Elastic wave behavior across linear slip interfaces

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A model for an imperfectly bonded interface between two elastic media is proposed. Displacement across this surface is not required to be continuous. The displacement discontinuity, or slip, is taken to be linearly related to the stress traction which is continuous across the interface. For isotropic interface behavior, there are two complex frequency dependent interface compliances, η_N and η_T, where the component of the slip normal to the interface is given by η_N times the normal stress and the component tangential to the interface is given by η_T times the shear stress and is in the same direction. Reflection and transmission coefficients for harmonic plane waves incident at arbitrary angles upon a plane linear slip interface are computed in terms of the interface compliances. These coefficients are frequency dependent even when the compliances are real and frequency independent. Examples of the effects of buried slip interfaces on reflection coefficient spectra and on Love-wave dispersion relations are presented.

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INTRODUCTION

A perfectly bonded interface is a surface across which both traction and displacement are continuous. Thus when solving harmonic wave problems in the neighborhood of a perfectly bonded interface between two different elastic media, wave solutions in one medium must be matched with those in the second medium through interface conditions. In general, there are six scalar equations relating the traction vector and the displacement vector on one side to the corresponding components on the other side. These conditions provide the values of the arbitrary constants in the general wave solutions for each medium.

A generalization of this concept is that of an imperfectly bonded interface for which the displacement across a surface need not be continuous. Some applications of such a generalization to elastodynamic problems are the study of composite media, crack detection, and seismic wave propagation.

Imperfect bonding is taken here to mean that the traction is continuous across the interface but that the small displacement field is not. The small vector difference in the displacement, is assumed to depend linearly on the traction vector. The dependence may be real and frequency independent corresponding to an elastic spring condition or it may be complex and frequency dependent corresponding to a viscoelastic spring condition. This interface condition, called a "linear slip condition," replaces the condition of continuous displacement.

The next section shows how linear slip conditions can be put on a firm footing within the theory of elasticity and how the notion of isotropy simplifies the general slip condition.

Subsequent sections consider plane wave reflection and transmission coefficients at plane linear slip interfaces for SH waves and for P and SV waves and propagation through stratified media containing slip interfaces. Also included in these sections is an analysis for the case of SH waves showing that a thin low impedance layer perfectly bonded between two half spaces gives rise to plane wave reflection and transmission coefficients that approach those derived for a linear slip boundary as the thickness to wavelength ratio approaches zero.

In the last section, two cases of elastic wave propagation in the presence of slip interfaces are presented. The first explores the effect of a buried slip interface on the reflection coefficient at normal incidence. In the second case dispersion curves for Love waves are derived and the effect of the buried slip interface is discussed.

I. LINEAR SLIP CONDITIONS

Consider a smooth, in general, curved surface between two elastic regions, across which the small displacement need not be continuous. The stress traction which is continuous across the interface, is assumed to be related to the discontinuity of displacement at each point.

Let the origin of a rectangular coordinate system be at a point on that surface so that x_1 and x_3 are directions tangential to the interface and x_2 is normal to the surface. With $\mathbf{u}$ denoting displacement, let $\Delta\mathbf{u}=\mathbf{u}(0,0^+,0)-\mathbf{u}(0,0^-,0)$ be the displacement discontinuity vector at the point with possible time dependence suppressed. The traction vector, $\mathbf{t}$, at that point on the interface, has components $\tau_{21},\tau_{22},\tau_{23}$, and is the force per unit area that the material on the $+x_2$ side of the interface exerts on the $-x_2$ side. Assume that $\mathbf{t}$ is an analytic function of $\Delta\mathbf{u}$ at each point subject to the requirements that $\Delta\mathbf{u}$ vanishes at a point if and only if $\mathbf{t}$ vanishes at that point. This relation may be expressed as a power series in $\Delta\mathbf{u}$ which is shown symbolically as

$$\mathbf{t}=\mathrm{F}(\Delta\mathbf{u})=\mathrm{k}\,\Delta\mathbf{u}+O(\Delta u_i\Delta u_j)\,, \tag{1}$$

and neglecting quadratic and higher order terms in components of $\Delta\mathbf{u}$ gives a linear relation between $\mathbf{t}$ and $\Delta\mathbf{u}$ through the "boundary stiffness matrix" k which has dimensions stress/length.

