Mur–Nédélec finite element schemes for Maxwell’s equations

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Abstract

We study mapped mass-lumped edge elements for approximating a three-dimensional scattering problem. The elements are mapped Mur–Nédélec second kind elements and the scheme is designed to handle anisotropic, inhomogeneous scatterers with moderately complicated geometry. A novel aspect of the scheme is the use of an unusual choice of the magnetic field finite element that results in a very efficient discrete curl operator. Using a non-standard dispersion analysis, we show that the phase accuracy of the cubic edge element does not deteriorate disastrously when the mesh is deformed. Our numerical experiments (in two dimensions) confirm that the scheme performs well on a distorted mesh and that it is efficient compared to the standard Yee scheme. © 1999 Published by Elsevier Science S.A. All rights reserved.

1. Introduction

The approximation of electrically large scattering problems (problems in which the scatterer is large compared to the wavelength of the incident field) requires the use of a phase accurate numerical scheme. One way to obtain such a scheme is to use higher order elements or even spectral methods [4,7,14]. Another approach uses least squares methods (see e.g. [7]) but we shall not pursue methods of this type here. Despite using such methods, the discrete problem is often rather large and so the scheme needs to be cheap to compute. In this context it can be useful to use an explicit time-stepping strategy and this requires that the discrete scheme be ‘mass-lumped’ so that the appropriate matrices are diagonal or nearly diagonal. For example, the scheme given in [4] used the cubic mass-lumped elements of [10] to obtain a sixth-order accurate phase error. However, the scheme is limited to orthogonal grids and isotropic material properties. Unfortunately, materials are often anisotropic, and so the numerical scheme must be able to deal with anisotropy in a transparent way. Once the scheme can handle anisotropic material properties, a mapping technique can be used to obtain a mapped grid that fits the geometry of a scatterer.

We now summarize the choices made in arriving at the scheme outlined in this paper.

Element geometry. Since we want to use accurate quadrature rules to enforce mass lumping, we choose to use a hexahedral reference element.

Finite elements space (Electric field): (see Section 3.1 for details). We use a cubic edge finite element space for the electric field. We do not use the Nédélec–Whitney edge elements [10] since the degrees of freedom are staggered as in the Yee scheme. Hence, at a quadrature point, not all components of the solution are available without interpolation. This implies that simulating anisotropic materials is difficult if the elements are to be mass-lumped (there is no problem if the elements are not mass-lumped). Instead, we elect to use the second

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family of edge elements due to Mur and Nédélec [9,11]. This family has the disadvantage that hexahedral elements of this type possess spurious modes. Note that provided any jumps in material properties (permeability, permittivity and conductivity) occur at mesh faces, the method can handle the resulting discontinuities in the solution.

**Finite elements space (magnetic field):** (see Section 3.2 for details). The usual choice would be the cubic Nédélec [11] divergence conforming element. However, this element results in a discrete curl matrix that varies from element to element. Thus, the curl matrix must either be stored or computed at each time step. To avoid this cost we propose to use a discontinuous cubic element transformed like the electric field space. This has more degrees of freedom than the standard divergence conforming space but the discrete curl operator is independent of element geometry and can be computed very rapidly.

**Time stepping:** (see Section 3.3 for details). To obtain a rapid time stepping algorithm we use the simple leap-frog scheme. By using quadrature based mass-lumping (see e.g. [4]) we obtain a partially mass-lumped scheme by which we mean that only those degrees of freedom at a particular quadrature point are coupled so the appropriate coupling matrix is thus block diagonal. We can’t use the more efficient corrected leap-frog scheme of [4] because of the use of an absorbing layer to terminate the computation.

**Geometric modeling:** (see Section 3 for details). To fit geometric boundaries (for example the surface of the scatterer or boundaries between different materials) we use mapped hexahedral elements.

**Absorbing boundary condition:** (see Section 2.2 for details). The scattering problem we will shall solve is posed on an infinite domain exterior to the scatterer. Hence, the computational domain must be terminated. We use the Bérenger perfectly matched absorbing layer technique [1], modified according to Cangellaris and Zhao [16].

The dispersion and numerical results in the paper are in two space dimensions. This is to simplify the coding and analysis of the scheme. However, the scheme is described in three dimensions since it is intended to be used in 3D. Now that we have obtained excellent results in two dimensions (as shown in this paper), we are developing the code for three dimensions.

The outline of this paper is as follows. First, in Section 2, we detail exactly the problem and in Section 3 we give details of the finite element method. In Section 4 we discuss the dispersion relation for the cubic finite element scheme paying particular attention to the effects of mesh distortion on the phase error. Unfortunately, a complete error analysis of the finite element method with the perfectly matched layer is currently not possible since the continuous perfectly matched layer is not well understood theoretically at present. In Section 5 we present some numerical results showing the flexibility of our scheme, and finally in Section 6 we make some conclusions from our work.

### 2. The scattering problem

In this section we outline the continuous problem we shall approximate. Then we give some details of the construction of a perfectly matched layer (PML) to truncate the problem. Finally, we give the actual problem we propose to solve by the finite element method.

#### 2.1. The true continuous problem

We suppose that we have a bounded scatterer consisting of a perfectly conducting body covered with a conducting, anisotropic, inhomogeneous material. The perfectly conducting body is contained in the domain $D$ and has boundary $\Gamma_i$. The entire scatterer is contained in the hexahedron $[-L_1, L_1] \times [-L_2, L_2] \times [-L_3, L_3]$.

Initially, no electromagnetic field is present, but at some time $t > 0$ a current source produces an electromagnetic field that scatters off of the coated conductor. We wish to compute this electromagnetic field consisting of the electric field
\( E = E(x, t) = \begin{pmatrix} E_1(x_1, x_2, x_3, t) \\ E_2(x_1, x_2, x_3, t) \\ E_3(x_1, x_2, x_3, t) \end{pmatrix} \)

and magnetic field
\( H = H(x, t) = \begin{pmatrix} H_1(x_1, x_2, x_3, t) \\ H_2(x_1, x_2, x_3, t) \\ H_3(x_1, x_2, x_3, t) \end{pmatrix} \)

which satisfy the Maxwell system:
\[
\begin{align*}
\epsilon \frac{\partial E}{\partial t} + \sigma E - \nabla \times H &= -J, \\
\mu \frac{\partial H}{\partial t} + \nabla \times E &= 0,
\end{align*}
\]

in \( \mathbb{R}^3 \setminus \mathcal{D} \),

(2.1)

where \( J(x, t) \) describes the known current source.

