Finite Element Methods for an Acoustic Well-Logging Problem Associated with a Porous Medium Saturated by a Two-Phase Immiscible Fluid

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A mathematical model is presented concerning wave propagation in a domain that arises in geophysical well-logging problems. The domain consists of a borehole Ω_f surrounded by a porous medium Ω_p. Ω_f is filled with a compressible inviscid fluid, and Ω_p is saturated by a two-phase immiscible fluid. Absorbing boundary conditions for artificial boundaries and boundary conditions on the interface between Ω_f and Ω_p are used. The existence and uniqueness theorems are stated for the resulting initial-boundary value problem. Stability and convergence estimates for a finite element method are also studied. © 1993 John Wiley & Sons, Inc.

I. INTRODUCTION

In this work we analyze the problem of numerical simulation of the wave field generated by a point source in an axisymmetric fluid-filled borehole Ω_f through a porous formation Ω_p saturated by two immiscible fluids. Wave propagation in the porous solid is described by the equations in [1], which allows us to include capillary effects and dissipation of energy due to the relative motion of the fluids with respect to the porous frame. For the artificial boundaries of Ω_f and Ω_p, absorbing boundary conditions are derived which allow the absorption of energy of waves arriving normally to the surfaces. On the contact surface between Ω_f and Ω_p, a boundary condition is used to take into account the mud-cake effects on the wave field. This boundary condition is a generalization of that suggested in [2] for the single-phase case. The special cases of open, sealed, and partially sealed interfaces are also treated in our work.

This article is related to numerous works on the subject. The propagation of waves in a porous solid saturated by a single-phase fluid was studied by Biot in several papers [3–5]. Generation of synthetic full-waveform acoustic logs has been attained via several techniques. In [6] the problem was tested under the assumption that the whole system $\Omega_f \cup \Omega_p$ is homogeneous. The solution was obtained using the so-called discrete wave number approach. The same approach was used in [7] using Biot's equations modified according to homogenization [8,9]. In [10] the same problem is solved, but it was assumed

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that the formation is an elastic solid, and the solution was computed using a finite difference method. In [11] the solution was computed using finite element techniques. In all previous works the solution is computed assuming that the formation is either an elastic solid or a porous solid saturated by a single-phase fluid. A finite element method for the approximation solution of the equations describing the propagation of waves in Ω_p was presented in [12].

The organization of the article is as follows. In Sec. II we present our model with the corresponding partial differential equations and the initial and boundary conditions. In Sec. III we derive the weak form of the problem and give results on existence, uniqueness, and regularity of the solution. In Sec. IV an explicit finite element procedure is presented and results on stability and convergence are stated. Finally, in Sec. V we give a derivation of the absorbing boundary condition for the artificial boundary of Ω_p.

II. MODELING OF THE PROBLEM

We shall consider the propagation of waves in a fluid-filled borehole Ω_f surrounded by a porous medium Ω_p which is saturated by two immiscible, viscous, compressible fluids. These two immiscible fluids may be considered as a nonwetting fluid (oil) and a wetting fluid (water). Ω_f is filled with a third kind of fluid (liquid mud). A compressional point source is excited at a point on the centerline of the borehole, and we will investigate the wave propagations of the fluid in Ω_f and of oil and water together with solid in Ω_p. The whole system is assumed isotropic and radially symmetric around the z axis, located at the center of the borehole. Naturally, cylindrical coordinates (r, θ, z) are chosen to describe the model:

$$\begin{aligned}
\Omega &= \{(r,\theta,z):\ 0 \le r \le R_p,\ 0 \le \theta \le 2\pi,\ 0 \le z \le Z_B\},\\
\Omega_f &= \{(r,\theta,z):\ 0 \le r \le R_f,\ 0 \le \theta \le 2\pi,\ 0 \le z \le Z_B\},\\
\Omega_p &= \{(r,\theta,z):\ R_f \le r \le R_p,\ 0 \le \theta \le 2\pi,\ 0 \le z \le Z_B\}.
\end{aligned}$$

Let the artificial top and bottom boundaries of Ω_f be denoted by Γ_1, those of Ω_p by Γ_{2T} and Γ_{2B}, the artificial outer lateral boundary of Ω_p by Γ_{2L}, and the common boundary between Ω_p and Ω_f by Γ_3. See Fig. 1 for a vertical cross section of Ω for any fixed angle.

Let $u_1 = (u_{1r}, 0, u_{1z})$ denote the fluid displacement in Ω_f. Also, let $u_2^s = (u_{2r}^s, 0, u_{2z}^s)$ be the solid displacement in Ω_p, $\tilde{u}_2^o = (\tilde{u}_{2r}^o, 0, \tilde{u}_{2z}^o)$ the average oil displacement in Ω_p, and $\tilde{u}_2^w = (\tilde{u}_{2r}^w, 0, \tilde{u}_{2z}^w)$ the average water displacement in Ω_p. Denote by $\phi = \phi(x)$ the effective porosity in Ω_p The relative oil and water displacements with respect to the solid frame in Ω_p are then written as

$$u_2^\iota = \phi(x)(\tilde{u}_2^\iota - u_2^s) = (u_{2r}^\iota, 0, u_{2z}^\iota), \qquad \iota = o, w.$$

Here, the subsequently, "ι" stands for scripts "o" for oil, "w" for water, and even sometimes "s" for solid. By the assumption of radial symmetry, the strain tensor $\varepsilon(u_2^s)$ in Ω_p is given in cylindrical coordinates [13] by

$$\begin{aligned}
\varepsilon_{rr}(u_2^s) &= \frac{\partial u_{2r}^s}{\partial r}, \qquad \varepsilon_{\theta\theta}(u_2^s) = \frac{u_{2r}^s}{r}, \qquad \varepsilon_{zz}(u_2^s) = \frac{\partial u_{2z}^s}{\partial z},\\
\varepsilon_{rz}(u_2^s) &= \frac{1}{2}\left(\frac{\partial u_{2r}^s}{\partial z} + \frac{\partial u_{2z}^s}{\partial r}\right),
\end{aligned}$$

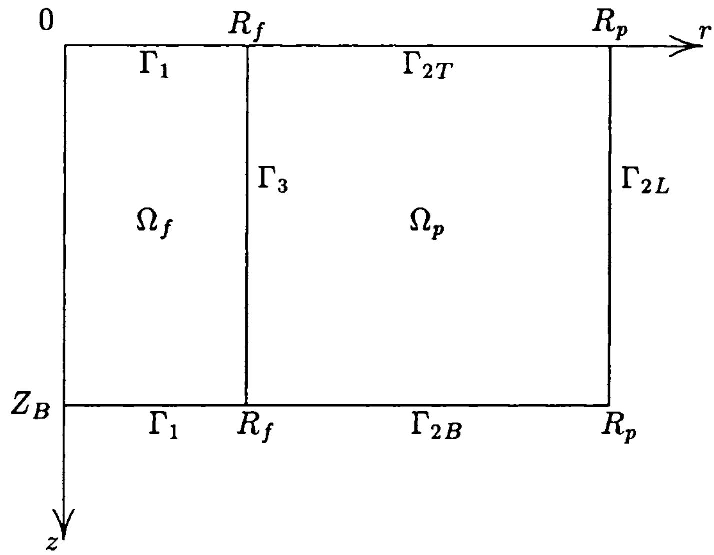

FIG. 1.

$$\varepsilon_{r\theta}(u_2^s) = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_{2r}^s}{\partial\theta} + \frac{\partial u_{2\theta}^s}{\partial r} - \frac{u_{2\theta}^s}{r}\right) = 0,$$

$$\varepsilon_{\theta z}(u_2^s) = \frac{1}{2}\left(\frac{\partial u_{2\theta}^s}{\partial z} + \frac{1}{r}\frac{\partial u_{2z}}{\partial\theta}\right) = 0.$$

Also,

$$\nabla \cdot u_2^s = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = \frac{1}{r}\frac{\partial(ru_{2r}^s)}{\partial r} + \frac{\partial u_{2z}^s}{\partial z}.$$

In the following, "$-$" and "Δ" shall be used to denote the reference quantities associated with the initial equilibrium state and the change in the corresponding quantity with respect to the initial reference quantity, respectively. Let $S_o = S_o(x) = \overline{S}_o + \Delta S_o$ and $S_w = S_w(x) = \overline{S}_w + \Delta S_w$ denote the oil and water saturations in Ω_p, respectively. Here, ΔS_ι represents change in the corresponding saturation with respect to reference saturation $\overline{S}_\iota$ associated with the initial equilibrium, for $\iota = o, w$. Since we are assuming that Ω_p is saturated completely by a mixture of oil and water,

$$S_o + S_w = 1.$$

Similarly, denote by $\tau_{ij} = \tau_{ij}(u_2^s, u_2^o, u_2^w) = \overline{\tau}_{ij} + \Delta\tau_{ij}$ the total stress tensor in the bulk material, and by $\sigma_{ij} = \sigma_{ij}(u_2^s, u_2^o, u_2^w) = \overline{\sigma}_{ij} + \Delta\sigma_{ij}$ the stress tensor in the solid part of Ω_p. Also, let $p_o = \overline{p}_o + \Delta p_o$ and $p_w = \overline{p}_w + \Delta p_w$ be the oil and water pressures. Assume that $\overline{p}_w = 0$. The capillary relation then takes the form [14]:

$$\begin{aligned} p_c = p_c(S_o) &= (\overline{p}_o + \Delta p_o) - (\overline{p}_w + \Delta p_w) \\ &= p_c(\overline{S}_o) + \Delta p_o - \Delta p_w \geq 0, \end{aligned}$$

where the capillary pressure p_c depends only on the (oil) saturation. For practical reasons, we shall always assume that $\overline{S}_o$ satisfies

$$0 < \overline{S}_m \leq \overline{S}_o \leq \overline{S}_M < 1,$$

where $\overline{S}_m$ and $\overline{S}_M$ denote the residual saturations of oil and water, respectively. Then set

$$\beta = \frac{p_c(\overline{S}_o)}{p_c'(\overline{S}_o)}.$$

Since $p_c' > 0$, β is non-negative. Let

$$\sigma_\iota = -\phi S_\iota \Delta p_\iota, \qquad \iota = o, w,$$
$$\sigma = \sigma_o + \sigma_w.$$