If a positive definite displacement discontinuity energy density function, U_I, of dimension energy/area = force/length is to be associated with a slip interface, then, under the constraint that U_I vanish when $\Delta\mathbf{u}$ vanish, U_I is of the form

$$U_I=L_j\Delta u_j+\tfrac{1}{2}K_{ij}\Delta u_i\Delta u_j+\text{3rd order terms.} \tag{2}$$

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An energy equation relating the rate of work done at a slip interface S_I, to the rate of increase of U_I, neglecting third order terms, gives

$$\dot{W} = \int_{S_I} \mathbf{t}\cdot\Delta\dot{u}\, dS_I = \int_{S_I} \dot{U}_I\, dS_I$$
$$= \int_{S_I} (L_j \Delta\dot{u}_j + K_{ij}\Delta u_i \Delta\dot{u}_j) dS_I, \tag{3}$$

and as this must hold for any part of the slip interface, the integrands may be equated giving

$$\mathbf{t} = \mathbf{L} + \Delta\mathbf{u}\mathsf{K} \tag{4}$$

which, to conform with Eq. (1), implies $\mathbf{L}=0$ and $\mathsf{k}=\mathsf{K}$, a positive definite, symmetric matrix.

If the boundary stiffness matrix is to be invariant with respect to inversion of the x_2 axis, it may be shown[1] that the off diagonal terms k_{21} and k_{23} between normal and tangential directions must vanish. A plane boundary with such a stiffness matrix will in general scatter a nonnormal incident plane wave in a homogeneous medium to three transmitted and three reflected waves, the P, SV, and SH waves. Only if the incident wave displacement is totally normal (a normally incident P wave) or totally tangential (a normally incident shear wave) will some of these waves not appear. In this case a normally incident shear wave will be scattered to shear waves that are out of polarity with the incident wave.

If there is rotational symmetry about the x_2 axis, then it may be shown that $k_{13}=0$ and $k_{11}=k_{33}$. This leaves but two independent stiffnesses, the normal stiffness, $k_{22} = k_N$ and the tangential stiffness, $k_{11}=k_{33}=k_T$. Some effects on wave behavior of such a "transverse isotropic" linear slip interface will be considered in subsequent sections.

It will be more convenient to characterize slip in terms of compliances instead of stiffnesses, where the compliance matrix is the inverse of the stiffness matrix. For the transverse isotropic slip interface to be considered, we may write

$$\Delta\mathbf{u} = \begin{bmatrix} \eta_T & 0 & 0 \\ 0 & \eta_N & 0 \\ 0 & 0 & \eta_T \end{bmatrix} \mathbf{t}, \tag{5}$$

where the compliances $\eta_N = k_N^{-1}$ and $\eta_T = k_T^{-1}$ have dimension length/stress. The vanishing of either or both of these compliances now leads to the usual perfectly bonded interface conditions.

In addition, as real elastic parameters may be generalized to complex frequency dependent viscoelastic parameters via the harmonic elastic–viscoelastic analogy,[2] so may the slip boundary compliances be generalized allowing the modeling of a linear viscoelastic slip interface.

II. PLANE INTERFACES, PLANE WAVE REFLECTION AND REFRACTION

Consider two homogeneous, isotropic, linear elastic half spaces in contact along a plane interface, denoted by $x_2=0$. Elasticmedium 1, with density ρ_1, compressional wave speed $\alpha_1 = (\lambda_1 + 2\mu_1/\rho_1)^{1/2}$, and shear wave speed $\beta_1 = (\mu_1/\rho_1)^{1/2}$, occupies the region $x_2<0$ and medium 2, with density ρ_2, compressional wave speed α_2, and shear wave speed β_2, occupies the region $x_2>0$. Assume an incident harmonic plane wave of frequency ω and unit amplitude, whose propagation vector lies in the x_1, x_2 plane, impinges on the interface in medium 1. We may consider separately the two uncoupled cases, one of an incident SH wave, giving rise to antiplane strain solutions and the other of either an incident P or SV wave giving rise to plane wave solutions.

A. Incident SH waves

This is the simplest case to discuss. It exhibits the effects of a slip condition most clearly as there is but one slip condition for this problem which influences the value of the one reflection and one transmission coefficient present in the general wave solutions. All displacements in both media have but one nonzero component, $u_3(x_1, x_2)$, which assumes the form

$$u_3 = \exp i\omega[(x_1 \sin\phi_1 + x_2\cos\phi_1)/\beta_1 - t]$$
$$+ R\exp i\omega[(x_1\sin\phi_1 - x_2\cos\phi_1)/\beta_1 - t], \quad x_2>0$$
$$u_3 = T\exp i\omega[(x_1\sin\phi_2 + x_2\cos\phi_2)/\beta_2 - t], \quad x_2>0 \tag{6}$$
$$\omega\sin\phi_1/\beta_1 = \omega\sin\phi_2/\beta_2 = k_1.$$

R is known as the reflection coefficient and T, the transmission coefficient. In all subsequent equations the $\exp i(k_1x_1 - \omega t)$ dependence of the wave fields will be suppressed.