In Eq. (2.1) the constitutive parameters \( \epsilon \) (permittivity), \( \sigma \) (conductivity) and \( \mu \) (permeability) are symmetric, real, piecewise smooth matrix functions of position. The matrices \( \epsilon \) and \( \mu \) are uniformly positive definite and \( \sigma \) is positive semi-definite. Outside the hexahedron \([-L_1, L_1] \times [-L_2, L_2] \times [-L_3, L_3]\) is assumed to be free space so the coefficients are scalar and \( \epsilon = \epsilon_0 > 0 \), \( \mu = \mu_0 > 0 \) and \( \sigma = 0 \). At \( t = 0 \) there is no electromagnetic field present so
\[
E(x, 0) = 0 \quad \text{and} \quad H(x, 0) = 0
\]

for all \( x \) outside \( \mathcal{D} \).

On the surface of the perfectly conducting scatterer \( \Gamma_{\nu} \) the total field has vanishing tangential component so
\[
\nu \times E = 0
\]

where \( \nu \) is the unit outward normal to \( \Gamma_{\nu} \).

The difficulty with the scattering problem we have just outlined (which is well-posed, see e.g. [8]) is that it is to be solved on all of \( \mathbb{R}^3 \setminus \mathcal{D} \). Thus we need to truncate the problem, which we do using a version of the Bérenger PML [1].

### 2.2 Perfectly matched layers

Here, for the convenience of the reader, we outline the use of Zhao–Cangellaris’s modified perfectly matched layer [16]. We shall utilize the change of variables approach due to Chew and Weedon [2] (see also [5,6,12]). This is easiest done in the frequency domain. Formally, we can move to the frequency domain by letting
\[
E = \mathbf{\hat{E}}(x) \exp(-i\omega t), \quad H = \mathbf{\hat{H}}(x) \exp(-i\omega t),
\]

and substituting these expressions in (2.1) assuming we are in free space (so \( \epsilon = \epsilon_0 \), \( \mu = 0 \), \( J = 0 \) and \( \sigma = 0 \)). We obtain the well known time-harmonic system
\[
\begin{align*}
i\omega \mathbf{\epsilon} \mathbf{\hat{E}} - \nabla \times \mathbf{\hat{H}} &= 0, \\
i\omega \mu \mathbf{\hat{H}} + \nabla \times \mathbf{\hat{E}} &= 0.
\end{align*}
\]

We want to absorb waves incident from the region containing the scatterer
\[
\{(x_1, x_2, x_3) | x_i < L_i, \ i = 1, 2, 3\}
\]

so we introduce the complex coordinates \( \mathbf{\hat{x}}_l, \ l = 1, 2, 3 \), defined by
\[
\mathbf{\hat{x}}_l = \begin{cases} 
  x_l & \text{if } |x_l| < L_l \\
  x_l + \int_{|x|}^{L_l} \frac{i\sigma(s)}{\omega} ds & \text{otherwise},
\end{cases}
\]
where \( \sigma_l, l = 1, 2, 3 \) are non-negative functions that will be given later. For now we assume that \( \sigma_l(s) = 0 \) if \( |s| < L, l = 1, 2, 3. \)

Using these stretched coordinates, the time harmonic Maxwell system becomes

\[
-i\omega \varepsilon_0 \hat{E} - \nabla \times \hat{H} = 0, \quad -i\omega \mu_0 \hat{H} + \nabla \times \hat{E} = 0.
\]

where \( \nabla \times \) is the curl in the variables \( (\hat{x}_1, \hat{x}_2, \hat{x}_3) \).

Now, we want to transform back to real coordinates using the chain rule which implies that

\[
\frac{\partial}{\partial \hat{x}_l} = \frac{1}{1 + i\sigma_l/\omega} \frac{\partial}{\partial x_l}, \quad l = 1, 2, 3,
\]

and so we define \( \gamma_l = 1 + i\sigma_l/\omega, l = 1, 2, 3. \) Looking in detail at the resulting equation for \( \hat{E} \) we find that

\[
-i\omega \varepsilon_0 \hat{E}_1 - \frac{1}{\gamma_2} \frac{\partial \hat{H}_2}{\partial x_2} + \frac{1}{\gamma_3} \frac{\partial \hat{H}_3}{\partial x_3} = 0,
\]

\[
-i\omega \varepsilon_0 \hat{E}_2 - \frac{1}{\gamma_3} \frac{\partial \hat{H}_3}{\partial x_3} + \frac{1}{\gamma_1} \frac{\partial \hat{H}_1}{\partial x_1} = 0,
\]

\[
-i\omega \varepsilon_0 \hat{E}_3 - \frac{1}{\gamma_1} \frac{\partial \hat{H}_1}{\partial x_1} + \frac{1}{\gamma_2} \frac{\partial \hat{H}_2}{\partial x_2} = 0.
\]

This system no longer involves a curl like the original Maxwell system. The idea of Zhao and Cangellaris is to transform also the dependent variables. Thus we define \( \hat{E}_l = \gamma_l \hat{E}_l \) and \( \hat{H}_l = \gamma_l \hat{H}_l \) for \( l = 1, 2, 3. \) Using these variables (taking into account that \( \gamma_l \) is independent of \( x_k \) if \( k \neq l \)) shows that

\[
-i\omega \tilde{E} - \nabla \times \hat{H} = 0, \quad -i\omega \tilde{H} + \nabla \times \hat{E} = 0,
\]

where \( \tilde{E} = \varepsilon_0 A \) and \( \tilde{H} = \mu_0 A \) and

\[
A = \begin{pmatrix}
\frac{\gamma_2}{\gamma_3} & 0 & 0 \\
\gamma_1 & \gamma_2 & 0 \\
0 & \gamma_1 & \gamma_2 \\
\end{pmatrix}.
\]

Now, we can introduce the auxiliary Bérenger variables

\[
E^*_l = \frac{\gamma_m \gamma_n}{\gamma_l} E_l, \quad m \neq l, n \neq l, m \neq n,
\]

\[
H^*_l = \frac{\gamma_m \gamma_n}{\gamma_l} H_l, \quad m \neq l, n \neq l, m \neq n.
\]

The modified Maxwell system in the perfectly matched layer is thus

\[
-i\omega \varepsilon_0 E^* - \nabla \times \hat{H} = 0, \\
-i\omega \mu_0 H^* + \nabla \times \hat{E} = 0,
\]

\[
(-i\omega)^2 \hat{E} - i\omega C E - D \hat{E} = (-i\omega)^2 E^* - i\omega GE^*, \\
(-i\omega)^2 \hat{H} - i\omega C \hat{H} + D \hat{H} = (-i\omega)^2 H^* - i\omega GH^*.
\]

where the matrices \( C, D \) and \( G \) are given by

\[
C = \begin{pmatrix}
\sigma_2 + \sigma_3 & 0 & 0 \\
0 & \sigma_1 + \sigma_3 & 0 \\
0 & 0 & \sigma_1 + \sigma_2
\end{pmatrix}, \quad D = \begin{pmatrix}
\sigma_2 \sigma_3 & 0 & 0 \\
0 & \sigma_1 \sigma_3 & 0 \\
0 & 0 & \sigma_1 \sigma_2
\end{pmatrix}.
\]
and

\[
G = \begin{pmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{pmatrix}.
\]