Then,

$$\tau_{ij} = \sigma_{ij} + \delta_{ij}\sigma.$$

Following [1], the stress–strain relations in Ω_p can be obtained in the form:

$$\begin{aligned}
\Delta\tau_{rr}(u_2^s, u_2^o, u_2^w) &= 2N\varepsilon_{rr}(u_2^s) + \lambda_c \nabla \cdot u_2^s + B_1 \nabla \cdot u_2^o + B_2 \nabla \cdot u_2^w, \\
\Delta\tau_{\theta\theta}(u_2^s, u_2^o, u_2^w) &= 2N\varepsilon_{\theta\theta}(u_2^s) + \lambda_c \nabla \cdot u_2^s + B_1 \nabla \cdot u_2^o + B_2 \nabla \cdot u_2^w, \\
\Delta\tau_{zz}(u_2^s, u_2^o, u_2^w) &= 2N\varepsilon_{zz}(u_2^s) + \lambda_c \nabla \cdot u_2^s + B_1 \nabla \cdot u_2^o + B_2 \nabla \cdot u_2^w, \\
\Delta\tau_{rz}(u_2^s) &= 2N\varepsilon_{rz}(u_2^s), \\
\Delta\tau_{r\theta} &= \Delta\tau_{\theta z} = 0, \\
((\overline{S}_o + \beta)\Delta p_o - \beta\Delta p_w)(u_2^s, u_2^o, u_2^w) &= -B_1 \nabla \cdot u_2^s - M_1 \nabla \cdot u_2^o - M_3 \nabla \cdot u_2^w, \\
\overline{S}_w \Delta p_w(u_2^s, u_2^o, p_2^w) &= -B_2 \nabla \cdot u_2^s - M_3 \nabla \cdot u_2^o - M_2 \nabla \cdot u_2^w.
\end{aligned} \tag{2.1}$$

In the above expressions, the coefficients N, λ_c, B_1, B_2, M_1, M_2, and M_3 are assumed to be functions of r and z alone. A method was shown in [15] to determine the above elastic coefficients in terms of the properties of the solid and individual fluid phases and the capillary pressure function.

Since the characteristic time needed for a change in saturations for the present case is at least three orders of magnitude greater than the time needed for a change in pressures, the saturations may be assumed to be independent of time for the analysis of dynamic behavior.

The strain energy density $W_p(\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \varepsilon_{rz}, -\nabla \cdot u_2^o, -\nabla \cdot u_2^w)$ in Ω_p is given [1] by

$$\begin{aligned}
W_p(\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \varepsilon_{rz}, \nabla \cdot u_2^o, \nabla \cdot u_2^w) = \frac{1}{2}[&\Delta\tau_{rr}\varepsilon_{rr} + \Delta\tau_{\theta\theta}\varepsilon_{\theta\theta} + \Delta\tau_{zz}\varepsilon_{zz} \\
&+ 2\Delta\tau_{rz}\varepsilon_{rz} + ((\overline{S}_o + \beta)\Delta p_o - \beta\Delta p_w) \\
&\cdot (-\nabla \cdot u_2^o) + (\overline{S}_w \Delta p_w)(-\nabla \cdot u_2^w)].
\end{aligned} \tag{2.2}$$

Recall that W_p is a positive-definite quadratic form in $\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \varepsilon_{rz}, -\nabla \cdot u_2^o$, and $-\nabla \cdot u_2^w$. Set

$$\hat{Y} = (\Delta\tau_{rr}, \Delta\tau_{\theta\theta}, \Delta\tau_{zz}, \Delta\tau_{rz}, (\overline{S}_o + \beta)\Delta p_o - \beta\Delta p_w, \overline{S}_w\Delta p_w),$$
$$\hat{Z} = (\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \varepsilon_{rz}, -\nabla \cdot u_2^o, -\nabla \cdot u_2^w).$$

Then, due to (2.1) and (2.2), the following relation holds

$$W_p(r,z) = \frac{1}{2}[\hat{E}_p,\hat{Z},\hat{Z}]_e, \tag{2.3}$$

for a symmetric, positive-definite matrix $\hat{E}_p(r,z) \in R^{6\times 6}$ associated with W_p, where $[\cdot,\cdot]_e$ denotes the usual scalar product in R^n.

For any self-adjoint matrix $D(r,z) \in R^{n\times n}$, let $\lambda_{\min}(D(r,z))$ and $\lambda_{\max}(D(r,z))$ denote the minimum and maximum eigenvalues of $D(r,z)$ and set

$$\lambda_{\min}(D) = \inf_{r,z}\{\lambda_{\min}(D(r,z))\},\ \lambda_{\max}(D) = \sup_{r,z}\{\lambda_{\max}(D(r,z))\}.$$

We are interested in the case such that

$$0 < \lambda_{\min}(\hat{E}_p) \leq \lambda_{\max}(\hat{E}_p) < \infty,$$

and consequently,

$$\begin{aligned} W_p(r,z) \geq \frac{\lambda_{\min}(\hat{E}_p)}{2}\big[&(\varepsilon_{rr})^2 + (\varepsilon_{\theta\theta})^2 + (\varepsilon_{zz})^2 \\ &+(\varepsilon_{rz})^2 + (\nabla\cdot u_2^o)^2 + (\nabla\cdot u_2^w)^2\big] \\ \geq \frac{\lambda_{\min}(\hat{E}_p)}{4}\big[&(\varepsilon_{rr})^2 + (\varepsilon_{\theta\theta})^2 + (\varepsilon_{zz})^2 \\ &+2(\varepsilon_{rz})^2 + (\nabla\cdot u_2^o)^2 + (\nabla\cdot u_2^w)^2\big]. \end{aligned} \tag{2.4}$$

Let ρ_s, ρ_o, and ρ_w represent the mass densities of solid, oil, and water in Ω_p, respectively. In Ω_p the mass density ρ of the bulk material is defined by

$$\rho = (1-\phi)\rho_s + \phi(\rho_o S_o + \rho_w S_w).$$

Because we are interested in investigating wave propagations in the low-frequency range, the relative flow oil inside the pores can be assumed to be of laminar type so that the relative microvelocities v_i^o and v_i^w satisfy

$$v_{2i}^o = a_{ij}\frac{\partial u_{2j}^o}{\partial t} + b_{ij}\frac{\partial u_{2j}^2}{\partial t}, \qquad v_{2i}^w = c_{ij}\frac{\partial u_{2j}^o}{\partial t} + d_{ij}\frac{\partial u_{2j}^w}{\partial t},$$

and then the components of the matrices $g_\alpha = \overline{g}_\alpha$, $\alpha = 1,2,3$, are defined by

$$\begin{aligned} g_{1_{ij}} &= \rho_o S_o \int_{\Omega_p} a_{ki}a_{kj}dx + \rho_w S_w \int_{\Omega_p} c_{ki}c_{kj}dx, \\ g_{2_{ij}} &= \rho_o S_o \int_{\Omega_p} b_{ki}b_{kj}dx + \rho_w S_w \int_{\Omega_p} d_{ki}d_{kj}dx, \\ g_{3_{ij}} &= \rho_o S_o \int_{\Omega_p} a_{ki}b_{kj}dx + \rho_w S_w \int_{\Omega_p} c_{ki}d_{kj}dx. \end{aligned}$$

The microgeometric coefficient matrices a, b, and c depend on tortuosity matrices of the pores. Since the porous medium is assumed isotropic, tortuosity matrices reduce to scalar multiples of the identity matrix [5,16] so that $g_{\alpha_{ij}}$ can be given by

$$g_{\alpha_{ij}} = g_\alpha \delta_{ij}, \qquad \alpha = 1,2,3,$$

where δ_{ij} is the Kronecker symbol. Assume that

$$\rho - \frac{\overline{g}_2(\rho_o \overline{S}_o)^2 + \overline{g}_1(\rho_w \overline{S}_w)^2 - 2\overline{g}_3 \rho_o \overline{S}_o \rho_w \overline{S}_w}{\overline{g}_1 \overline{g}_2 - \overline{g}_3^2} > 0, \qquad (r,\theta,z) \in \Omega_p. \tag{2.5}$$

Denote by μ_ι, $\iota = o, w$, the viscosities of oil and water, and by $k_{r\iota}$, $\iota = o, w$, the relative permeability of oil and water, respectively, and by k the absolute permeability of the porous medium.

