)}$$

is approximately equal to the constant $\hat{Q}$ in the range of frequencies where the equations are solved, which makes this model convenient for geophysical applications. Values of $\hat{Q}$ range from $\hat{Q} = 20$ for highly dissipative materials to about $\hat{Q} = 1000$ for almost elastic ones. Note that $G_r(\omega) > 0, G_i(\omega) > 0$, a fact that is used in the following sections.

3 Variational formulation

The set of equations (2.66)-(2.71) need to be written in variational form if the finite element method is to be used. Here the changes are accounted for.

3.1 Variational formulation for Biot’s equations

For $X \subset \mathbb{R}$ with boundary $\partial X$, let $(\cdot, \cdot)_X$ and $(\cdot, \cdot)_{\partial X}$ denote the complex $L^2(X)$ and $L^2(\partial X)$ inner products for scalar, vector, or matrix valued functions. Also, for $s \in \mathbb{R}$, $\| \cdot \|_{s,X}$ and $| \cdot |_{s,X}$ will denote the usual norm and seminorm for the Sobolev space $H^s(X)$. 
4. UNIQUENESS OF THE SOLUTION OF THE VARIATIONAL PROBLEMS

In addition, if \( X = \Omega \) or \( X = \Gamma \), the subscript \( X \) may be omitted such that \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\Omega} \) or \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\Gamma} \).

Consider that our computational domain is an interval \( \Omega = (0, 1) \) in the positive \( z \)-direction with boundary \( \Gamma \) and assume that there exist a solution of Biot’s equations (2.68), (2.69) with the boundary condition (2.71). Next, test (2.68) against \( v \in H^1(\Omega^B) \), use integration by parts and apply the boundary condition (2.71). Also, test (2.69) against \( w \in L^2(\Omega^B) \). Thus the following variational formulation is obtained:

Find \((u_s^x, u_f^x) \in H^1(\Omega^B) \times L^2(\Omega^B)\) such that

\[
-\omega^2(\rho_b u_s^x, v)_{\Omega^B} - \omega^2(\rho_f u_f^x, v)_{\Omega^B} + (G \nabla u_s^x, \nabla v)_{\Omega^B} + \langle i\omega (\alpha G)^{1/2} u_s^x, v \rangle_{\Gamma^B} = 0,
\]

\[
 v \in H^1(\Omega),
\]

\[
-\omega^2(\rho_f u_s^x, w)_{\Omega^B} - \omega^2(g_0 u_f^x, w)_{\Omega^B} + i\omega(\eta/k_0) u_s^x, w)_{\Omega^B} = \left(\eta/k_0 L_0 E_x, w\right)_{\Omega^B}, \quad w \in L^2(\Omega^B)
\]

In (3.1) \( \nabla \) stands for \( \frac{\partial}{\partial x} \).

3.2 Variational formulation for Maxwell equations

Existence and uniqueness of the solution of equations (2.66)-(2.67) with boundary conditions (2.70) has been proven in [22]. Then, test (2.66) against \( \phi \in L^2(\Omega) \) and test (2.67) against \( \psi \in H^1(\Omega) \) and integrate by parts using boundary condition (2.70). The obtained variational formulation reads:

Find \((E_x^s, H_y^s) \in H^1(\Omega) \times L^2(\Omega)\) such that

\[
(\partial E_x^s/\partial z, \phi) + (i\omega \mu H_y^s, \phi) = 0, \quad \phi \in L^2(\Omega),
\]

\[
(\sigma_c E_x^s, \psi) + \langle a E_x^s, \psi \rangle - (H_y^s, \partial \psi/\partial z) = (-J_T, \psi), \quad \psi \in H^1(\Omega),
\]

where \( J_T = \sigma_s E_x^p \).

4 Uniqueness of the solution of the variational problems

The usual mechanism is to prove that the unique possible solution with no sources is the null one.
4. UNIQUENESS OF THE SOLUTION OF THE VARIATIONAL PROBLEMS

4.1 Biot Part

Set $E_x = 0$ in (3.2). Take $v = u_x^s$, $w = u_x^f$ in (3.1)-(3.2) to obtain

$$\begin{aligned}
-\omega^2 (\rho_b u_x^s, u_x^s)_{\Omega_B} - \omega^2 (\rho_f u_x^f, u_x^f)_{\Omega_B} + (G \nabla u_x^s, \nabla u_x^s)_{\Omega_B} \\
+ (i \omega (\alpha G)^{1/2} u_x^s, u_x^s)_{\Gamma_B} = 0,
\end{aligned}
$$