The values of R, T are then found from the two nontrivial interface conditions in this problem that relate the values of u_3 and $\tau_{23} = \rho\beta^2 \partial u_3/\partial x_2$, the only nonvanishing component of the traction across the interface. The conditions that τ_{23} is continuous across $x_2=0$ and that $\Delta u_3 = \eta_T\tau_{23}$ give the following two equations:

$$i\omega Z_1(1-R) = i\omega Z_2 T, \quad T-(1+R) = \eta_T i\omega Z_1(1-R), \tag{7}$$
$$Z_i = \rho_i\beta_i\cos\phi_i, \quad i=1,2,$$

which give values for R and T

$$R = \frac{Z_1 - Z_2 - i\omega\eta_T Z_1 Z_2}{Z_1 + Z_2 - i\omega\eta_T Z_1 Z_2}, \quad T = \frac{2Z_1}{Z_1 + Z_2 - i\omega\eta_T Z_1 Z_2}. \tag{8}$$

Clearly as the compliance $|\eta_T| \to 0$, the case of perfect bonding is approached and R, T approach their conventional values.[3] As $|\eta_T|^{-1} \to 0$, the case of a free surface is approached and $R\to 1$, $T\to 0$ as expected. For $\beta_2 > \beta_1$, when ϕ_1 is greater than the critical angle, Z_2 is positive imaginary. At critical incidence, $\cos\phi_2$ and hence Z_2 vanish giving $R=1$ and $T=2$ the usual result. This is because τ_{23} and hence Δu_3 vanish.

For η_T complex, both the real and imaginary parts of η_T must be positive which guarantees that $|R| \le 1$. For a pure viscous slip interface, $\Delta\dot{u}_3$ is proportional to τ_{23}. Letting the viscous compliance be ξ_T, gives, in the frequency domain

$$-i\omega\Delta u_3 = \xi_T\tau_{23}, \tag{9}$$

which says that the boundary compliance, η_T, may be written

$$\eta_T = i\xi_T/\omega . \tag{10}$$

Substituting this value into Eq. (8) gives for R, T

$$R = \frac{Z_1 - Z_2 + \xi_T Z_1 Z_2}{Z_1 + Z_2 + \xi_T Z_1 Z_2}, \quad T = \frac{2Z_1}{Z_1 + Z_2 + \xi_T Z_1 Z_2}, \tag{11}$$

showing that for this case R and T are again real and frequency independent.

Two special cases of interest are when; (1) both half-spaces have identical properties, and (2) when half-space 2 is rigid.

The first case, which implies $Z_1 = Z_2 = Z$, yields

$$R = \frac{-i\omega\eta_T Z}{2 - i\omega\eta_T Z}, \quad T = \frac{2}{2 - i\omega\eta_T Z}, \tag{12}$$

giving a measure as to how well bonded a crack or an interface is in an otherwise homogeneous body.

When half-space 2 is rigid, the displacement in medium 2 is assumed to vanish and the second of Eq. (7) gives

$$R = -\frac{1 + i\omega\eta_T Z_1}{1 - i\omega\eta_T Z_1}. \tag{13}$$

It is easy to visualize the type of physical mechanism and the associated assumptions giving rise to linear slip behavior for this simple case of SH waves. Consider the situation of a single homogeneous, isotropic layer of thickness h, density ρ', and shear wave speed β', and thus of shear modulus, $\mu' = \rho'\beta'^2$, located between the two half-spaces, and assume perfect bonding on both of the interfaces at $x_2 = 0$ and $x_2 = h$. The general solution for all x_2, x_3 is

$$\begin{aligned}
&u_3 = \exp(ik_{s1}x_2) + R' \exp(-ik_{s1}x_2), \quad x_2 < 0, \\
&u_3 = A' \cos(k_s' x_2) + B' \sin(k_s' x_2), \quad 0 < x_2 < h, \\
&u_3 = T' \exp[ik_{s2}(x_2 - h)], \quad x_2 > h, \\
&k_{si} = \frac{\omega}{\beta_i} \cos\phi_i = \left(\frac{\omega^2}{\beta_i^2} - k_1^2\right)^{1/2}, \\
&k_s' = \frac{\omega}{\beta'} \cos\phi' = \left(\frac{\omega^2}{\beta'^2} - k_1^2\right)^{1/2}.
\end{aligned} \tag{14}$$