Finally, in the borehole Ω_f, let $\rho = \rho_f(r,z)$ and $A_f = A_f(r,z)$ denote the mass density and the incompressibility modulus of the fluid in Ω_f, which are assumed to be bounded above and below by positive constants:

$$0 < \rho_{f^*} \leq \rho_f(r,z) \leq \rho_f^* < \infty,$$
$$0 < A_{f^*} \leq A_f(r,z) \leq A_f^* < \infty.$$

In order to formulate a differential system, we shall adopt standard equations of motion for a compressible, inviscid, inhomogeneous fluid in the borehole Ω_f, and the generalized Biot's dynamic equations [1] for wave propagation in the porous medium Ω_p saturated by two immiscible, viscous, compressible fluids. Set $J = (0,T)$ and let

$$u_1^0 = (r,z) = (u_{1r}^0, 0, u_{1z}^0), \qquad v_1^0(r,z) = (v_{1r}^0, 0, v_{1z}^0), \qquad f_1(r,z,t) = (f_{1r}, 0, f_{1z}),$$

for $(r,\theta,z) \in \Omega_f$, $(r,\theta,z,t) \in \Omega_f \times J$, and

$$u_2^{\iota,0}(r,z) = \left(u_{2r}^{\iota,0}, 0, u_{2z}^{\iota,0}\right), \qquad v_2^{\iota,0}(r,z) = \left(v_{2r}^{\iota,0}, 0, v_{2z}^{\iota,0}\right), \qquad f_2^\iota(r,z,t) = (f_{2r}^\iota, 0, f_{2z}^\iota),$$

for $(r,\theta,z) \in \Omega_p$, $(r,\theta,z,t) \in \Omega_p \times J$, $\iota = s, o, w$, be given initial and inhomogeneous data which are radially symmetric around the z axis.

We are then interested in solving the following initial-boundary value problem: Find

$$u(r,z,t) = (u_1, u_2^s, u_2^o, u_2^w), \qquad t \in J,$$

such that

$$\text{(i)} \qquad \rho_f \frac{\partial^2 u_{1r}}{\partial t^2} - \frac{\partial}{\partial r}(A_f \nabla \cdot u_1) = f_{1r}(r,z,t),$$

$$\text{(ii)} \qquad \rho_f \frac{\partial^2 u_{1z}}{\partial t^2} - \frac{\partial}{\partial z}(A_f \nabla \cdot u_1) = f_{1z}(r,z,t), \quad \text{for} \quad (r,\theta,z,t) \in \Omega_f \times J,$$

and

$$\text{(iii)} \qquad \rho \frac{\partial^2 u_{2r}^s}{\partial t^2} + \rho_o \overline{S}_o \frac{\partial^2 u_{2r}^o}{\partial t^2} + \rho_w \overline{S}_w \frac{\partial^2 u_{2r}^w}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}(r \Delta\tau_{rr}(u_2^s, u_2^o, u_2^w))$$
$$- \frac{\partial}{\partial z}(\Delta\tau_{rz}(u_2^s)) + \frac{1}{r}\Delta\tau_{\theta\theta}(u_2^s, u_2^o, u_2^w) = f_{2r}^s(r,z,t),$$

$$\text{(iv)} \qquad \rho \frac{\partial^2 u_{2z}^s}{\partial t^2} + \rho_o \overline{S}_o \frac{\partial^2 u_{2z}^o}{\partial t^2} + \rho_w \overline{S}_w \frac{\partial^2 u_{2z}^w}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}(r \Delta\tau_{rz}(u_2^s))$$

$$-\frac{\partial}{\partial z}(\Delta\tau_{zz}(u_2^s,u_2^o,u_2^w)) = f_{2z}^s(r,z,t), \tag{2.6}$$

(v) $$\rho_o\overline{S}_o\frac{\partial^2 u_{2r}^s}{\partial t^2} + \overline{g}_1\frac{\partial^2 u_{2r}^o}{\partial t^2} + \overline{g}_3\frac{\partial^2 u_{2r}^w}{\partial t^2} + \frac{\overline{S}_o^2\mu_o}{kk_{ro}}\frac{\partial u_{2r}^o}{\partial t} + \frac{\partial}{\partial r}(((\overline{S}_o+\beta)\Delta p_o - \beta\Delta p_w)(u_2^s,u_2^o,u_2^w)) = f_{2r}^o(r,z,t),$$

(vi) $$\rho_o\overline{S}_o\frac{\partial^2 u_{2z}^s}{\partial t^2} + \overline{g}_1\frac{\partial^2 u_{2z}^o}{\partial t^2} + \overline{g}_3\frac{\partial^2 u_{2z}^w}{\partial t^2} + \frac{\overline{S}_o^2\mu_o}{kk_{ro}}\frac{\partial u_{2z}^o}{\partial t} + \frac{\partial}{\partial z}(((\overline{S}_o+\beta)\Delta p_o - \beta\Delta p_w)(u_2^s,u_2^o,u_2^w)) = f_{2z}^o(r,z,t),$$

(vii) $$\rho_w\overline{S}_w\frac{\partial^2 u_{2r}^s}{\partial t^2} + \overline{g}_3\frac{\partial^2 u_{2r}^o}{\partial t^2} + \overline{g}_2\frac{\partial^2 u_{2r}^w}{\partial t^2} + \frac{\overline{S}_w^2\mu_w}{kk_{rw}}\frac{\partial u_{2r}^w}{\partial t} + \frac{\partial}{\partial r}(\overline{S}_w\Delta p_w(u_2^s,u_2^o,u_2^w)) = f_{2r}^w(r,z,t),$$

(viii) $$\rho_w\overline{S}_w\frac{\partial^2 u_{2z}^s}{\partial t^2} + \overline{g}_3\frac{\partial^2 u_{2z}^o}{\partial t^2} + \overline{g}_2\frac{\partial^2 u_{2z}^w}{\partial t^2} + \frac{\overline{S}_w^2\mu_w}{kk_{rw}}\frac{\partial u_{2z}^w}{\partial t} + \frac{\partial}{\partial z}((\overline{S}_w\Delta p_w(u_2^s,u_2^o,u_2^w)) = f_{2z}^w(r,z,t),$$

$$\text{for} \quad (r,\theta,z,t)\in\Omega_p\times J,$$

with boundary conditions

(i) $$-A_f\nabla\cdot u_1 = \sqrt{\rho_f A_f}\,\frac{\partial u_1}{\partial t}\cdot\nu_f, \quad \text{on} \quad \Gamma_1\times J,$$

(ii) $$\left(-\Delta\tau\nu_p\cdot\nu_p, -\Delta\tau\nu_p\cdot\chi_p^1, -\Delta\tau\nu_p\cdot\chi_p^2, ((\overline{S}_o+\beta)\Delta p_o - \beta\Delta p_w), \overline{S}_w\Delta p_w\right)^t = B\left(\frac{\partial u_2^s}{\partial t}\cdot\nu_p, \frac{\partial u_2^s}{\partial t}\cdot\chi_p^1, \frac{\partial u_2^s}{\partial t}\cdot\chi_p^2, \frac{\partial u_2^o}{\partial t}\cdot\nu_p, \frac{\partial u_2^w}{\partial t}\cdot\nu_p\right)^t, \quad \text{on} \quad \Gamma_2\times J,$$

(iii) $$\Delta\tau\nu_p + A_f\nabla\cdot u_1\nu_f = 0, \quad \text{on} \quad \Gamma_3\times J, \tag{2.7}$$

(iv) $$(u_2^s + \overline{S}_o u_2^o + \overline{S}_w u_2^w)\cdot\nu_p + u_1\cdot\nu_f = 0, \quad \text{on} \quad \Gamma_3\times J,$$

(v) $$A_f\nabla\cdot u_1 = -\Delta p_\iota + m_\iota\frac{\partial u_2^\iota}{\partial t}\cdot\nu_p, \qquad \iota = o,w, \quad \text{on} \quad \Gamma_3\times J,$$

and initial conditions

(i) $$u_1(r,z,0) = u_1^0(r,z), \qquad (r,\theta,z)\in\Omega_f,$$

(ii) $$\frac{\partial u_1}{\partial t}(r,z,0) = v_1^0(r,z), \qquad (r,\theta,z)\in\Omega_f, \tag{2.8}$$

(iii) $$u_2^\iota(r,z,0) = u_2^{\iota,0}, \qquad (r,\theta,z)\in\Omega_p, \qquad \iota = s,o,w,$$

(iv) $$\frac{\partial u_2^\iota}{\partial t}(r,z,0) = v_2^{\iota,0}(r,z), \qquad (r,\theta,z)\in\Omega_p, \qquad \iota = s,o,w.$$

In the above, $\nu_i = (\nu_{ir},\nu_{i\theta},\nu_{iz}) = (\nu_{ir},0,\nu_{iz})$, $i = f,p$, denotes the unit outward normal along $\partial\Omega_i$ and χ_p^m, $m = 1,2$, orthogonal unit tangent vectors along $\partial\Omega_p$ chosen in a canonical fashion as follows:

on Γ_{2L}, $\nu_p = (1,0,0)$, $\chi_p^1 = (0,1,0)$, $\chi_p^2 = (0,0,1)$,

on Γ_{2T}, $\nu_p = (0,0,-1)$, $\chi_p^1 = (1,0,0)$, $\chi_p^2 = (0,1,0)$,

on Γ_{2B}, $\nu_p = (0,0,1)$, $\chi_p^1 = (1,0,0)$, $\chi_p^2 = (0,1,0)$,

on Γ_3, $\nu_p = (-1,0,0)$, $\chi_p^1 = (0,0,1)$, $\chi_p^2 = (0,1,0)$.