Now, we can move formally back into the time domain to obtain (we drop the tilde and denote the variables as \(E, H, E^*\) and \(H^*\)) the following time dependent system:

\[
\begin{align*}
\epsilon_0 \frac{\partial E^*}{\partial t} - \nabla \times H &= 0, \\
\mu_0 \frac{\partial H^*}{\partial t} + \nabla \times E &= 0, \\
\frac{\partial^2 E}{\partial t^2} + C \frac{\partial E}{\partial t} + DE &= \frac{\partial^2 E^*}{\partial t^2} + C \frac{\partial E^*}{\partial t}, \\
\frac{\partial^2 H}{\partial t^2} + C \frac{\partial H}{\partial t} + DH &= \frac{\partial^2 H^*}{\partial t^2} + C \frac{\partial H^*}{\partial t}.
\end{align*}
\]

These equations describe the electromagnetic field in the PML. To completely specify the scattering problem, we must add the equations near the scatterer which we do in the next section.

2.3. The truncated continuous problem

We can now write down the actual problem we shall approximate. We will solve for the electromagnetic field in the domain

\[
\Omega = (\mathbb{R}^3) \cap \left([-L_1 - \delta_1, L_1 + \delta_1] \times [-L_2 - \delta_2, L_2 + \delta_2] \times [-L_3 - \delta_3, L_3 + \delta_3]\right)
\]

where \(\delta_i > 0, i = 1, 2, 3\), is the thickness of the PML in the \(x_i\) directions (see Fig. 1 for a 2D cross-section of the domain). Let \(\Gamma_\omega\) denote the boundary of the box \([-L_1 - \delta_1, L_1 + \delta_1] \times [-L_2 - \delta_2, L_2 + \delta_2] \times [-L_3 - \delta_3, L_3 + \delta_3]\). We use the Maxwell–Bérenger system (just the Maxwell system near the scatterer) to compute an approximate scattered field \((E_\omega, H_\omega)\) and an auxiliary field \((E^*, H^*)\) which satisfy

Fig. 1. This shows the major elements of the computational domain in two dimensions. The scatterer is obtained in the shaded region and surrounded by an inhomogeneous, anisotropic conductor within the dashed line. The rest of the box contains free space except for the outer annulus near \(\Gamma_\omega\) which is the perfectly matched layer.
\begin{equation}
\begin{aligned}
\epsilon \frac{\partial E^*}{\partial t} + \sigma E^* - \nabla \times H_a = -J, \\
\mu \frac{\partial H^*}{\partial t} + \nabla \times E_a = 0, \\
\frac{\partial^2 E_a}{\partial t^2} + C \frac{\partial E_a}{\partial t} + DE_a = \frac{\partial^2 E^*}{\partial t^2} + G \frac{\partial E^*}{\partial t} \\
\frac{\partial^2 H_a}{\partial t^2} + C \frac{\partial H_a}{\partial t} + DH_a = \frac{\partial^2 H^*}{\partial t^2} + G \frac{\partial H^*}{\partial t}
\end{aligned}
\right\} \text{ in } \Omega.
\end{equation}

The boundary conditions are
\begin{equation*}
\nu \times E_a = 0 \quad \text{on } \Gamma_\nu \quad \text{and} \quad \nu \times E_a = 0 \quad \text{on } \Gamma_\sigma.
\end{equation*}

Here, we could have used an absorbing boundary condition on \( \Gamma_\sigma \) but for simplicity we follow Bérenger [1] and use a perfectly conducting boundary condition. Finally, the initial condition is
\begin{equation*}
E_a = 0, \quad E^* = 0, \quad H_a = 0, \quad \text{and} \quad H^* = 0 \quad \text{for } t \leq 0.
\end{equation*}

Unfortunately, we cannot say if this boundary value problem has a solution or if the solution \((E_a, H_a)\) is close to the exact scattered field \((E, H)\). However, our numerical results are quite promising.

To complete this section, we write down a variational formulation for the continuous problem suitable for discretization by finite elements. This is rather normal since we do not know the correct spaces for the problem. Let
\begin{equation*}
H_0(\text{curl}; \Omega) = \{ u \in (L^2(\Omega))^3 \mid \nabla \times u \in (L^2(\Omega))^3 \quad \text{and} \quad \nu \times u = 0 \text{ on } \partial \Omega \}
\end{equation*}
and let
\begin{equation*}
(u, v) = \int_{\Omega} u \cdot v \, dV.
\end{equation*}

We suppose that at each time \( t \) the field \( E_a(t) \in H_0(\text{curl}; \Omega) \) and \( E^*(t) \in H_0(\text{curl}; \Omega) \). In addition, \( H_a(t) \in (L^2(\Omega))^3 \) and \( H^*(t) \in (L^2(\Omega))^3 \). All fields are assumed to be sufficiently differentiable in time and to satisfy
\begin{equation}
\frac{d}{dt} (eE^*, \phi) + (\sigma E^*, \phi) - (H_a, \nabla \times \phi) = -(J, \phi), \quad \forall \phi \in H_0(\text{curl}; \Omega),
\end{equation}
\begin{equation}
\frac{d^2}{dt^2} (E_a, \psi) + \frac{d}{dt} (CE_a, \psi) + (DE_a, \psi) = \frac{d^2}{dt^2} (E^*, \psi) + \frac{d}{dt} (GE^*, \psi), \quad \forall \psi \in H_0(\text{curl}; \Omega),
\end{equation}
\begin{equation}
\frac{d}{dt} (\mu H^*, \xi) + (\xi, \nabla \times E_a) = 0, \quad \forall \xi \in (L^2(\Omega))^3,
\end{equation}
\begin{equation}
\frac{d^2}{dt^2} (H_a, \eta) + \frac{d}{dt} (CH_a, \eta) + (DH_a, \eta) = \frac{d^2}{dt^2} (H^*, \eta) + \frac{d}{dt} (GH^*, \eta), \quad \forall \eta \in (L^2(\Omega))^3.
\end{equation}

3. An efficient discrete problem

This is the most important section of our paper because it justifies in detail our choice of elements and mass-lumping technique.