Satisfying displacement and traction boundary conditions at $x_2 = 0$ and $x_2 = h$ gives

$$\begin{aligned}
&R' = \frac{(Z_1 - Z_2)\cos(k_s'h) - i(Z_1 Z_2 - Z'^2)[\sin(k_s'h)/Z']}{(Z_1 + Z_2)\cos(k_s'h) - i(Z_1 Z_2 + Z'^2)[\sin(k_s'h)/Z']}, \\
&T' = \frac{2Z_1}{(Z_1 + Z_2)\cos(k_s'h) - i(Z_1 Z_2 + Z'^2)[\sin(k_s'h)/Z']}, \\
&Z' \equiv \rho'\beta' \cos\phi' = k_s'\mu'/\omega .
\end{aligned} \tag{15}$$

Allowing the layer to be thin compared to a wavelength and of low impedance, $\rho\beta$, relative to medium 1, so that

$$\begin{aligned}
&\cos(k_s'h) = 1 + O[(k_s'h)^2], \\
&(1/Z')\sin(k_s'h) = (\omega h/\mu')\{1 + O[(k_s'h)^2]\}, \\
&|Z'/Z_1| = \epsilon \ll 1,
\end{aligned} \tag{16}$$

enables Eq. (14) to have the form

$$\begin{aligned}
&R' = R + O(\epsilon^2) + O[(k_s'h)^2], \\
&T' = T + O(\epsilon^2) + O[(k_s'h)^2],
\end{aligned} \tag{17}$$

where R, T are given by Eq. (8) with η_T replaced by h/μ'. Thus, physically, some knowledge of two of three unknowns, η_T, h, and μ' gives an indication of the third. Information about η_T, tangential compliance, may be inferred by measurement of reflection and/or transmission coefficients. Information about interface thickness may be inferred from knowledge of the polishing procedure used in preparing surfaces to be bonded, and information about μ' may be inferred from knowledge of the interstitial material between the two partially bonded elastic regions.

B. Incident P or SV waves

This is the plane strain problem for which all displacements lie in the x_1, x_2 plane, with components $u_i(x_1, x_2)$, $i = 1, 2$. The incident wave field in medium 1 is of the form

$$\begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix} = \begin{bmatrix} \sin\theta_1 \\ \\ \cos\theta_1 \end{bmatrix} e^{i\omega x_2 \cos\theta_1/\alpha_1}, \tag{18}$$

if the incident wave is a P wave, or

$$\begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix} = \begin{bmatrix} -\cos\phi_1 \\ \\ \sin\phi_1 \end{bmatrix} e^{i\omega x_2 \cos\phi_1/\beta_1}, \tag{19}$$

if the incident wave is an SV wave. In either case, the reflected field in medium 1 is of the form

$$\begin{aligned}
\begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix} &= R_p \begin{bmatrix} \sin\theta_1 \\ \\ -\cos\theta_1 \end{bmatrix} e^{-i\omega x_2 \cos\theta_1/\alpha_1} \\
&+ R_s \begin{bmatrix} \cos\phi_1 \\ \\ \sin\phi_1 \end{bmatrix} e^{-i\omega x_2 \cos\phi_1/\beta_1},
\end{aligned} \tag{20}$$

and the transmitted field in medium 2 is of the form

$$\begin{aligned}
\begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix} &= T_p \begin{bmatrix} \sin\theta_2 \\ \\ \cos\theta_2 \end{bmatrix} e^{i\omega x_2 \cos\theta_2/\alpha_2} \\
&+ T_s \begin{bmatrix} -\cos\phi_2 \\ \\ \sin\phi_2 \end{bmatrix} e^{i\omega x_2 \cos\phi_2/\beta_2},
\end{aligned} \tag{21}$$

where

$$\omega\frac{\sin\theta_1}{\alpha_1} = \omega\frac{\sin\phi_1}{\beta_1} = \omega\frac{\sin\theta_2}{\alpha_2} = \omega\frac{\sin\phi_2}{\beta_2} = k_1 . \tag{22}$$

There are four interface conditions to determine the four constants R_p, R_s, T_p, T_s. The two nontrivial stress conditions are that τ_{22} and τ_{21} are continuous across the interface where the stress components are given by

$$\tau_{22}=\rho\left[\alpha^2\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)-2\beta^2\frac{\partial u_1}{\partial x_1}\right], \tag{23}$$

$$\tau_{21}=\rho\beta^2\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right).$$

Note that $\partial u_1/\partial x_1+\partial u_2/\partial x_2=0$ for any shear wave and $=i\omega/\alpha$ for any compressional wave on the interface $x_2=0$.