Also, $\tau\nu_p$ denotes the surface traction on $\partial\Omega_p$, and $\tau\nu_p \cdot \nu_p$ and $\tau\nu_p \cdot \chi_p^m$, $m = 1,2$, are the normal and two tangent components of $\tau\nu_p$ on $\partial\Omega_p$.

Explanations of the above boundary conditions are as follows:

Condition (2.7.i), which is an absorbing boundary condition, is the equation of momentum on Γ_1 and, consequently, waves arriving normally at Γ_1 pass through completely.

Condition (2.7.ii) is another absorbing boundary condition on the artificial boundary Γ_2 whose effect is to absorb the energy of waves arriving normally to the boundary Γ_2. The definition of the positive definite, symmetric matrix B and the derivation of the boundary condition are given in Sec. V.

Condition (2.7.iii) corresponds to the continuity of the normal total stress and the vanishing of tangential stresses along Γ_3.

Condition (2.7.iv) comes from the continuity of the normal displacement on Γ_3;

$$u_1 \cdot \nu_f + \left[(1-\phi)u_2^s + \phi\left(\overline{S}_o\tilde{u}_2^o + \overline{S}_w\tilde{u}_2^w\right)\right] \cdot \nu_p = 0,$$

which in turn implies that

$$u_1 \cdot \nu_f + \left[u_2^s + \phi\left(\overline{S}_o(\tilde{u}_2^o - \tilde{u}_2^s) + \overline{S}_w(\tilde{u}_2^w - \tilde{u}_2^s)\right)\right] \cdot \nu_p = 0.$$

Condition (2.7.v) states that the acoustic fluid-flow velocities across the borehole wall are related to the acoustic pressure differences between the borehole fluid in Ω_f and oil and water in Ω_p by simple surface impedance functions $m_\iota = m_\iota(z)$, $\iota = o, w$. The surface impedances on Γ_3 represents the effect of the mud-cake on the wave field. (See [2] for more physical explanations.) The boundary conditions given by (2.7.v) implies that the capillary pressure on the interface Γ_3 is the difference between the two acoustic fluid velocity components in the normal direction multiplied by their corresponding impedances:

$$\Delta p_c\left(m_o\frac{\partial u_2^o}{\partial t} - m_w\frac{\partial u_2^w}{\partial t}\right) \cdot \nu_p .$$

The well-posedness of the differential system is guaranteed under the following condition on the oil and water impedances:

$$4\overline{S}_w(\beta + \overline{S}_o)m_o - \beta^2 m_w \geq 0. \tag{2.9}$$

Only the case $0 < m_{\iota *} \leq m_\iota(z) \leq m_\iota^* < \infty$ is described for our model in this section. The special cases in which $m_\iota = 0$ or ∞ will be analyzed at the end of Sec. III.

III. THE EXISTENCE AND UNIQUENESS RESULTS

For $i = f, p$ let $(\cdot,\cdot)_i$ and $\|\cdot\|_{0,\Omega_i}$ denote the inner product and norm in $L^2(\Omega_i)$. For any $\Gamma \subset \partial\Omega_i$ let $\langle\cdot,\cdot\rangle_\Gamma$ denote the inner product in $L^2(\Gamma)$. The inner product and norm in

$[L^2(\Omega_i)]^3$ are denoted as follows: for $\varphi = (\varphi_r, \varphi_\theta, \varphi_z)$ and $\psi = (\psi_r, \psi_\theta, \psi_z)$,

$$(\varphi, \psi)_i = (\varphi_r, \psi_r)_i + (\varphi_\theta, \psi_\theta)_i + (\varphi_z, \psi_z)_i$$

and

$$\|\varphi\|_{0,\Omega_i} = [(\varphi, \varphi)_i]^{1/2}.$$

Set

$$\tilde{H}(\mathrm{div}, \Omega_i) = \{\varphi = (\varphi_r, \varphi_\theta, \varphi_z) \in H(\mathrm{div}, \Omega_i) : \ \varphi_\theta = 0\},$$

which is a closed subspace of $H(\mathrm{div}, \Omega_i)$. Considering the cylindrical symmetry, we set

$$\begin{aligned}[\tilde{H}^1(\Omega_p)]^3 &= \left\{\varphi = (\varphi_r, \varphi_\theta, \varphi_z) \in [H^1(\Omega_p)]^3 : \ \varphi_\theta = 0, \frac{\partial \varphi_r}{\partial \theta} = \frac{\partial \varphi_z}{\partial \theta} = 0\right\} \\ &= \left\{\varphi = (\varphi_r, \varphi_\theta, \varphi_z) \in [H^1(\Omega_p)]^3 : \ \varphi_\theta = 0, \varepsilon_{r\theta}(\varphi) = \varepsilon_{\theta z}(\varphi) = 0\right\},\end{aligned}$$

which is a closed subspace of $[H^1(\Omega_p)]^3$. If $\varphi \in [\tilde{H}^1(\Omega_p)]^3$,

$$\begin{aligned}\|\varphi\|_{1,\Omega_p} = \Bigg[\int_{\Omega_p} \Bigg[&(\varphi_r)^2 + (\varphi_z)^2 + \left(\frac{\partial \varphi_r}{\partial r}\right)^2 + \left(\frac{\varphi_r}{r}\right)^2 + \left(\frac{\partial \varphi_z}{\partial z}\right)^2 \\ &+ \left(\frac{\partial \varphi_z}{\partial r}\right)^2 + \left(\frac{\partial \varphi_r}{\partial z}\right)^2\Bigg] r\, dr\, d\theta\, dz\Bigg]^{1/2}.\end{aligned}$$

Now let us introduce a separable Hilbert space

$$\tilde{V} = \tilde{H}(\mathrm{div}, \Omega_f) \times [\tilde{H}^1(\Omega_p)]^3 \times \tilde{H}(\mathrm{div}, \Omega_p) \times \tilde{H}(\mathrm{div}, \Omega_p)$$

under the natural norm

$$\|v\|_{\tilde{V}} = \left[\|v_1\|^2_{\tilde{H}(\mathrm{div},\Omega_f)} + \|v_2\|^2_{1,\Omega_p} + \|v_3\|^2_{\tilde{H}(\mathrm{div},\Omega_p)} + \|v_4\|^2_{\tilde{H}(\mathrm{div},\Omega_p)}\right]^{1/2}.$$

Then the space V of admissible test function can be chosen as follows:

$$V = \{v = (v_1, v_2, v_3, v_4) \in \tilde{V} : \ (v_2 + \overline{S}_o v_3 + \overline{S}_w v_4 - v_1) \cdot \nu_p = 0 \quad \text{on} \quad \Gamma_3\},$$

where $\overline{S}_o$ and $\overline{S}_w$ are initially given saturations. Here, the boundary condition (2.7.*iv*) is strongly imposed. Notice that V is a closed subspace of $\tilde{V}$ (with the same norm).

The weak formulation of problem (2.6)–(2.8) is given as follows:

Find $u = (u_1, u_2^s, u_2^o, u_2^w) \in V$ such that

$$\begin{aligned}&\left(\rho_f \frac{\partial^2 u_1}{\partial t^2}, v_1\right)_f + \left(A \frac{\partial^2 (u_2^s, u_2^o, u_2^w)}{\partial t^2}, (v_2, v_3, v_4)\right)_p \\ &+ \left(C \frac{\partial (u_2^s, u_2^o, u_2^w)}{\partial t}, (v_2, v_3, v_4)\right)_p + \Lambda(u, v) \\ &+ \left\langle \sqrt{\rho_f A_f} \frac{\partial u_1}{\partial t} \cdot \nu_f, v_1 \cdot \nu_f \right\rangle_{\Gamma_1} \\ &+ \left\langle B\left(\frac{\partial u_2^s}{\partial t} \cdot \nu_p, \frac{\partial u_2^s}{\partial t} \cdot \chi_p^1, \frac{\partial u_2^s}{\partial t} \cdot \chi_p^2, \frac{\partial u_2^o}{\partial t} \cdot \nu_p, \frac{\partial u_2^w}{\partial t} \cdot \nu_p\right)^t,\right.\end{aligned} \tag{3.1}$$

$$\left(v_2 \cdot \nu_p, v_2 \cdot \chi_p^1, v_2 \cdot \chi_p^2, v_3 \cdot \nu_p, v_4 \cdot \nu_p\right)^t\Big\rangle_{\Gamma_2}$$

$$+\left\langle m_o(\beta + \overline{S}_o)\frac{\partial u_2^o}{\partial t} \cdot \nu_p, v_3 \cdot \nu_p\right\rangle_{\Gamma_3} - \left\langle m_w\beta\frac{\partial u_2^w}{\partial t} \cdot \nu_p, v_3 \cdot \nu_p\right\rangle_{\Gamma_3}$$

$$+\left\langle m_w\overline{S}_w\frac{\partial u_2^w}{\partial t} \cdot \nu_p, v_4 \cdot \nu_p\right\rangle_{\Gamma_3}$$

$$= (f_1, v_1)_f + \left((f_2^s, f_2^o, f_2^w), (v_2, v_3, v_4)\right)_p, \qquad v = (v_1, v_2, v_3, v_4) \in V, \qquad t \in J,$$

where $\Lambda(v, w)$ is the symmetric, bilinear form on $\tilde{V}$ defined by

$$\begin{aligned}\Lambda(v, w) = {} & (A_f\nabla \cdot v_1, \nabla \cdot w_1)_f + (\Delta\tau_{rr}(v_2, v_3, v_4), \varepsilon_{rr}(w_2))_p \\ & + (\Delta\tau_{\theta\theta}(v_2, v_3, v_4), \varepsilon_{\theta\theta}(w_2))_p + (\Delta\tau_{zz}(v_2, v_3, v_4), \varepsilon_{zz}(w_2))_p \\ & + 2(\Delta\tau_{rz}(v_2), \varepsilon_{rz}(w_2))_p \\ & + (((\overline{S}_o + \beta)\Delta p_o - \beta\Delta p_w), -\nabla \cdot v_3)_p \\ & + (\overline{S}_w\Delta p_w, -\nabla \cdot v_4)_p, \quad \text{for} \quad v, w \in \tilde{V},\end{aligned}$$

and the symmetric, positive-definite [due to (2.5)] mass matrix $A(r, z)$ and the non-negative dissipation matrix $C(r, z)$ are given by

$$A = \begin{bmatrix} \rho I & \rho_o\overline{S}_o I & \rho_w\overline{S}_w I \\ \rho_o\overline{S}_o I & \overline{g}_1 I & \overline{g}_3 I \\ \rho_w\overline{S}_w I & \overline{g}_3 I & \overline{g}_2 I \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{(\overline{S}_o)^2\mu_o}{kk_{ro}} I & 0 \\ 0 & 0 & \frac{(\overline{S}_w)^2\mu_w}{kk_{rw}} I \end{bmatrix},$$

I being the 3×3 identity matrix.