We suppose that \( \Omega \) is covered by a mesh denoted by \( M_h \) of elements of maximum diameter \( h \) consisting of regular, quasi-uniform curvilinear hexahedra. Thus
\begin{equation*}
\Omega = \bigcup_{i=1}^{N} K_i
\end{equation*}
where each \( K_i \) is a curvilinear hexahedron. One important restriction on the mesh is that it is assumed to be orthogonal (i.e. consisting of right hexahedra) in the PML layer.
Each element $K_i \in M_h$ is obtained by mapping from the reference element $\hat{K} = [-1, 1]^3$ to $K_i$ using a smooth invertible transformation $F_i$. In our 2D code $F_i$ is a bilinear transformation, but more exotic (for example bi-cubic) transformations are possible. We denote by $DF_i$ the Jacobian matrix of partial derivatives of $F_i$ and by $J_i = \det(DF_i)$. We assume $J_i > 0$ which simply requires that the mappings to be oriented appropriately. We shall use the notation that for any matrix $A$, the matrix $A^T$ is its transpose and $A^{-T}$ is the transpose of the inverse of $A$.

We define $Q_m$ to be the space of polynomials of degree at most $m$ in $x_1$, $m$ in $x_2$ and $m$ in $x_3$. In this paper we shall consider two finite element spaces built on $M_h$:

- Piecewise tri-linear ($m = 1$),
- Piecewise tri-cubic ($m = 3$).

These spaces are two instances of a general class of elements defined by Nédélec [11] which we define next (extending the original definition to curvilinear grids).

3.1. Choice of the electric field spaces and mass lumping

For each degree $m$, the finite element space for the electric field quantities is

$$U_h^{(m)} = \{ u_h \in H_0(\text{curl}; \Omega) \mid u_h|_{K_i} \circ F_i = (DF_i)^{-T} \hat{u} \text{ for some } \hat{u} \in Q_m^3 \},$$

(3.1)

3.1.1. The problem of the mass matrix

Assuming that we have a suitable space $V_h^{(m)}$ for the magnetic field (we discuss this choice soon), the finite element equation for the auxiliary electric field corresponding to (2.3) is to find $E^*_h \in U_h^{(m)}$ such that

$$\frac{d}{dt} (\epsilon E^*_h, \phi_h) + (\sigma E^*_h, \phi_h) - (H_{a,h}, \nabla \times \phi_h) = -(J, \phi_h), \quad \forall \phi_h \in U_h^{(m)},$$

where $H_{a,h} \in V_h^{(m)}$ is an approximation to $H_a$. Now, we focus on the time derivative term. This is evaluated element by element by mapping to the reference element. In view of the relationship between the finite element function on $K_i$ and $\hat{K}$ in (3.1) we have

$$\frac{d}{dt} (\epsilon E^*_h, \phi_h) = \sum_{i=1}^{N} \frac{d}{dt} \int_{K_i} \epsilon E^*_h \cdot \phi_h \, dV$$

$$= \sum_{i=1}^{N} \frac{d}{dt} \int_{K_i} J_i \epsilon (DF_i^{-T} \hat{E}^*_h) \cdot (DF_i^{-T} \hat{\phi}) \, dV$$

$$= \sum_{i=1}^{N} \frac{d}{dt} \int_{K} J_i ((DF_i^{-1} \epsilon DF_i^{-T}) \hat{E}^*_h) \cdot \hat{\phi} \, dV$$

This equality shows that even when $\epsilon$ is diagonal (or constant!) we shall get a non-diagonal mass matrix.

3.1.2. A partial mass-lumping technique

Here, we outline how to obtain a block diagonal mass matrix with small diagonal blocks. Let $\xi_l$, $l = 1, \ldots, m + 1$, be the Gauss-Lobatto quadrature points on $[-1, 1]$ (the resulting quadrature scheme is exact for polynomials of degree $2m$). The set $\hat{\Sigma} = \{ (\xi_l, \xi_p, \xi_q) \mid 1 \leq l, p, q \leq m + 1 \}$ is unisolvent for $Q_m$ on $\hat{K}$. Hence, the three vector degrees of freedom at each point in $\hat{\Sigma}$ are degrees of freedom for the space of vector functions $Q_m^3$ used in constructing $U_h^{(m)}$. Actually, the basis functions on $\hat{K}$ will be of the form

$$\hat{\phi}_{l,p,q}^{(1)} = (\hat{\phi}_{l,p,q}, 0, 0), \quad \hat{\phi}_{l,p,q}^{(2)} = (0, \hat{\phi}_{l,p,q}, 0), \quad \hat{\phi}_{l,p,q}^{(3)} = (0, 0, \hat{\phi}_{l,p,q}),$$

where $\hat{\phi}_{l,p,q}$ is the classical Lagrange basis function associated with the point $(\xi_l, \xi_p, \xi_q)$. Our mass-lumping technique is to use quadrature at the Gauss-Lobatto points to approximate volume integrals. On a particular element $K_i$ we calculate the contribution to the mass matrix by the following approximation:
\[
\int_K ((DF_i^{-1} e(DF_i^{-T}) \hat{\phi}_{i,p,q}^{(r)}, \hat{\phi}_{i',p',q'}^{(r')}) \cdot \hat{\psi}_{i_1,i_2,i_3} \, dV = \sum_{t_1=1}^{m+1} \sum_{t_2=1}^{m+1} \sum_{t_3=1}^{m+1} [J_i,((DF_i^{-1} e(DF_i^{-T}) \hat{\phi}_{i,p,q}^{(r)}), \hat{\phi}_{i',p',q'}^{(r')})](\xi_{t_1}, \xi_{t_2}, \xi_{t_3}) \hat{w}_{t_1,t_2,t_3},
\]

where \( \hat{w}_{t_1,t_2,t_3} \) is the Gauss–Lobatto quadrature weight at \( (\xi_{t_1}, \xi_{t_2}, \xi_{t_3}) \). This formula shows that if \( \hat{\phi}_{i,p,q}^{(r)} \) and \( \hat{\phi}_{i',p',q'}^{(r')} \) are associated with different quadrature points the above contribution will be zero. However, if \( (l, p, q) = (l', p', q') \), we shall in general get a non-zero contribution from degrees of freedom at the given point (regardless of \( r \) and \( r' \)). This produces local mass-matrices defined by the interactions of the degrees of freedom around a given point in the quadrature. If the mesh is in fact orthogonal (for example in the PML) the equations decouple completely and the method is mass lumped in the standard sense.