The most general slip conditions consistent with a transverse isotropic boundary are

$$\Delta\begin{bmatrix}u_1\\ \\ u_2\end{bmatrix}=\begin{bmatrix}u_1\\ \\ u_2\end{bmatrix}_{x_2\to0^+}-\begin{bmatrix}u_1\\ \\ u_2\end{bmatrix}_{x_2\to0^-}=\begin{bmatrix}\eta_T & 0\\ \\ 0 & \eta_N\end{bmatrix}\begin{bmatrix}\tau_{21}\\ \\ \tau_{22}\end{bmatrix}_{x_2=0}, \tag{24}$$

and Eqs. (23) and (24), making use of Eqs. (18)–(22) give a set of equations on the coefficients

$$\mathrm{A}\begin{bmatrix}R_p\\ R_s\\ T_p\\ T_s\end{bmatrix}=\mathrm{B}, \tag{25}$$

with $\mathrm{B}=\mathrm{B}_p$ for an incident P wave and $\mathrm{B}=\mathrm{B}_s$ for an incident SV wave. A, B_p, and B_s are given by

$$\mathrm{A}=\begin{bmatrix}-p_1 & \gamma_1\cos\phi_1 & p_2 & \gamma_2\cos\phi_2\\ \gamma_1\cos\theta_1 & q_1 & \gamma_2\cos\theta_2 & -q_2\\ -\sin\theta_1 & -\cos\phi_1 & \sin\theta_2-i\omega\eta_T\gamma_2\cos\theta_2 & -\cos\phi_2+i\omega\eta_T q_2\\ \cos\theta_1 & -\sin\phi_1 & \cos\theta_2-i\omega\eta_N p_2 & \sin\phi_2-i\omega\eta_N\gamma_2\cos\phi_2\end{bmatrix}, \tag{26}$$

$$\gamma_i=2\rho_i\beta_i\sin\phi_i=2\rho_i\beta_i^2k_1/\omega,$$
$$p_i=\rho_i\alpha_i-\gamma_i\sin\theta_i=\rho_i\alpha_i(1-2\beta_i^2k_1^2/\omega^2),$$
$$q_i=\rho_i\beta_i\cos^2\phi_i-\tfrac{1}{2}\gamma_i\sin\phi_i=\rho_i\beta_i(1-2\beta_i^2k_1^2/\omega^2),$$
$$1-2\beta_i^2k_1^2/\omega^2\equiv\cos2\phi_i$$

$$\mathrm{B}_p=\begin{bmatrix}-A_{11}\\ A_{21}\\ -A_{31}\\ A_{41}\end{bmatrix},\quad \mathrm{B}_s=\begin{bmatrix}A_{12}\\ -A_{22}\\ A_{32}\\ -A_{42}\end{bmatrix},$$

and from these equations, the coefficients R_p, R_s, T_p, T_s may be obtained.

In the case of normal incidence, k_1 vanishes, and for an incident P wave, R_s and T_s vanish and for an incident S wave, R_p and T_p vanish. In both cases the nonvanishing R and T are given by

$$R=-\frac{Z_1-Z_2-i\omega\eta Z_1Z_2}{Z_1+Z_2-i\omega\eta Z_1Z_2},\quad T=\frac{2Z_1}{Z_1+Z_2-i\omega\eta Z_1Z_2}, \tag{27}$$

where $Z_i=\rho_i\beta_i$ and $\eta=\eta_T$ for a normally incident shear wave and $Z_i=\rho_i\alpha_i$ and $\eta=\eta_N$ for a normally incident compressional wave. Note that the sign of R in Eq. (27) is the opposite of the sign R for SH waves. This is because for SH waves the positive displacement vector was in the $+x_3$ direction regardless of whether the wave was upgoing or downgoing. This is not the case here as seen from Eqs. (18)–(21).

As in the previous section, it has been assumed that there is a thin layer between the two half-spaces in order to examine the physical mechanism involved in the assumption of linear slip. It may be shown that the reflection coefficients approach those from the linear slip boundary theory with η_T and η_N replaced by h/μ' and $h/(K'+4\mu'/3)$, respectively, where K' is the layer bulk modulus. This occurs under the assumptions that the layer impedances are much less than the half-space impedances and the layer thickness is much less than a wavelength.