Recalling Korn's second inequality [17–19] and owing to (2.2) and (2.4), one can get the following estimate:

$$\begin{aligned}\Lambda(v, v) \geq {} & A_{f*}\|\nabla \cdot v_1\|_{0,\Omega_f}^2 + \frac{\lambda_{\min}(\hat{E}_p)}{2}\int_{\Omega_p}\Big[(\varepsilon_{rr}(v_2))^2 + (\varepsilon_{\theta\theta}(v_2))^2 \\ & + (\varepsilon_{zz}(v_2))^2 + 2(\varepsilon_{rz}(v_2))^2 \\ & + (\nabla \cdot v_3)^2 + (\nabla \cdot v_4)^2\Big] r\, dr\, d\theta\, dz \\ \geq {} & C_1\|v\|_{\tilde{V}}^2 - C_2\Big(\|v_1\|_{0,\Omega_f}^2 + \|v_2, v_3, v_4\|_{0,\Omega_p}^2\Big), \quad v \in \tilde{V}.\end{aligned}$$

Next, set

$$Q_n^2 = \left\|\frac{\partial^n f_1}{\partial t^n}\right\|_{L^2(J;[L^2(\Omega_f)]^3)}^2 + \left\|\frac{\partial^n (f_2^s, f_2^o, f_2^w)}{\partial t^n}\right\|_{L^2(J;[L^2(\Omega_p)]^9)}^2,$$

$$\begin{aligned}G_0^2 = {} & \|u_1^0\|_{2,\Omega_f}^2 + \left\|\left(u_2^{s,0}, u_2^{o,0}, u_2^{w,0}\right)\right\|_{2,\Omega_p}^2 + \|v^0\|_{\tilde{V}}^2 \\ & + \|f_1(0)\|_{0,\Omega_f}^2 + \|(f_2^s(0), f_2^o(0), f_2^w(0))\|_{0,\Omega_p}^2 + 1.\end{aligned}$$

The well-posedness of problem (2.6)–(2.8) follows from Theorem 3.1.

THEOREM 3.1. *Let* $f = (f_1, f_2^s, f_2^o, f_2^w)$, $u^0 = (u_1^0, u_2^{s,0}, u_2^{o,0}, u_2^{w,0})$ *and* $v^0 = (v_1^0, v_2^{s,0}, v_2^{o,0}, v_2^{w,0})$ *be given data satisfying* $G_0 < \infty$, $Q_i < \infty$, $i = 0, 1$. *Assume that* Γ_3 *is of class* C^m *for some integer* $m \geq 2$. *Also, assume that*

$$\begin{aligned}
&\text{support}(u_1^0) \cap \Omega_f \subset\subset \Omega_f,\\
&\text{support}(v_1^0) \cap \Omega_f \subset\subset \Omega_f,\\
&\text{support}\left(u_2^{s,0}, u_2^{o,0}, u_2^{w,0}\right) \subset\subset \Omega_p,\\
&\text{support}\left(v_2^{s,0}, v_2^{o,0}, v_2^{w,0}\right) \subset\subset \Omega_p.
\end{aligned}$$

Then there exists a unique solution $u(r, z, t)$ *of problem (2.5)–(2.7) such that* u, $\partial u/\partial t \in L^\infty(J, V)$; $\partial^2 u_1/\partial t^2 \in L^\infty(J, [L^2(\Omega_f)]^3)$; *and* $\partial^2(u_2^s, u_2^o, u_2^w)/\partial t^2 \in L^\infty(J, [L^2(\Omega_p)]^9)$.

The proof will be omitted since it is quite similar to the corresponding one given in [20].

In the case in which the contact surface Γ_3 between Ω_f and Ω_p is known to be Lipschitz continuous, the following existence and uniqueness theorem holds; the proof is similar to that of Theorem 4.1 in [20].

THEOREM 3.2. *Let* $f = (f_1, f_2^s, f_2^o, f_2^w)$ *be given and such that* $Q_i < \infty$, $i = 0, 1$. *Assume that* $u^0 = v^0 = 0$ *and that* Γ_3 *is Lipschitz continuous. Then there exists a unique solution* $u(r, z, t)$ *of problem (2.5)–(2.7) such that* u, $\partial u/\partial t \in L^\infty(J, V)$; $\partial^2 u_1/\partial t^2 \in L^\infty(J, [L^2(\Omega_f)]^3)$; *and* $\partial^2(u_2^s, u_2^o, u_2^w)/\partial t^2 \in L^\infty(J, [L^2(\Omega_p)]^9)$.

Finally, let us indicate the modifications needed to treat the cases of open, sealed, and partially sealed interface Γ_3.

Case 1. $m_w = m_o = 0$

Condition (2.7.v) changes into

$$A_f \nabla \cdot u_1 = -\Delta p_w = -\Delta p_o, \qquad \Gamma_3 \times J,$$

which is the continuity of fluid pressures along the interface Γ_3. Such a condition is analyzed in [21], and for the single-phase case it is shown to be energy-flux preserving. Moreover, the capillary pressure on the interface vanishes. The boundary integral terms on Γ_3 in the weak formulation (3.1) should disappear and the trial function space V remains unchanged.

Case 2. $m_w = 0, 0 < m_o < \infty$

Condition (2.7.v) change into

$$A_f \nabla \cdot u_1 = -\Delta p_w = -\Delta p_o + m_o \frac{\partial u_2^o}{\partial t} \cdot \nu_p, \qquad \Gamma_3 \times J,$$

which states the continuity of fluid pressures between borehole fluid and water along Γ_3. The boundary integral terms on Γ_3 becomes

$$\left\langle m_o(\beta + \overline{S}_o) \frac{\partial u_2^o}{\partial t} \cdot \nu_p, v_3 \cdot \nu_p \right\rangle_{\Gamma_3}$$

with the same trial function space V. In this case, the capillary pressure on the interface affects only the oil velocity.

Case 3. $m_w = 0,\ m_o = \infty$

Condition (2.7.v) is modified as follows:

$$A_f \nabla \cdot u_1 = -\Delta p_w, \qquad u_2^o \cdot \nu_p = 0, \qquad \Gamma_3 \times J,$$

and (2.7.iv) as follows:

$$(u_2^s + \bar{S}_w u_2^w) \cdot \nu_p + u_1 \cdot \nu_f = 0, \qquad \Gamma_3 \times J.$$

In addition, the test function space should be

$$V = \{v = (v_1, v_2, v_3, v_4) \in \tilde{V} : v_3 \cdot \nu_p = 0, \\ (v_2 + \bar{S}_w v_4) \cdot \nu_p + v_1 \cdot \nu_f = 0 \quad \text{on} \quad \Gamma_3\}.$$

In this case, boundary integral terms on Γ_3 in (3.1) shall vanish.

Case 4. $0 < m_w < \infty,\ m_o = \infty$

Condition (2.7.v) should be altered into

$$A_f \nabla \cdot u_1 = -\Delta p_w + m_w \frac{\partial u_2^w}{\partial t} \cdot \nu_p, \qquad u_2^o \cdot \nu_p = 0, \qquad \Gamma_3 \times J,$$

while (2.7.iv) changes to

$$(u_2^s + \bar{S}_w u_2^w) \cdot \nu_p + u_1 \cdot \nu_f = 0, \qquad \Gamma_3 \times J.$$

Then, the space V should be

$$V = \{v = (v_1, v_2, v_3, v_4) \in \tilde{V} : v_3 \cdot \nu_p = 0, \\ (v_2 + \bar{S}_w v_4) \cdot \nu_p + v_1 \cdot \nu_f = 0 \quad \text{on} \quad \Gamma_3\}.$$

The boundary integral terms on Γ_3 in (3.1) should become

$$\left\langle m_w \bar{S}_w \frac{\partial u_2^w}{\partial t} \cdot \nu_p, v_4 \cdot \nu_p \right\rangle_{\Gamma_3}.$$

Case 5. $m_w = m_o = \infty$

Condition (2.7.v) becomes

$$u_2^w \cdot \nu_p = u_2^o \cdot \nu_p = 0, \qquad \Gamma_3 \times J,$$

and (2.7.iv) converts to

$$u_2^s \cdot \nu_p + u_1 \cdot \nu_f = 0, \qquad \Gamma_3 \times J.$$

The trial function space is given by

$$V = \{v = (v_1, v_2, v_3, v_4) \in \tilde{V} : v_3 \cdot \nu_p = v_4 \cdot \nu_p = 0,$$

$$v_2 \cdot \nu_p + v_1 \cdot \nu_f = 0 \quad \text{on} \quad \Gamma_3\}.$$

In the weak formulation (3.1), boundary integral terms on Γ_3 should disappear.