### 3.2. Choice of the magnetic field spaces and discrete curl matrices

The finite element space for the magnetic field is denoted \( V_h^{(m)} \). Since the space will be discontinuous subspace of \( (L_2(\Omega))^3 \) mass-lumping is not a problem—at most only degrees of freedom associated with a given element will be coupled in the mass matrix. However, the choice of the space can have a major effect on the efficiency of the method. We focus on the discrete curl matrix.

Since we will be concerned with the curl of functions in \( U_h^{(m)} \), first note that an elementary computation shows that if \( \nabla \times \psi \in H_\text{curl}(\Omega) \),

\[
(\nabla \times \psi) \circ F_i = \frac{1}{\det(DF_i)} DF_i \tilde{\nabla} \times \tilde{\psi}.
\]

The usual choice for the space for the magnetic fields \( V_h^{(m)} \) is motivated as follows. Since \( \mu \partial H^*/\partial t = -\nabla \times E_a \) and \( \nabla \cdot (\nabla \times E_a) = 0 \) it is natural to choose the space \( V_h^{(m)} \subset H(\text{div}; \Omega) \) where

\[
H(\text{div}; \Omega) = \{ u \in (L^2(\Omega))^3 \mid \nabla \cdot u \in L^2(\Omega) \}.
\]

This is the choice for example in [10] and is the standard ‘face’ finite element space at lowest order. For the corresponding finite element functions, we are thus forced to transform as follows in order to preserve the divergence. If \( \tilde{\psi} \in H(\text{div}; \tilde{K}) \) and \( \psi \in H(\text{div}; K) \) we have

\[
\psi \circ F_i = \frac{1}{\det(DF_i)} DF_i \tilde{\psi}.
\]

In this case, a suitable choice for \( V_h^{(m)} \subset H(\text{div}; \Omega) \) would be

\[
V_h^{(m)} = \{ u \in H(\text{div}; \Omega) \mid \det(DF_i)DF_i^{-1}u|_{K_i} \circ F_i \in [Q_3]^3 \}
\]

Using the relation defining \( V_h^{(m)} \)

\[
H_a,h|_{K_i} \circ F_i = \frac{1}{J_i} DF_i \tilde{H}
\]

for some \( \tilde{H} \in Q_3^3 \) and the corresponding relation (3.2) for \( \phi_a \), we can change variables in the local curl integral as follows:

\[
\int_{K_i} H_a,h \cdot \nabla \times \phi_a \, dV = \int_K \frac{1}{\det(DF_i)} ((DF_i^\top DF_i) \tilde{H}) \cdot \tilde{\nabla} \times \tilde{\phi} \, dV.
\]

From this equality we can see that using the standard choice for \( V_h^{(m)} \) implies the need to store (or compute at every time step), for each degree of freedom of the mesh, a curl stencil coming from the presence of \( 1/\det(DF_i)DF_i^\top DF_i \) in the integrals. The number of interactions of the degrees of freedom can provide, for higher order finite elements, rather large stencils. Hence this approach can be time consuming or memory intensive.

We can avoid this storage, and decrease the size of the stencil, by using the following identity
This result would hold at the discrete level if \((V_h^{(m)}) \subset H(\text{curl}; \Omega)\), which is not the case for the natural choice of \(V_h^{(m)}\) discussed above. We have tried taking \(V_h^{(m)} = U_h^{(m)}\), but a dispersion analysis shows that the resulting scheme is very dispersive.

Another way to get relation (3.3) is to take \(V_h^{(m)} \subset [L^2(\Omega)]^3\) and to use the \(H(\text{curl}; K_i)\) transform on each hexahedron of the mesh. In other words, we shall set

\[
V_h^{(m)} = \{u \in [L^2(\Omega)]^3 \mid DF_i^T u |_{K_i} \circ F_i = \tilde{u} \text{ for some } \tilde{u} \in Q_m^3\}.
\]

An important consequence of this definition is that by taking the degrees of freedom of \(V_h^{(m)}\) at the same points as those of \(U_h^{(m)}\), we shall get the optimal sparsity for the curl matrix. Actually, all the diagonal interactions between the basis function will be zero. Hence, sparsity will increase with the order of the method. For instance, for \(m = 2\), \(\dim[Q_m]_3 = 81\) and so, the number of possible interactions between the basis functions is \(81 \times 81 = 6561\). The ‘natural’ choice of \(V_h \subset H(\text{div}, \Omega)\) gives 1188 non-zero interactions and the new choice 432 only.

The local curl matrix is now entirely independent of the element. Thus, a single local curl matrix can be computed and used repeatedly to accumulate the curl terms during the iteration. So, the new method not only uses a curl stencil which does not depend on the mesh, but also provides sparse stencils, which will speed up dramatically the algorithm.

### 3.3. Summary of the discrete scheme

The elements we shall use are

\[
U_h^{(m)} = \{u_h \in H_0(\text{curl}; \Omega) \mid u_h|_K \circ F_K = (DF_K)^{-T} \tilde{u} \text{ for some } \tilde{u} \in Q_m^3\},
\]

\[
V_h^{(m)} = \{u_h \in [L^2(\Omega)]^3 \mid u_h|_K \circ F_K = (DF_K)^{-T} \tilde{u} \text{ for some } \tilde{u} \in Q_m^3\},
\]

for \(m = 1\) and \(m = 3\). Corresponding to these spaces are discrete inner products computed by Gauss–Lobatto on the reference element quadrature. For a given element \(K \in M_h\) we define

\[
(u, v)_{e,h,K} = \sum_{j=1}^{(m+1)^3} (DF_K^{-T} \tilde{u})(\tilde{a}_j) \cdot (\varepsilon \circ F_K)(\tilde{a}_j) \cdot (DF_K^{-T} \tilde{v})(\tilde{a}_j) |J_K| |\tilde{a}_j| \tilde{w}_j
\]

where \(\tilde{a}_j, \tilde{w}_j, j = 1 \ldots (m + 1)^3\) are the Gauss–Lobatto points and weights in \(K\). Then

\[
(u, v)_{e,h} = \sum_{i=1}^{N} (u, v)_{e,h,K_i}.
\]

Similar definitions hold for \((u, v)_{u,h}\) and \((u, v)_{\varepsilon,h}\). These are 2m order accurate approximations to the continuous inner products provided \(u, v, \varepsilon\) and \(\mu\) are smooth on each element.

Note that since we have used the positions of the degrees of freedom as quadrature points, the coercivity of these bilinear forms is obvious. Hence, the matrices resulting from the finite element method are invertible.