The case of a fluid filled crack may be approximated by letting $\eta_N=0$ and $\eta_T\neq0$, which is equivalent to requiring the normal displacement to be continuous. The limiting case of $\eta_N=0, \eta_T\to\infty$ is equivalent to requiring that the shear stress across the interface vanish (two conditions), the normal stress be continuous, and normal component of the displacement discontinuity vanish. For such a crack between half-spaces of identical properties, Eqs. (25) and (26) reduce to

$$\begin{bmatrix}-p & \gamma\cos\phi & p & \gamma\cos\phi\\ \gamma\cos\theta & q & 0 & 0\\ 0 & 0 & \gamma\cos\theta & q\\ \cos\theta & -\sin\phi & \cos\theta & \sin\phi\end{bmatrix}\begin{bmatrix}R_p\\ R_s\\ T_p\\ T_s\end{bmatrix}=\begin{bmatrix}p\\ \gamma\cos\theta\\ 0\\ \cos\theta\end{bmatrix}\ \text{or}\ \begin{bmatrix}\gamma\cos\theta\\ -q\\ 0\\ \sin\phi\end{bmatrix}. \tag{28}$$

For pure viscous slip in shear i.e., $\eta_T=i\xi/\omega$, and $\eta_N=0$, Eqs. (25) and (26) for the reflection coefficients become frequency independent and give real values for R_p, R_s, T_p, T_s as long as all angles, θ_i, ϕ_i are real.

III. SLIP INTERFACES WITHIN STRATIFIED REGIONS

The matrix method of Thomson and Haskell[4,5] for the analysis of wave propagation in stratified elastic media is very easily adapted to include the case of linear transverse isotropic slip between any two elastic homogeneous layers. In this method, for plane strain propagation through isotropic layers, a transfer matrix, $\mathrm{Q}(a,b)$ is found which relates stresses and displacements at $x_1=a$ to corresponding stresses and displacements at b, where a and b are values of x_1 within the same homogeneous layer. Letting $\mathbf{Y}(x_1)$ be the "vector" $[\tau_{22}(x_1), \tau_{12}(x_1), u_1(x_1), u_2(x_1)]^T$, this relationship may be denoted

$$\mathbf{Y}(a)=\mathrm{Q}(a,b)\mathbf{Y}(b), \tag{29}$$

and if there are a set of n horizontal layers with boundaries $a_0, a_1, a_2, \ldots, a_n$ then the relationship may be extended to many or all of the layers by matrix multiplication by noticing that perfect bonding at an interface a_i means that $Y(a_i^+)=Y(a_i^-)$ giving

$$\mathbf{Y}(a_n)=\mathrm{Q}_n(a_n,a_{n-1})\cdot\cdots\cdot\mathrm{Q}_1(a_1,a_0)\mathbf{Y}(a_0). \tag{30}$$

For the case of a transverse isotropic linear slip boundary at a_i, $\mathbf{Y}(a_i^+)\neq\mathbf{Y}(a_i^-)$ but are related as

$$\mathbf{Y}(a_i^+)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&\eta_T&1&0\\\eta_N&0&0&1\end{bmatrix}\mathbf{Y}(a_i^-)\equiv(\mathrm{I}+\mathrm{S}_i)\mathbf{Y}(a_i^-), \tag{31}$$

and this matrix, $\mathrm{I}+\mathrm{S}_i$, then, is included in the product of Eq. (30) between Q_{i+1}, and Q_i giving the transfer function across the n elastic layers with slip between the ith and $(i+1)$th layers. Clearly, $(\mathrm{I}+\mathrm{S}_i)^{-1}=\mathrm{I}-\mathrm{S}_i$ as it must according to the sign convention of Eq. (24).

IV. EXAMPLES

A. Sounding a slip interface

Consider an acoustic half-space separated by an elastic layer from another elastic half-space and assume the layer is imperfectly bonded to the elastic half-space. It is of interest to relate the reflection coefficient for an acoustic plane wave launched at the elastic system with the compliances associated with the imperfectly bonded interface. To simplify this example, consider only the case of normal incidence, see Fig. 1, which involves only P waves speeds, densities and the normal compliance, η_N at $x_2=0$. Note that for an elastic incident medium the exact same analysis could be carried out with an incident shear wave. Let the elastic layer, medium 1 occupy the region $-H<x_2<0$ and let it be imperfectly bonded to the elastic half-space, medium 2, at $x_2=0$, occupying the region $x_2>0$. The acoustic half-space, medium 0, occupies the region $x_2<-H$ and the usual interface conditions apply at $x_2=-H$.