(4.1)

$$\begin{aligned}
-\omega^2 (\rho_f u_x^s, u_x^f)_{\Omega_B} - \omega^2 (g_0 u_x^f, u_x^f)_{\Omega_B} + i \omega (\eta / k_0) u_x^f, u_x^f)_{\Omega_B} = 0.
\end{aligned}
$$

(4.2)

Take imaginary part in (4.2) to see that

$$\|u_x^f\|_{0, \Omega_B} = 0,$$

(4.3)

i.e.,

$$u_x^f = 0, \text{ in } L^2(\Omega_B).$$

(4.4)

Thus, (4.2) reduces to

$$\begin{aligned}
-\omega^2 (\rho_b u_x^s, u_x^s)_{\Omega_B} + (G \nabla u_x^s, \nabla u_x^s)_{\Omega_B} + (i \omega (\alpha G)^{1/2} u_x^s, u_x^s)_{\Gamma_B} = 0.
\end{aligned}
$$

(4.5)

Take imaginary part in (4.5) to see that

$$(G, \nabla u_x^s, \nabla u_x^s)_{\Omega_B} + (\omega (\alpha G)^{1/2} u_x^s, u_x^s)_{\Gamma_B} = 0.$$

(4.6)

Since each term in (4.6) is positive, we conclude that

$$\|\nabla u_x^s\|_{0, \Omega_B} = 0, \quad |u_x^s|_{0, \Gamma_B} = 0.$$

(4.7)

Thus $u_x^s \in H^1_0(\Omega_B)$ so that satisfies Poicare’s inequality

$$\|u_x^s\|_{1, \Omega_B} \leq C(\Omega_B) \|\nabla u_x^s\|_{0, \Omega_B}.$$

(4.8)

Now (4.7) and (4.8) implies that

$$\|u_x^s\|_{1, \Omega_B} = 0,$$

(4.9)

so that

$$u_x^s = 0, \text{ in } H^1(\Omega_B).$$

(4.10)

The result is summarized in the following theorem.

**Theorem 4.1.** For any $\omega > 0$, uniqueness holds for the solution of (3.1)-(3.2).
4.2 Maxwell Part

Setting \( J_T = 0, \psi = E^s_x \) and \( \phi = H_y^s \) in (3.3)-(3.4), the following equations are obtained:

\[
\frac{\partial E^s_x}{\partial z}, H^s_y + (i\omega \mu H^s_y, H^s_y) = 0, \quad (4.11)
\]

\[
(\sigma_c E^s_x, E^s_x) + (aE^s_x, E^s_x) - (H^s_y, \frac{\partial E^s_x}{\partial z}) = 0. \tag{4.12}
\]

Conjugate (4.11) and use it to write the third term of (4.12) in terms of the magnetic field, then

\[
(\sigma_c E^s_x, E^s_x) + (aE^s_x, E^s_x) - i\omega(\mu H^s_y, H^s_y) = 0 \quad (4.13)
\]

is obtained. The real part of this equation is

\[
(\sigma_c E^s_x, E^s_x) + \langle \text{Re}(a) E^s_x, E^s_x \rangle = 0. \tag{4.14}
\]

As both terms are positive, it turns out that \( ||E^s_x||_0 = 0 \), and then \( E^s_x = 0 \) in \( L^2(\Omega) \). It is then straightforward to see, by taking imaginary part of (4.13) that \( ||H^s_y||_0 = 0 \), and therefore \( H^s_y = 0 \) in \( L^2(\Omega) \). This result can be then stated in the following theorem:

**Theorem 4.2.** For any \( \omega > 0 \), uniqueness holds for the solution of (3.3)-(3.4)

5 The Finite Element Procedure

5.1 Biot Part

Let \( \mathcal{T}^h \) be a uniform partition of \( \Omega \) into subinterval of length \( h \) and let \( \mathcal{T}^{B,h} \) be the part of the partition contained in \( \Omega^B \). Let \( V^h \subset H^1(\Omega^B), W^h \subset L^2(\Omega^B) \) be finite element spaces consisting of \( C^0 \)-piecewise linear and piecewise constants over the partition \( \mathcal{T}^{B,h} \). The finite element procedures is then formulated as follows:

Find \( (u^s_x, u^f_x) \in V^h \times W^h \) such that

\[
-\omega^2(\rho_b u^s_x, v)_{\Omega^B} - \omega^2(\rho_f u^f_x, v)_{\Omega^B} + (G \nabla u^s_x, \nabla v)_{\Omega^B}
+ \langle i\omega(\alpha G)^{1/2} u^s_x, v \rangle_{\Gamma_B} = 0, \quad v \in V^h,
\]

\[
-\omega^2(\rho_f u^s_x, w)_{\Omega^B} - \omega^2(g_0 u^f_x, w)_{\Omega^B} + i\omega(\frac{\eta}{k_0}, \frac{L_0}{k_0} u^f_x, w)_{\Omega^B}
= \langle \frac{\eta}{k_0} L_0 E^h_x, w \rangle_{\Omega^B}, \quad w \in W^h. \tag{5.2}
\]
5. THE FINITE ELEMENT PROCEDURE

Uniqueness of the solution of the discrete problem (5.1)-(5.2) follows with the same argument that for the continuous case, since it can be shown that $u^{s,h} \in H_0^1(\Omega^B)$ and we can apply Poincaré’s inequality as before.

The usual steps to transform equations (5.1)-(5.1) in a system of linear equations can be now undertaken. Let $z_j, j = 1,...,N$ be the nodes of the finite element partition $\mathcal{T}^h$ and let $N^B$, with $N^B < N$ be those corresponding to the subsurface. Then if $h_j = z_{j+1} - z_j$, $j = 1,...,N-1$, set the basis functions for $V^h$ as

$$\psi_j(z) = \begin{cases} 
\frac{z - z_{j-1}}{h_j} & \text{if } z_{j-1} \leq z < z_j \\
\frac{z_j - z}{h_j} & \text{if } z_j \leq z < z_{j+1}
\end{cases} \quad j = 1,...,N^B, \quad (5.3)$$

and the basis functions for $W^h$ as

$$\xi_l(z) = \begin{cases} 
1 & \text{if } z_l \leq z < z_{l+1} \\
0 & \text{elsewhere}
\end{cases} \quad l = 1,...,N^B - 1. \quad (5.4)$$

Then write in equations (5.1)-(5.2) $u^s = \sum_{j=1}^{N^B} u_j^s \psi_j(z)$, $u^f = \sum_{l=1}^{N^B-1} u_l^f \xi_l(z)$, $v = \psi_j(z)$, $j = 1,...,N^B$, $w = \xi_l(z), \ l = 1,...,N^B-1$ to get a system of $2N^B - 1$ equations in the $N^B$ $u_j^s$ and $N^B - 1$ $u_l^f$ unknowns. It turns out that the choice of piecewise constant functions for the approximation of the fluid displacements makes it possible to write the unknown fluid displacement coefficients in terms of the solid ones; therefore the final system is a tridiagonal one in the $N^B$ unknowns of the solid displacement.

It must be noticed that the right hand side of this system involves the total electric field $E^h_{ex}$, so that the secondary electric field must be calculated previously to this procedure. Once the $u_j^s$ are obtained, the $u_l^f, \ l = 1,...,N^B-1$ can be straightforwardly calculated.

It must be noticed that all the physical parameters appearing in equations (5.1)-(5.2) must be assigned to each of the $N^B-1$-computational cells so that the resulting tridiagonal matrix is correctly written.

5.2 Maxwell Part

Let $N^h \subset H^1(\Omega), M^h \subset L^2(\Omega)$ be finite element spaces consisting of $C^0$-piecewise linear and piecewise constants over the partition $\mathcal{T}^h$. The finite element procedure to solve Maxwell equations is then formulated as follows:
Find \((E^{s,h}_x, H^{s,h}_y) \in N^h \times M^h\) such that

\[
\left(\frac{\partial E^{s,h}_x}{\partial z}, \phi \right) + (i\omega \mu H^{s,h}_y, \phi) = 0, \quad \phi \in M^h, \tag{5.5}
\]

\[
(\sigma_c E^{s,h}_x, \psi) + \langle aE^{s,h}_x, \psi \rangle - (H^{s,h}_y, \frac{\partial \psi}{\partial z}) = (-J_T, \psi), \quad \psi \in N^h. \tag{5.6}
\]

Uniqueness of the solution of the discrete problem (5.5)-(5.6) follows with a similar argument of that for the continuous case.

The linear system of equations associated with (5.5)-(5.6) can be obtained following the steps outlined above for the case of Biot’s equations (5.1)-(5.2). Notice however that in this case the number of nodes to be considered is \(N\) (not \(N^B\) as above), because the air layer must be taken into account. A tridiagonal system for the \(N\) unknowns \(e_j\) of the secondary electric field is obtained, and the \(N-1\) unknowns \(h_l\) of the magnetic field can be calculated from them. This is however not necessary, because only the total electric field, calculated as \(E^{h}_x(z) = E^{p}(z) + \sum_{j=1}^{N} e_j \psi_j(z)\) is needed to be entered in the source term of equation (5.2).