Next, we detail the fully discrete finite element scheme derived from Eqs. (2.3)–(2.6). To discretize in time we use a simple leapfrog scheme based on the time step \(\Delta t > 0\). For initial conditions we take all the discrete inner products to vanish at \(t = 0\) and \(t = \Delta t/2\). Suppose we have the electric fields \(E_h^{(m)} \in U_h^{(m)}\) and \(E_h^{(m)} \in U_h^{(m)}\) at \(t = t_n = n \Delta t\) and the magnetic fields \(B_h^{(m)} \in V_h^{(m)}\) and \(H_h^{(m+1/2)} \in V_h^{(m)}\) at \(t = t_{n+1/2} = (n + 1/2) \Delta t\). We compute updated fields by first finding the auxiliary electric field \(E_h^{(m+1)} \in U_h^{(m)}\) by solving

\[
\frac{1}{\Delta t} ((H_h^{(m+1)} - H_h^{(m)}), \phi_h)_{e,h} + \frac{1}{2} ((E_h^{(m+1)} + E_h^{(m)}), \phi_h)_{e,h} - (H_h^{(m+1/2)}, \nabla \times \phi_h)_h = -(J_h^{(m+1/2)}, \phi_h)_h
\]

for every \(\phi_h \in U_h^{(m)}\). Having computed the auxiliary variable by solving the above equation for \(E_h^{(m+1)}\) we can then update the physical variable \(E_h^{(m+1)}\) using
\[
\frac{1}{(\Delta t)^2} ((E_{n+1}^{a,h} - 2E_{n}^{a,h} + E_{n-1}^{a,h}), \psi_{h})_{h} + \frac{1}{2 \Delta t} (C(E_{n+1}^{a,h} - E_{n}^{a,h}), \psi_{h})_{h} + (DE_{n}^{a,h}, \psi_{h})_{h}
\]
\[
= \frac{1}{(\Delta t)^2} ((E_{n+1}^{*,n} - 2E_{n}^{*,n} + E_{n-1}^{*,n}), \psi_{h})_{h} + \frac{1}{2 \Delta t} (G(E_{n+1}^{*,n} - E_{n}^{*,n-1}), \psi_{h})_{h}
\]
for every \( \psi_{h} \in \mathcal{U}_{h}^{(m)} \).

Next, we update the magnetic field in a similar way. We first solve for the auxiliary variable \( H_{n+3/2}^{*,a} \) using
\[
\frac{1}{(\Delta t)^2} ((H_{n+3/2}^{*,n} - H_{n+1/2}^{*,n}), \xi_{h})_{\mu,h} + (\nabla \times E_{n+1}^{a,h}, \xi_{h})_{h} = 0
\]
for \( \xi_{h} \in \mathcal{V}_{h}^{(m)} \). Finally, we update \( H_{n+3/2}^{*,a} \) using
\[
\frac{1}{(\Delta t)^2} ((H_{n+3/2}^{*,n+1/2} - 2H_{n+1/2}^{*,n+1/2} + H_{n-1/2}^{*,n+1/2}), \chi_{h})_{h} + \frac{1}{2 \Delta t} (C(H_{n+3/2}^{*,n} - H_{n-1/2}^{*,n}), \chi_{h})_{h} + (DH_{n}^{*,n+1/2}, \chi_{h})_{h}
\]
\[
= \frac{1}{(\Delta t)^2} ((H_{n+3/2}^{*,n+3/2} - 2H_{n+1/2}^{*,n+3/2} + H_{n-1/2}^{*,n+1/2}), \chi_{h})_{h} + \frac{1}{2 \Delta t} (G(H_{n+3/2}^{*,n+3/2} - H_{n+1/2}^{*,n-1/2}), \chi_{h})_{h}
\]
for every \( \chi_{h} \in \mathcal{V}_{h}^{(m)} \). This completes updating all the fields. Summarizing our derivation:

- The matrices associated with the bilinear forms \((\cdot, \cdot)_{\mu,h}, (\cdot, \cdot)_{\nu,h}\) and \((\cdot, \cdot)_{h}\) are all block diagonal with the same block structure. This is because of the use of quadrature at the points used to define the degrees of freedom. Hence, only degrees of freedom associated with a particular quadrature point are coupled. The fact that these matrices are block diagonal with small blocks allows us to solve the linear system rapidly, and we term the system 'partially mass-lumped'.
- By virtue of the previous comment, the Bérenger equations (3.5) and (3.5) guarantee that \( H_{n+1/2}^{*,a} = H_{n+1/2}^{*,a,n} \) and \( E_{n+1/2}^{*,a} = E_{n+1/2}^{*,a,n} \) on all elements outside the PML. Thus, there is no need to solve the auxiliary equations away from the layer, and storage for the auxiliary variables need not be allocated. Indeed away from the PML the system becomes precisely a mass-lumped edge element approximation to the standard Maxwell system.

4. Dispersion analysis

Here, we present the results of a dispersion analysis of the linear and cubic element. The results are for the semi-discrete scheme without time discretization.

The dispersion relation for the method we propose on a uniform rectilinear grid was derived in [3]. If the wave vector is defined by \( \textbf{k} = (k_{1}, k_{2}, k_{3}) \) and the discrete angular frequency is \( \omega_{n} \) the linear scheme \((m = 1)\) has the same 'physical' dispersion relation as the Yee scheme
\[
\omega_{h}^{2} = 4 \sin^{2}\left(\frac{k_{1} h}{2}\right) + 4 \sin^{2}\left(\frac{k_{2} h}{2}\right) + 4 \sin^{2}\left(\frac{k_{3} h}{2}\right)
\]
but in addition has parasitic modes
\[
\omega_{h}^{2} = 4 \sin^{2}\left(\frac{k_{1} h}{2}\right) + 4 \sin^{2}\left(\frac{k_{2} h}{2}\right),
\]
\[
\omega_{h}^{2} = 4 \sin^{2}\left(\frac{k_{2} h}{2}\right) + 4 \sin^{2}\left(\frac{k_{3} h}{2}\right),
\]
\[
\omega_{h}^{2} = 4 \sin^{2}\left(\frac{k_{3} h}{2}\right) + 4 \sin^{2}\left(\frac{k_{1} h}{2}\right).
\]
In the same way the cubic scheme \(m = 3\) has a sixth order accurate 'physical' dispersion relation
\[
\omega_{h}^{2} = \omega_{1,h}(k_{1}) + \omega_{2,h}(k_{2}) + \omega_{3,h}(k_{3})
\]
where
Fig. 2. The unit cells for dispersion analysis: (a) a unit-cell made of two trapezoids, (b) a unit-cell made of four quadrilaterals.

\[
\omega_{i,k}^2(k) = k^2 \left( 1 - \frac{h^6 k^6}{302400} + O(h^8 k^8) \right), \quad i = 1, 2, 3,
\]

together with some parasitic modes.