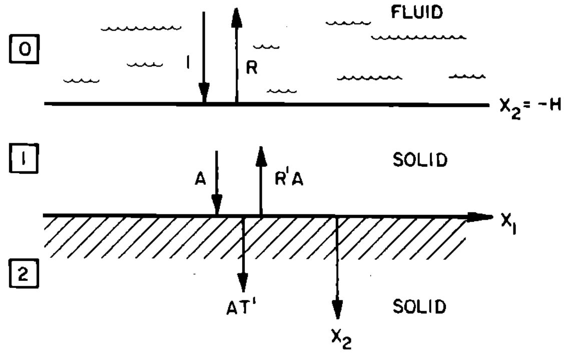

FIG. 1. Reflection of a normally incident plane wave by an elastic layer over an elastic half-space. The layer is assumed to be imperfectly bonded to the underlaying half-space.

Satisfying the two conditions at $x_2=-H$ gives two equations on the reflection coefficient R and an additional parameter A. The coefficients depend on R', the plane wave reflection coefficient for the imperfect interface between elastic medium 1 and elastic medium 2, and thus we can solve for R as a function of R', giving[6]

$$R=\frac{r+\exp(2i\Omega)R'}{1+\exp(2i\Omega)rR'}, \tag{32}$$

$$\Omega=\omega H/\alpha_1, \quad r=(Z_1-Z_0)/(Z_1+Z_0).$$

R' and T' can be found by applying the 2 imperfect interface conditions for normal incidence, and then R' is given by Eq. (27). Fig. 2 shows $|R|$ as a function of the nondimensional frequency Ω for $Z_1=Z_2=3Z_0$ for various values of the nondimensional complance, $E=\eta\rho_1\alpha_1^2/H$.

For $E=0$, the layer and underlying half space would be one medium and R would equal r. However, for E real and nonzero, $R(0)=r$, $dR(0)/d\Omega=0$, and as $\Omega\to\infty$, $|R|$ approaches unity. It is the layer thickness that gives the reflection coefficient spectra the additional oscillatory structure. If the layer were pure viscous,

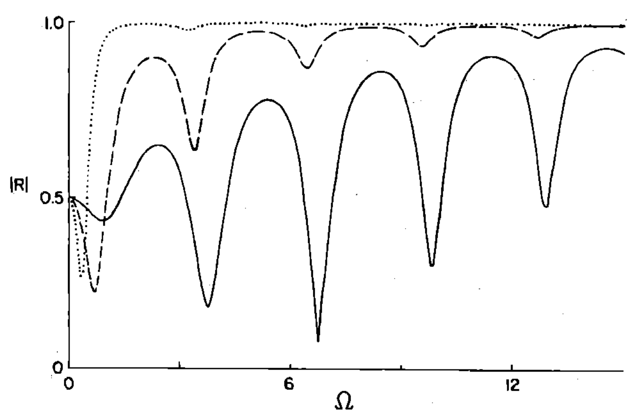

FIG. 2. Plane wave reflection coefficient amplitude spectra of the configuration of Fig. 1 for various values of E, the nondimensional normal compliance. Media 1 and 2 are taken to be the same with an impedance three times that of medium 0. The nondimensional frequency, Ω, is given in Eq. (32). The nondimensional normal complance, $E=\eta\rho_1\alpha_1^2/H$, is set equal to 0.2(——), 1.0(---), 5.0(····).

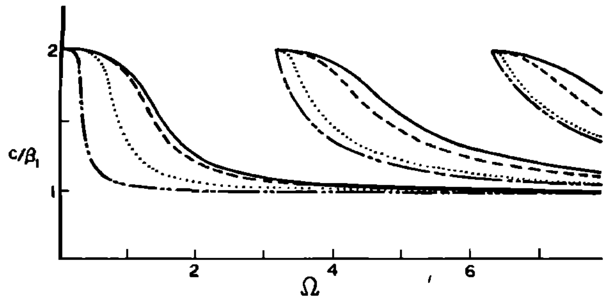

FIG. 3. Love-wave dispersion curves for the first three modes. The normalized phase speed, c/β_1, is given as a function of the nondimensional frequency, Ω, as given in Eq. (35). The shear speed ratio, β_2/β_1, is taken to be 2 and the shear modulus ratio, μ_2/μ_1, is taken to be 6. The nondimensional tangential compliance, E, also given in Eq. (35), is set equal to 0.0(——), 0.1(– – –), 1.0(· · · ·), 10.0(– · · – · ·).