In the above five cases, the conclusions of Theorems 3.1 and 3.2 remain valid.

IV. AN EXPLICIT FINITE ELEMENT PROCEDURE

For $0 < h < 1$, let $\tau_h^f = \tau_h^f(\Omega_f)$ and $\tau_h^p = \tau_h^p(\Omega_p)$ be quasiregular partitions of Ω_f and Ω_p with elements generated by the rotation around the z axis of rectangles in the (r,z) variables of diameter bounded by h. Set $\tau_h = \tau_h^f \cup \tau_h^p$. Since the boundary condition (2.7.iv) is strongly imposed on the finite element spaces to be used for the spatial discretization, the partitions τ_h^f and τ_h^p will be assumed to be compatible along the contact surface Γ_3 in the following sense. For any vertical cross section $\tau_h \cap \{\theta = \theta_0\}$ of τ_h, if R_f is a rectangle in $\tau_h^f \cap \{\theta = \theta_0\}$ such that one edge e of R_f is contained in Γ_3, then e is also an edge of some rectangle R_p in $\tau_h^p \cap \{\theta = \theta_0\}$. Let $P_{1,1}(r,z)$ denote the bilinear polynomials in the (r,z) variables and set

$$M_h(\Omega_p) = \{\varphi = (\varphi_r, 0, \varphi_z) \in C^0(\overline{\Omega}_p) : \ \varphi_r \in rP_{1,1}(r,z) \quad \text{and} \quad \varphi_z \in P_{1,1}(r,z)\}.$$

Then $M_h(\Omega_p) \subset [\tilde{H}^1(\Omega_p)]^3$. The r component of φ is multiplied by r in order to insure that all components of the strain tensor of φ remain polynomials in r and z. Morley [22] showed the following approximation property

$$\inf_{\varphi \in M_h(\Omega_p)} \left[\|v - \varphi\|_{0,\Omega_p} + h\|v - \varphi\|_{1,\Omega_p} \right] \leq Ch^s \|v\|_{s,\Omega_p}, \qquad s = 1,2. \tag{4.1}$$

Let $W_h(\Omega_i)$, $i = f, p$, be the vector part of the lowest-order mixed finite element space associated with τ_h^i defined by Morley [22]. Away from $r = 0$, the elements in $W_h(\Omega_i)$ are locally of the form $(ar^{-1} + br, 0, c + dz)$, while the innermost elements near $r = 0$ have the local form $(br, 0, c + dz)$. Globally the elements must lie in $H(\text{div}, \Omega_i)$, $i = f$ or p, as appropriate. Note that the divergence of each element is piecewise constant. It is also shown in [22] that

$$\text{(i)} \qquad \inf_{\varphi \in W_h(\Omega_i)} \|v - \varphi\|_{H(\text{div},\Omega_i)} \leq Ch[\|v\|_{1,\Omega_i} + \|\nabla \cdot v\|_{1,\Omega_i}],$$

$$v \in [H^1(\Omega_i)]^3, \qquad \nabla \cdot v \in H^1(\Omega_i),$$

$$\text{(ii)} \qquad \inf_{\varphi \in W_h(\Omega_i)} \|v - \varphi\|_{0,\Omega_i} \leq Ch\|v\|_{1\Omega_i}, \qquad v \in [H^1(\Omega_i)]^3. \tag{4.2}$$

Let

$$\tilde{V}_h = W_h(\Omega_f) \times M_h \times W_h(\Omega_p) \times W_h(\Omega_p)$$

and

$$V_h = \{v \in \tilde{V}_h : \ (v_2 + \overline{S}_o v_3 + \overline{S}_w v_4 - v_1) \cdot \nu_f = 0 \quad \text{on} \quad \Gamma_3\}.$$

Then $V_h \subset V$ and it follows from (4.1)–(4.2) that

$$\inf_{\varphi \in V_h} \left[\|v_1 - \varphi_1\|_{0,\Omega_f} + \|(v_2, v_3, v_4) - (\varphi_2, \varphi_3, \varphi_4)\|_{0,\Omega_f} \right]$$

$$\leq Ch\Big[\|v_1\|_{1,\Omega_f} + \|v_2, v_3, v_4\|_{1,\Omega_p}\Big] \quad (4.3)$$

for

$$v \in \left([\tilde{H}^1(\Omega_f)]^3 \times [\tilde{H}^1(\Omega_p)]^3 \times [\tilde{H}^1(\Omega_p)]^3 \times [\tilde{H}^1(\Omega_p)]^3\right) \cap V$$

and that

$$\inf_{\varphi\in V_h} \|v - \varphi\|_V \leq Ch[\|v_1\|_{1,\Omega_f} + \|\nabla \cdot v_1\|_{1,\Omega_f} + \|v_2\|_{2,\Omega_p}$$
$$+ \|v_3\|_{1,\Omega_p} + \|\nabla \cdot v_3\|_{1,\Omega_p} + \|v_4\|_{1,\Omega_p} + \|\nabla \cdot v_4\|_{1,\Omega_p}] \quad (4.4)$$

for

$$v \in \left([\tilde{H}^1(\Omega_f)]^3 \times [\tilde{H}^2(\Omega_p)]^3 \times [\tilde{H}^1(\Omega_p)]^3 \times [\tilde{H}^1(\Omega_p)]^3\right) \cap V$$

with

$$\nabla \cdot v_1 \in H^1(\Omega_f),$$
$$\nabla \cdot v_3 \in H^1(\Omega_p),$$

and

$$\nabla \cdot v_4 \in H^1(\Omega_p).$$

Let L be a positive integer, $\Delta t = T/L$, and $U^n = U(n\Delta t)$. Set

$$d_t U^n = (U^{n+1} - U^n)/\Delta t\,,$$
$$\partial U^n = (U^{n+1} - U^{n-1})/2\Delta t\,,$$
$$\partial^2 U^n = (U^{n+1} - 2U^n + U^{n-1})/(\Delta t)^2.$$

Because we want to use an explicit procedure, we will compute all integrals involving time derivative terms using the quadrature rule

$$\int_Q f(r,z) r\,dr\,d\theta\,dz \approx 2\pi h_r h_z \frac{f_1 r_1 + f_2 r_2 + f_3 r_3 + f_4 r_4}{4}, \quad (4.5)$$

with f_i denoting the value of f at the node a_i in the rectangle Q' (see Fig. 2). Here Q' denotes a cross section of Q for any fixed angle θ. Note that the rule (4.5) is exact if $rf(r,z)$ is bilinear.

For the elements in M_h, the rule (4.5) is the natural choice since the local degrees of freedom for any element $v(v_r, 0, v_z) \in M_h$ are the values of v_r and v_z at the nodes a_i, $1 \leq i \leq 4$. On the other hand, since the local degrees of freedom of a mixed Morley element $v = (v_r, 0, v_z)$ are the values of $v \cdot \nu_Q$ at the midpoints of each side of Q' (i.e., the values of v_r at the nodes a_5 and a_7 and of v_z at the nodes a_6 and a_8), such values being constant along the sides of Q', the mass-lumping quadrature rule (4.5) can be used for those elements as well.

Let $[v,w]_i$ and $|||v|||_{0,\Omega_i}$, $i = f, p$, denote the inner product $(v,w)_i$ and the norm $\|v\|_{0,\Omega_i}$ computed approximately using the quadrature rule (4.5). Also, let $\langle\langle v, w\rangle\rangle_\Gamma$ denote the inner product $\langle v, w\rangle_\Gamma$ computed using the corresponding analogue of (4.5) along Γ.