One observation is that the dispersion relations decouple in terms of one-dimensional results in each of the coordinate directions on an orthogonal grid. In the remainder of this section we shall present a two-dimensional dispersion analysis. This applies also to 3D if the grid is distorted in the \( x_1, x_2 \) plane only.

In particular, we are interested in the effects of grid distortion on the phase velocity of the numerical wave. We shall analyze wave propagation on a mesh made of translates of a particular ‘unit cell’. To analyze mesh distortion, we have employed two non-standard ‘unit cells’. The first is shown in Fig. 2(a). The unit cell is made of two trapezoidal elements. We vary \( \alpha \) (the length of the top edge of the trapezoid) and choose \( \beta \) so that the area of the trapezoid is exactly \( h^2 \) which ensures the same mesh density as we change the geometric parameters. The global mesh is obtained by translates of the unit cell in the obvious way.

The second unit cell is shown in Fig. 2(b). In this case we can choose the coordinates of the center point \((a, b)h\) so that none of the interior sides of the quadrilaterals in the unit cell are parallel. The dimensions of the cell are also chosen to preserve a mesh density of \( 1/h^2 \) elements per unit of area.

4.1. Trapezoidal cell

We start with results for the trapezoidal unit cell shown in Fig. 2(a). The phase velocities shown in Fig. 3 are computed for a mesh distortion parameter \( \alpha = 1/2 \). The graphs show that (at least for this value of \( \alpha \) and the directions considered) the phase velocity of both methods is asymptotically exact as \( h \to 0 \). In fact, the results

Fig. 3. Here, we show the phase velocity for waves along the \( x \)-axis \((k_x = 0)\), along the \( y \)-axis \((k_y = 0)\) and along the line \( k = k_1 \) as a function of the reciprocal number of grid points per wavelength. For linear elements the reciprocal number of grid points per wavelength is defined by \( 2\pi/(kh) \) while for cubic elements it is \( 6\pi/(kh) \) to allow for the greater number of degrees of freedom per cell. The trapezoid is given by \( \alpha = 1/2 \) (see Fig. 2(a)). The results are consistent with \( O(h^2) \) convergence for linear elements and \( O(h^4) \) convergence for cubic elements.
are consistent with an $O(h^2)$ convergence rate of convergence of the phase velocity for the linear scheme and $O(h^6)$ for the cubic scheme. This is the convergence rate for a non-uniform grid and suggests that mesh distortion does not destroy the rate of convergence of the phase velocity. Note that in plotting the results we have compensated for the larger number of unknowns per element in the cubic scheme since we have plotted the phase velocity as a function of reciprocal grid points per wavelength. In the case of bilinear elements this is the same as the number of elements per wavelength. But for the cubic elements there are more unknowns per element which is allowed for in our graphs.

From the results it seems that mesh distortion does not decrease the convergence rate of the phase velocity, and the cubic scheme handles mesh distortion better than the linear scheme.

The second result is for the fixed wave number corresponding to approximately ten grid points per wavelength (i.e. $|k|h = 6/10$ for the linear scheme and $|k|h = 18/10$ for the cubic scheme). We vary $\alpha$ from $3/10$ to 1 (see Fig. 4). The effect on the phase velocity error depends on the direction of propagation of the wave, but generally as $\alpha$ decreases from 1 the phase velocity error grows. However, provided $\alpha \geq 0.8$, the phase error is at most doubled for the directions of propagation we consider. Thus some distortion of the mesh can be tolerated by the method.

4.2. Quadrilateral unit cell

The second set of results is for the composite quadrilateral unit cell in Fig. 2(b). Again, we start by investigating mesh refinement holding the geometric parameters $(a, b)$ fixed at $a = 3/2$ and $b = 3/5$. Results are shown in Fig. 5. For the three wave directions shown in Fig. 2(b), the phase velocity converges to one as $h$
Fig. 6. Here, we show the phase velocity for waves along the x-axis ($k_x = 0$), along the y-axis ($k_y = 0$) and along the line $k_x = k_y$ as functions of the parameter $b$ (we choose $a = 3(b - 1)/2 + 1$. See Fig. 2(b)). For linear elements $|k|h = 6/10$, while for cubic elements $|k|h = 18/10$. This is approximately 10 grid points per wavelength in both cases. Recall that $a = b = 1$ is a square grid.

decreases, so the method is consistent despite the highly distorted grid. Again, the curves are consistent with $O(h^2)$ convergence for the linear scheme and $O(h^6)$ for the cubic scheme.

The final result of this section (Fig. 6) shows the effect of changing the grid on phase velocity. We take $a = 3(b - 1)/2 + 1$ and vary $b$ from 0.7 to 1.3. The wave number is chosen so that $|k|h = 6/10$ for the linear scheme and $|k|h = 18/10$ for the cubic scheme. Obviously, the most accurate phase velocity is when $a = b = 1$, but grid distortion does not completely destroy the phase velocity.

5. Numerical results

Here, we present some numerical results with the purpose of showing that, in accordance with the dispersion analysis, the method handles non-uniform meshes well. In addition we show that the PML works for these higher-order elements and that the method is efficient compared to the standard Yee scheme [15].

5.1. Simple cavity problem

The first problem is a very simple cavity problem. We choose this problem so we can compare our method to the standard Yee FDTD scheme easily, and so that we can evaluate the effects of mesh anisotropy in isolation. No PML or scatterer is present in this problem which is simple wave propagation in a box.

First, we describe the problem to be solved. The parameters $\varepsilon$ and $\mu$ are taken to be unity. The domain is $\Omega = [0, 12] \times [0, 12]$ and we compute the solution up to $t = 50$. Of course the wave speed in this case is unity, so we are integrating for over four transits of the domain. The boundary data is zero and we choose the applied current density to be

$$J(x, t) = \begin{cases} \frac{\gamma(t) \exp(-\gamma^2 t^2)}{2\gamma(t-a)^2 - 1} \exp(-\gamma^2 t^2), & 0 \leq t \leq c \\ 0, & \text{otherwise} \end{cases}$$

Here, $r = |x - x_0|$, $x_0 = (6, 6)$ and the Ricker pulse $g(t)$ is given by

$$g(t) = \begin{cases} 2\gamma(2\gamma(t-a)^2 - 1) \exp(-\gamma^2 t^2), & 0 \leq t \leq c \\ 0, & \text{otherwise} \end{cases}$$

where $\gamma = (\pi/b)^2$ and $a = 1.35$, $b = 1.31$ and $c = 3.48$. The function $g(t)$ is approximately band-limited and has a cutoff frequency corresponding to a wave-length of one [13]. When we quote the number of grid points per wavelength in future, it will be the number of grid points in the unit interval. The forcing function outlined above is a standard example in geophysics.