$\eta = i\xi/\omega$, R' in Eq. (32) would be a real frequency independent number and the amplitude spectra would be periodic in Ω with period π. This is similar to the case when layer 1 and half-space 2 have different elastic properties but are perfectly bonded.

In general, Eq. (29) can be easily inverted, and measuring $R(\omega)$ enables $R'(\omega)$ to be constructed. This can be used as data for trying to invert Eq. (28) to give estimates of η_N and Z_2/Z_1.

B. Love-wave dispersion

Horizontally polarized shear waves may propagate with a real phase velocity in an elastic layer with shear speed β_1 bonded to an elastic half space with shear speed β_2 provided $\beta_1 < \beta_2$. Letting the layer occupy the region $-H < x_2 < 0$ and the half-space occupy $x_2 > 0$, then displacements in the layer and half-space may be written

$$u_3 = A\cos[\omega(1/\beta_1^2 - 1/c^2)^{1/2}(x_2 + H)], \quad -H < x_2 < 0,$$
$$u_3 = B\exp[-\omega(1/c^2 - 1/\beta_2^2)^{1/2}x_2], \quad x_2 > 0, \qquad (33)$$

where c is the phase speed in the x_1 direction. This automatically satisfies the stress free condition at $x_2 = -H$. At $x_2 = 0$, the continuity of τ_{23} and the slip condition on the displacement u_3 yields two equations on A and B,

$$-A\mu_1\omega(1/\beta_1^2 - 1/c^2)^{1/2}\sin\omega(1/\beta_1^2 - 1/c^2)^{1/2}H = -B\mu_2\omega(1/c^2 - 1/\beta_2^2)^{1/2}, \qquad (34)$$
$$B - A\cos\omega(1/\beta_1^2 - 1/c^2)^{1/2}H = -\eta_T B\mu_2\omega(1/c^2 - 1/\beta_2^2)^{1/2},$$

which, to allow a nontrivial solution, must have a vanishing determinant. After some manipulation, this dispersion relation may be put in the form

$$\cos X - \left(\frac{\mu_1/\mu_2}{(\Omega^2 - X^2)^{1/2}} + E\right) X\sin X = 0,$$
$$X = \omega H(1/\beta_1^2 - 1/c^2)^{1/2}, \quad \Omega = \omega H(1/\beta_1^2 - 1/\beta_2^2)^{1/2}, \qquad (35)$$
$$E = \eta_T \mu_1/H.$$

The terms Ω and E are the nondimensional frequency and tangential compliance, respectively. Setting $E = 0$ gives the usual Love-wave dispersion relation and letting $E \to \infty$ implies that $\sin X = 0$ or X is an integer multiple of π, which is the dispersion relation for SH waves in a free plate.

In general, if $\Omega = N\pi + \delta$, $\delta < \pi$, then $N+1$ propagating modes at that frequency exist, with speeds specified by $X_n = n\pi + \epsilon_n(\Omega, E)$, $n = 0, 1, 2, \ldots, N$, where $\epsilon_n \leq \delta$ as $X \leq \Omega$. In addition $\pi/2 > \epsilon_0 > \epsilon_1 > \ldots > \epsilon_N$. All values of ϵ_n decrease monotonically with increasing E. As for conventional Love waves, mode n cutoff occurs at $\Omega = n\pi$ with $X = \Omega$, i.e., $c = \beta_2$ and with group velocity equal to β_2.

Figure 3 shows the dispersion curves for $\beta_2/\beta_1 = 2$, $\mu_2/\mu_1 = 6$ for various values of E. The general shape, the high-frequency value for c, and the low-frequency cutoff value $c = \beta_2$ remain independent of E but as E increases the phase speed decreases making the drop in speed from low-frequency cutoff to high frequency sharper for higher values of E. The group velocity, c_g, given by $c/[1 - (\omega/c)\partial c/\delta\omega]$ is always positive but less than c. With increasing E, each mode has a sharper and slower group velocity minimum.

CONCLUSION

The theory of a linear slip condition between two elastic media has been presented and the plane wave reflection coefficients for plane slip interfaces have been derived. The effects on wave behavior of such interfaces have been exhibited in several cases. Such a slip condition can exhibit characteristic signatures in the spectra of reflected waves and on the dispersion relations of various elastic wave modes.

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