The discrete-time explicit Galerkin procedure is defined as follows: Find $U^n \in V_h$, $n = 0, \ldots, L$, such that

$$[\rho_f \partial^2 U_1^n, v_1]_f + [A\partial^2 (U_2, U_3, U_4)^n, (v_2, v_3, v_4)]_p$$

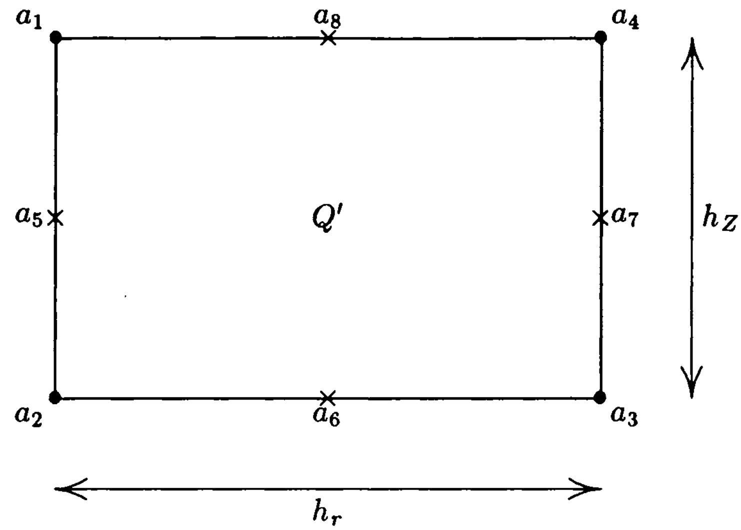

FIG. 2.

$$
\begin{aligned}
&+[C\partial(U_2,U_3,U_4)^n,(v_2,v_3,v_4)]_p+\Lambda(U^n,v)\\
&+\left\langle\left\langle\sqrt{\rho_f A_f}\,\partial U_1^n\cdot\nu_f,\,v_1\cdot\nu_f\right\rangle\right\rangle_{\Gamma_1}\\
&+\left\langle\left\langle B\left(\partial U_2^n\cdot\nu_p,\,\partial U_2^n\cdot\chi_p^1,\,\partial U_2^n\cdot\chi_p^2,\,\partial U_3^n\cdot\nu_p,\,\partial U_4^n\cdot\nu_p\right)^t,\right.\right.\\
&\left.\left.\left(v_2\cdot\nu_p,v_2\cdot\chi_p^1,\,v_2\cdot\chi_p^2,\,v_3\cdot\nu_p,\,v_4\cdot\nu_p\right)^t\right\rangle\right\rangle_{\Gamma_2}\\
&+\langle\langle m_o(\beta+\overline{S}_o)\partial U_3^n\cdot\nu_p,\,v_3\cdot\nu_p\rangle\rangle_{\Gamma_3}-\langle\langle m_w\beta\partial U_4^n\cdot\nu_p,\,v_3\cdot\nu_p\rangle\rangle_{\Gamma_3}\\
&+\langle\langle m_w\overline{S}_w\partial U_4^n\cdot\nu_p,\,v_4\cdot\nu_p\rangle\rangle_{\Gamma_3}\\
&=(f_1^n,v_1)_f+((f_2^{s,n},f_2^{o,n},f_2^{w,n}),(v_2,v_3,v_4))_p,\qquad v\in V_h,\qquad 1\le n\le L-1. \qquad (4.6)
\end{aligned}
$$

Note that the quadrature rule (4.5) is $O(h^2)$ correct. The stability and optimal order error estimates are guaranteed for the scheme (4.6) as stated in the following theorems. The proofs are omitted because the arguments are very similar to those in [11].

THEOREM 4.1. *Assume $f_1\in L^\infty(J,[L^2(\Omega_f)]^3)$ and $(f_2^s,f_2^o,f_2^w)\in L^\infty(J,[L^2(\Omega_p)]^9)$, and let $\Delta t\in(0,1]$ satisfy the stability condition*

$$\Delta t<\frac{h}{\tilde C}\min\left(\left(\frac{2\rho_{f*}}{A_f^*}\right)^{1/2},\left(\frac{2\lambda_{\min}(A)}{\lambda_{\max}(E_p)}\right)^{1/2}\right),$$

where $\tilde C$ is a constant independent of h such that the following inverse inequalities hold:

(i) $\|\nabla\cdot v\|_{0,\Omega}\le\tilde C|||v|||_{0,\Omega},\qquad v\in W_h(\Omega_i),\qquad i=f,p,$

(ii) $\left[\|\epsilon_{rr}(v)\|_{0,\Omega_p}^2+\|\epsilon_{\theta\theta}(v)\|_{0,\Omega_p}^2+\|\epsilon_{zz}(v)\|_{0,\Omega_p}^2+\|\epsilon_{rz}(v)\|_{0,\Omega_p}^2\right]^{1/2}\le\tilde C h^{-1}|||v|||_{0,\Omega_p},$
$v\in M_h.$

Then the solution $(U^n)_{0\le n\le L}$ *of the scheme (4.6) is stable; moreover,*

$$\max_{1\le N\le L-1}\Big[|||d_tU_1^N|||_{0,\Omega_f} + |||d_t(U_2,U_3,U_4)^N|||_{0,\Omega_p} + \|U^N\|_{\tilde V} + \|U^{N+1}\|_{\tilde V}\Big]$$
$$\le C\Big[|||d_tU_1^0|||_{0,\Omega_f} + |||d_t(U_2,U_3,U_4)^0|||_{0,\Omega_p} + \|U^0\|_{\tilde V} + \|U^1\|_{\tilde V}$$
$$+\|f_1\|_{L^\infty(J,[L^2(\Omega_f)]^3)} + \|(f_2^s,f_2^o,f_2^w)\|_{L^\infty(J,[L^2(\Omega_p)]^9)}\Big].$$

THEOREM 4.2. *Let Δt be given as Theorem 4.1. Then the following optimal order error estimate holds:*

$$\max_{1\le N\le L-1}\Big[|||d_t(u_1 - U_1)^N|||_{0,\Omega_f} + |||d_t(u_2^s - U_2, u_2^o - U_3, u_2^w - U_4)^N|||_{0,\Omega_p}$$
$$+\|(u - U)^N\|_{\tilde V} + \|(u - U)^{N+1}\|_{\tilde V}\Big]$$
$$\le C(u)\Big[|||d_t(u_1 - U_1)^0|||_{0,\Omega_f} + |||d_t(u_2 - U_2, u_3 - U_3, u_4 - U_4)^0|||_{0,\Omega_p}$$
$$+\|(u - U)^0\|_{\tilde V} + \|(u - U)^1\|_{\tilde V} + (\Delta t)^2 + h\Big].$$

V. DERIVATION OF THE ABSORBING BOUNDARY CONDITIONS

In this section, we will derive the absorbing boundary condition (2.7.*ii*) for the artificial boundary Γ_2 of Ω_p. We will mainly follow the ideas given in [11].

Fix the wave velocity $c > 0$. Denote by $u_2^{\iota,c} = (u_{2r}^{\iota,c}, 0, u_{2z}^{\iota,c})$, $\iota = s,o,w$, the displacements in solid, oil, and water in Ω_p whose wave fronts arrive normally to Γ_2 with velocity c. Next recall that [23]

$$\begin{aligned}
&\varepsilon_{rr}(u_2^{\iota,c}) = -\frac{1}{c}\frac{\partial u_{2r}^{\iota,c}}{\partial t}\nu_{pr}, \qquad \varepsilon_{\theta\theta}(u_2^{\iota,c}) = 0, \qquad \varepsilon_{zz}(u_2^{\iota,c}) = -\frac{1}{c}\frac{\partial u_{2z}^{\iota,c}}{\partial t}\nu_{pz},\\
&\varepsilon_{rz}(u_2^{\iota,c}) = -\frac{1}{2c}\left(\frac{\partial u_{2r}^{\iota,c}}{\partial t}\nu_{pz} + \frac{\partial u_{2z}^{\iota,c}}{\partial t}\nu_{pr}\right), \qquad \varepsilon_{r\theta}(u_2^{\iota,c}) = \varepsilon_{\theta z}(u_2^{\iota,c}) = 0,\\
&(r,\theta,z) \in \Gamma_2, \quad t \in J, \quad \text{for} \quad \iota = s,o,w.
\end{aligned} \tag{5.1}$$

Moreover,

$$\nabla \cdot u_2^{\iota,c} = -\frac{1}{c}\frac{\partial u_2^{\iota,c}}{\partial t}\cdot \nu_p, \quad \text{on} \quad \Gamma_2 \times J, \quad \text{for} \quad \iota = s,o,w. \tag{5.2}$$

Next, let us introduce the variables

$$\alpha_1^c = \frac{1}{c}\frac{\partial u_2^{s,c}}{\partial t}\cdot \nu_p, \qquad \alpha_2^c = \frac{1}{c}\frac{\partial u_2^{s,c}}{\partial t}\cdot \chi_p^1, \qquad \alpha_3^c = \frac{1}{c}\frac{\partial u_2^{s,c}}{\partial t}\cdot \chi_p^2,$$
$$\alpha_4^c = \frac{1}{c}\frac{\partial u_2^{o,c}}{\partial t}\cdot \nu_p, \qquad \alpha_5^c = \frac{1}{c}\frac{\partial u_2^{w,c}}{\partial t}\cdot \nu_p,$$

and set

$$\alpha^c = (\alpha_1^c, \alpha_2^c, \alpha_3^c, \alpha_4^c, \alpha_5^c)^t.$$

Combining the stress–strain relations (2.1) and (2.2) with (5.1) and (5.2) shows that the strain energy density W_p on Γ_2 can be written as a quadratic function

$$\Pi(\alpha^c) = W_p(\varepsilon(\alpha^c), -\nabla \cdot u_2^{o,c}, -\nabla \cdot u_2^{w,c})$$

in the form

$$\Pi(\alpha^c) = \frac{1}{2}(\alpha^c)^t \tilde{E}_p \alpha^c, \tag{5.3}$$

where $\tilde{E}_p \in R^{5\times 5}$ is the symmetric, positive-definite matrix given by

$$\tilde{E}_p = \begin{bmatrix} 2N+\lambda_c & 0 & 0 & B_1 & B_2 \\ 0 & N & 0 & 0 & 0 \\ 0 & 0 & N & 0 & 0 \\ B_1 & 0 & 0 & M_1 & M_3 \\ B_2 & 0 & 0 & M_3 & M_2 \end{bmatrix}.$$