For this problem, we do not have an easily computable exact solution. Instead, we compute a 'reference solution' using the method with a very fine grid. In this case, the reference solution is computed by using a
regular mesh of 96 × 96 elements (24 points per wavelength) and a time-step equal to \( \Delta t = 0.005 \) (which corresponds to the CFL limit divided by 4).

The meshes used are shown in Fig. 7. We have a coarse and fine regular mesh. In the coarse mesh there are two cells per wavelength, whereas in the fine mesh there are three cells per wavelength. The other two grids have the same number of cells as the coarse and fine grids respectively, but the mesh had been distorted.

The curves shown in Figs. 8–10 are time traces (or seismograms) of the magnetic field at the point (3, 3) in the domain with the source described earlier at the center. The time step is \( \Delta t = 0.041 \) for the coarse meshes and \( \Delta t = 0.0273 \) for the fine meshes. They both correspond to half of the stability coefficient for a regular mesh. We only show the fine grid results for \( 25 \leq t \leq 50 \) since up to that time the curves of the reference solution and the fine grid solution are almost indistinguishable.

The CPU time for the solution on a coarse grid is 15 s and 51 s for a fine grid using a DEC alphastation-500. We have also computed the results using the Yee scheme. The accuracy of our scheme with 2 elements per wavelength (coarse grid) is equivalent to the Yee scheme a grid of 180 × 180 cells which takes 17 s to run. That means that our code is slightly faster than the Yee scheme, but it can handle mesh distortion and anisotropy.

5.2. Scattering calculations

Here, we present some results of calculations of wave propagation in more complex media than in the previous example. We have no exact solutions for these problems. Instead, we present snapshots of the spatial variation of the magnetic field \( H \) at particular times. Some of these calculations use the PML and hence we first give some further details of our implementation of the PML.

The PML functions \( \sigma_1 \) and \( \sigma_2 \) are defined as follows:

\[
\sigma_i = \begin{cases} 
0 & \text{if } |x| \leq L_i \\
\sigma_0 \left( \frac{|x| - L_i}{\delta_i} \right)^2 & \text{otherwise}
\end{cases}
\]
Fig. 8. Time traces of the magnetic field at $x = (3, 3)$ for the regular coarse grid shown in Fig. 7(a). The reference solution is the dashed line.

for $l = 1, 2$ with

$$\sigma_0 = \frac{3c_0}{2\delta} \log(1/r)$$

where $\delta_l = \delta - 1$ is the width of the layer, $c_0 = 1$ and $r = 1/1000$. Using these choices, the amplitude of the waves reflected by the PML is roughly the same as that of the waves generated by the dispersion of the method.
Fig. 9. Time traces of the magnetic field at $x = (3, 3)$ for the regular coarse grid shown in Fig. 7(b). The reference solution is the dashed line. Comparing with Fig. 8, the mesh irregularity has had little effect on the quality of the solution.

5.2.1. First experiment—Anisotropic medium

This experiment is made on a square domain $30 \times 30$ with two subdomains:

- A disk of radius 5 at the center of the domain in which $\varepsilon = \mu = 1$.
- Outside of the disk of radius 5. Here, $\mu = 1$ but $\varepsilon$ is given by the matrix $M$ equal to

$$
M = \begin{pmatrix}
1 & 1/4 \\
1/4 & 2
\end{pmatrix}
$$

(5.1)
Fig. 10. Time traces of the magnetic field at $x = (3, 3)$ for the fine regular grid (top) and fine irregular grid (bottom). These are for $25 < t < 50$. The reference solution is the dashed line. Mesh irregularity has had little effect on the quality of the solution.

The source (which is the same as for the $12 \times 12$ experiments discussed earlier in this section) is located at the center of the domain.

The mesh shown in Fig. 11(a) contains 8384 elements, 201 920 degrees of freedom for $E$ and 134 144 degrees of freedom for $H$. The time step is $\Delta t = 0.0333$. The cost of one iteration in time is 0.2 s.

In Fig. 12 we show two snapshots for this experiment. The mesh is irregular at the center. In Fig. 12(b) the wave has not yet entered the anisotropic region. It should be circular and, despite the irregular grid, the wave is...
propagating correctly. In Fig. 12(c) the wave has entered the region of anisotropy which the method is designed to handle.

5.2.2. Second experiment—conducting scatterer

This experiment is made on a square domain $30 \times 30$ in the center of which is a perfectly conducting ogive of length equal to 8. In the whole domain $\epsilon = \mu = 1$. The PML is active in this experiment.

Fig. 12. In (a) and (b) we show snapshots of the magnetic field at $t = 5$ (a) and $t = 15$ (b) for the anisotropic medium experiment. At $t = 5$ the wave has not entered the inhomogeneous layer. At $t = 15$ (lower) the effect of propagating in an inhomogeneous medium is clearly evident.
The source (which has the same profile as in the previous experiments) is now at the point \((1, 10)\). The domain is surrounded by a PML of width equal to 1 ended by a closed boundary (the PML is roughly one wavelength wide). The PML is included in the snapshots.

The mesh shown in Fig. 11(b) contains 9948 elements, 239,656 degrees of freedom in \(E\) and 159,168 degrees of freedom in \(H\). From this, the PML use 1164 elements and 27,984 degrees of freedom in \(E\). The time step is \(\Delta t = 0.0333\). The cost of one iteration in time is 0.23 s of CPU time. The addition of PML costs about 15% of the CPU time.

Fig. 13. Here, we show two snapshots of the magnetic field scattered off of the ogive. The snapshots are at \(t = 15\) (a) and \(t = 25\) (b).

Fig. 14. Snapshots of the magnetic field at \(t = 15\) (a) and \(t = 25\) (b) for the coated ogive. Compared to Fig. 13 the coating has changed the scattered field noticeably.
The results in Fig. 13 clearly show that the PML is working well for this example and the method can be used to fit the mesh to a scatterer.

5.2.3. Third experiment—coated ogive

This experiment is the same as the previous one except that in the elements around the ogive $e = M$ where $M$ is the anisotropy defined in the first experiment (see (5.1)). The mesh is the same as before (see Fig. 11(b)). Comparing Fig. 13 with Fig. 14, we can see that the thin anisotropic coating has modified the scattered wave.

6. Conclusion

We have shown, by a mixture of non-standard dispersion analysis and numerical tests, that the two-dimensional, mapped, mass-lumped, cubic edge element scheme can handle material anisotropy and grid mapping.

The method now needs to be coded in $\mathbb{R}^3$. In addition, we need to further understand the spurious modes predicted (but not seen) in this method, and derive a rigorous error analysis of the mapped scheme.

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