6 Numerical Examples

7 Conclusions
Bibliography


[40] Lee, S., Cornillon, P. and Campanella, O., Propagation of ultrasound waves through frozen foods, *2002 Annual Meeting and Food Expo., Anaheim (CA)*

[41] Lee, S., Pyrak-Nolte, L. J., Cornillon, P. and Campanella, O., Characterization of frozen orange juice by ultrasound and wavelet analysis, to appear in *Journal of the Science of Food and Agriculture*


Appendix 1: Structure of the Fortran 77 code

The numerical algorithm has been structurated modularly, so as to make easier the task of modifying any of its portions. Subroutines communicate through common blocks; it has been avoided almost completely to use dummy arguments. All common blocks are defined in a file called variables, which is included in all program blocks, making almost all variables of global type.

On the other hand, the program is organized in such a way that the user needs to worry about only the structure of the model and its physical parameters, the finite element procedure remains 'hidden'. The main program, called electrosismica1d.f has the following structure:

```fortran
implicit none
include "variables"
integer info,j,jf
open(unit=99,file='output',status='unknown')

call lectura_de_datos
call geometria
call corriente_electrica
call inicializacion

c

c Loop principal para resolver en el dominio de las FRECUENCIAS

do jf=jw1,jw2
w=2.d0*pi*(float(jf)-.5d0)*delf
call modulo_complejo
call lhs_maxwell
call rhs_maxwell(jf)
call zgtsl(nodes,mccel,mcced,mcceu,rhs_m,info)
do j=1,nodes
epsi(j)=rhs_m(j)
enddo
```
call lhs_biot
call rhs_biot(jf)
nodesb=nodes-vecnodes(1)+1
call zgtsl(nodesb,biotl,biotd,biotu,rhs_b,info)
do j=1,nodesb
us(j)=rhs_b(j)
enddo

Now a brief description of each subroutine

lectura_de_datos: The input file is read.

geometria: The finite element grid is generated.

corriente_electrica: The time structure of the electric current density source and its Fourier transform are calculated.

inicializacion: Some vectors are given the initial value zero.

modulo_complejo: The shear modulus $G$ is made complex.

lhs_maxwell: The left hand side for the tridiagonal system of equations for the electric field is calculated.

rhs_maxwell: The right hand side is calculated

zgtsl: The system is solved

lhs_biot: The left hand side for the tridiagonal system of equations for the solid displacements is calculated.
rhs_biot: The right hand side is calculated

zgtsl: The system is solved

desplazamiento_fluido: The fluid displacements are calculated from the already obtained solid ones.

guarda_desplazamientos: Both solid and fluid displacements are kept to be transformed Fourier back into the time domain.

output: Results are stored.
9 Appendix 2: Sample input file

100.d0  \( z \) coordinate of the Earth’s surface (mts)
1100.d0  \( z \) coordinate of the domain bottom (mts)
3  number of different layers
300.d0  width of each layer (mts)
300.d0
400.d0
2  number of different rocks
1  number of different fluids
1.d0  electrical conductivity first rock (Siemens/m)
.20d0  porosity first rock
1680.d0  density first rock (kg/m\(^3\))
2725.d0  estimated velocity for first shear wave
1.d-12  permeability first rock (m\(^2\))
1.d-9  electrokinetic coupling constant for first rock
100.d0  quality factor first rock (dimensionless)
.01d0  electrical conductivity second rock
.20d0  porosity second rock
1970.d0  density first rock
2930.d0  estimated velocity for second shear wave
1.d-12  permeability second rock
1.d-14  electrokinetic coupling constant second rock
100.d0  quality factor second rock
1000.d0  density first fluid (kg/m\(^3\))
1.d-3  viscosity first fluid (kg/(m s))
1  rock type for first layer
1  fluid type for first layer
2  rock type for second layer
1  fluid type for second layer
1  rock type for third layer
1  fluid type for third layer
2  source type (2=ricker)
30.d0  source central freq. (Hz)
0.75d0  simulation duration (sec)
9. APPENDIX 2: SAMPLE INPUT FILE

00.d0  f1 - filter parameter - (Hz)
00.d0  f2 - filter parameter - (Hz)
56.d0  f3 - filter parameter - (Hz)
60.d0  f4 - filter parameter - (Hz)
1    number of receivers
0.0d0  position of the receivers (w.r.t. surface)
1    number of snapshots
0.10d0 times for the snapshots (sec)