Now the momentum equations on Γ_2 can be written as

$$\begin{aligned}
&\text{(i)} && c\left[\rho \frac{\partial u_2^{s,c}}{\partial t} + \rho_o \overline{S}_o \frac{\partial u_2^{o,c}}{\partial t} + \rho_w \overline{S}_w \frac{\partial u_2^{w,c}}{\partial t}\right]^t = -\tau\nu_p, \\
&\text{(ii)} && c\left[\rho_o \overline{S}_o \frac{\partial u_2^{s,c}}{\partial t} + \overline{g}_1 \frac{\partial u_2^{o,c}}{\partial t} + \overline{g}_3 \frac{\partial u_2^{w,c}}{\partial t}\right]^t = [(\overline{S}_o + \beta)\Delta p_o - \beta\Delta p_w]\nu_p, \\
&\text{(iii)} && c\left[\rho_w \overline{S}_w \frac{\partial u_2^{s,c}}{\partial t} + \overline{g}_3 \frac{\partial u_2^{o,c}}{\partial t} + \overline{g}_2 \frac{\partial u_2^{w,c}}{\partial t}\right]^t = \overline{S}_w \Delta p_w \nu_p, \\
& && (r,\theta,z) \in \Gamma_2, \qquad t \in J.
\end{aligned} \tag{5.4}$$

Equations (5.4) can be rewritten in terms of the new variables α_i^c, $1 \le i \le 5$, in the following manner. First, we get one relation on taking the inner product of (5.4.*i*) with ν_p. We then take the inner product of (5.4.*ii*) and (5.4.*iii*) with χ_p^m, $m = 1,2$, in order to get the relationships

$$\begin{aligned}
\frac{\partial u_2^{o,c}}{\partial t} \cdot \chi_p^m &= -c \frac{\overline{g}_2 \rho_o \overline{S}_o - \overline{g}_3 \rho_w \overline{S}_w}{\overline{g}_1 \overline{g}_2 - \overline{g}_3^2} \alpha_{m+1}^c, \\
\frac{\partial u_2^{w,c}}{\partial t} \cdot \chi_p^m &= -c \frac{\overline{g}_1 \rho_w \overline{S}_w - \overline{g}_3 \rho_o \overline{S}_o}{\overline{g}_1 \overline{g}_2 - \overline{g}_3^2} \alpha_{m+1}^c, \qquad m = 1,2.
\end{aligned}$$

Using the above identities, we get two relations from the inner product of (5.4.*i*) with χ_p^m, $m = 1,2$, while the other two relations can be obtained from the inner product of (5.4.*ii*) and (5.4.*iii*) with ν_p. Thus

$$c^2 \tilde{A} \alpha^c = -\mathcal{F}, \tag{5.5}$$

where

$$\mathcal{F} = \left(\tau\nu_p \cdot \nu_p, \tau\nu_p \cdot \chi_p^1, \tau\nu_p \cdot \chi_p^2, -[(\overline{S}_o + \beta)\Delta p_o - \beta\Delta p_w], -\overline{S}_w \Delta p_w\right)^t$$

and $\tilde{A} \in R^{5\times 5}$ is the symmetric, positive-definite matrix defined by

$$\tilde{A} = \begin{pmatrix} \rho & 0 & 0 & \rho_o\overline{S}_o & \rho_w\overline{S}_w \\ 0 & \hat{\rho} & 0 & 0 & 0 \\ 0 & 0 & \hat{\rho} & 0 & 0 \\ \rho_o\overline{S}_o & 0 & 0 & \overline{g}_1 & \overline{g}_3 \\ \rho_w\overline{S}_w & 0 & 0 & \overline{g}_3 & \overline{g}_2 \end{pmatrix},$$

$$\hat{\rho} \equiv \rho - \frac{\overline{g}_2(\rho_o\overline{S}_o)^2 + \overline{g}_1(\rho_w\overline{S}_w)^2 - 2\overline{g}_3\rho_o\overline{S}_o\rho_w\overline{S}_w}{\overline{g}_1\overline{g}_2 - \overline{g}_3^2}.$$

Furthermore, a combination of (2.1), (2.2), (5.1), and (5.2) gives

$$-\mathcal{F} = \tilde{E}_p\alpha^c = \frac{\partial\Pi}{\partial\alpha^c}. \tag{5.6}$$

Let $S = \tilde{A}^{-1/2}\tilde{E}_p\tilde{A}^{-1/2}$ and $\overline{\alpha}^c = \tilde{A}^{1/2}\alpha^c$. From (5.5) and (5.6) it follows that

$$S\overline{\alpha}^c = c^2\overline{\alpha}^c. \tag{5.7}$$

The strain energy density on Γ_2 is then written in the form

$$\overline{\pi}(\overline{\alpha}^c) \equiv \pi(\alpha^c) = \frac{1}{2}(\overline{\alpha}^c)^t S\overline{\alpha}^c. \tag{5.8}$$

Let c_i, $1 \leq i \leq 5$, be the five positive wave velocities satisfying (5.7) such that

$$\det(S - c^2 I) = 0.$$

Two velocities are given by

$$c_2 = c_3 = \left(\frac{N}{\hat{\rho}}\right)^{1/2}.$$

c_1, c_4, and c_5 have more complicated forms in terms of the mass and stiffness coefficients of Ω_p. Here c_2 and c_3 correspond to the shear modes of propagation, while c_1, c_4, and c_5 correspond to the compressional modes of propagation (see [1] also). Let M_i, $1 \leq i \leq 5$, be the set of orthonormal eigenvectors corresponding to c_i, $1 \leq i \leq 5$, and let M be the matrix whose ith row is M_i. Let $D = \mathrm{diag}\{c_1^2, c_2^2, c_3^2, c_4^2, c_5^2\}$. Then,

$$S = M^t DM. \tag{5.9}$$

Let

$$\alpha = \left(\frac{\partial u_2^s}{\partial t}\cdot\nu_p, \frac{\partial u_2^s}{\partial t}\cdot\chi_p^1, \frac{\partial u_2^s}{\partial t}\cdot\chi_p^2, \frac{\partial u_2^o}{\partial t}\cdot\nu_p, \frac{\partial u_2^w}{\partial t}\cdot\nu_p\right)^t$$

denote a general velocity on the surface Γ_2 due to the simultaneous normal arrival of waves of velocities c_i, $1 \leq i \leq 5$. Then,

$$\overline{\alpha} \equiv \tilde{A}^{1/2}\alpha = \sum_{i=1}^{5}[M_i, \tilde{A}^{1/2}\alpha]_e M_i.$$

Let $\overline{\alpha}^{c_i}$ be the component of

$$\frac{1}{c_i}\tilde{A}^{1/2}\alpha = \frac{1}{c_i}\overline{\alpha}$$

along M_i, i.e.,

$$\overline{\alpha}^{c_i} = \frac{1}{c_i}[M_i, \tilde{A}^{1/2}\alpha]_e M_i, \qquad 1 \leq i \leq 5. \tag{5.10}$$

Analogously to (5.7) and (5.8),

$$S\overline{\alpha}^{c_i} = c_i^2 \overline{\alpha}^{c_i}, \tag{5.11}$$

and

$$\overline{\pi}(\overline{\alpha}^{c_i}) = \frac{1}{2}(\overline{\alpha}^{c_i})^t S\overline{\alpha}^{c_i}. \tag{5.12}$$

From (5.5), (5.6), (5.10), and (5.12) it follows that

$$\tilde{A}^{1/2}\frac{\partial \overline{\pi}}{\partial \overline{\alpha}^{c_i}} = \tilde{A}^{1/2} S\overline{\alpha}^{c_i} = -\mathcal{F}_i, \tag{5.13}$$

where $\mathcal{F}_i$ denotes the force on Γ_2 corresponding to $\overline{\alpha}^{c_i}$.

We will neglect the interaction energy among the different types of waves arriving normally to Γ_2. The total strain energy density and the total force can then be regarded as the sum of their partial strain energies and of partial forces corresponding to each $\overline{\alpha}^{c_i}$. Thus

$$\overline{\pi} = \overline{\pi}(\overline{\alpha}) = \sum_{i=1}^{5} \overline{\pi}(\overline{\alpha}^{c_i}),$$

$$\mathcal{F} = \sum_{i=1}^{5} \mathcal{F}_i = -\tilde{A}^{1/2} \sum_{i=1}^{5} S\overline{\alpha}^{c_i}.$$

Now,

$$S\overline{\alpha}^{c_i} = -[M_i, \tilde{A}^{-1/2}\mathcal{F}]_e M_i, \qquad 1 \leq i \leq 5, \tag{5.14}$$

since

$$\tilde{A}^{-1/2}\mathcal{F} = \sum_{i=1}^{5} [M_i, \tilde{A}^{-1/2}\mathcal{F}]_e M_i.$$

A combination of (5.10), (5.11), and (5.14) therefore leads to

$$c_i[M_i, \tilde{A}^{1/2}\alpha]_e M_i = -[M_i, \tilde{A}^{-1/2}\mathcal{F}]_e M_i, \qquad 1 \leq i \leq 5.$$

Equivalently,

$$-M\tilde{A}^{-1/2}\mathcal{F} = D^{1/2} M \tilde{A}^{1/2}\alpha.$$

From the above equation and (5.9), we see that

$$-\mathcal{F} = \tilde{A}^{1/2} S^{1/2} \tilde{A}^{1/2}\alpha = [(\tilde{A}^{-1}\tilde{E}_p)^t]^{1/2} \tilde{A}\alpha = B\alpha,$$

where B is the symmetric, positive-definite matrix given by

$$B = [(\tilde{A}^{-1}\tilde{E}_p)^t]^{1/2} A.$$

Therefore we arrive at the absorbing condition (2.7.*ii*).

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