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REFLECTION OF ELASTIC WAVES FROM PERIODICALLY STRATIFIED MEDIA WITH INTERFACIAL SLIP*

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Abstract

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A periodically stratified elastic medium can be replaced by an equivalent homogeneous transverse isotropic medium in the long wavelength limit. The case of a homogeneous medium with equally spaced parallel interfaces along which there is imperfect bonding is a special instance of such a medium. Slowness surfaces are derived for all plane wave modes through the equivalent medium and reflection coefficients for a half-space of such a medium are found. The slowness surface for the SH mode is an ellipsoid. The exact solution for the reflection of SH-waves from a half-space with parallel slip interfaces is found following the matrix method of K. Gilbert applied to elastic waves. Explicit results are derived and in the long wavelength limit, shown to approach the results for waves in the equivalent homogeneous medium. Under certain conditions, a half-space of a medium with parallel slip interfaces has a reflection coefficient independent of the angle of incidence and thus acts like an acoustic reducing mirror. The method for the reflection of P- and SV-waves is fully outlined, and reflection coefficients are shown for a particular example. The solution requires finding the eigenvalues of a 4×4 transfer matrix, each eigenvalue being associated with a particular wave. At higher frequencies, unexpected eigenvalues are found corresponding to refracted waves for which shear and compressional parameters are completely coupled. The two eigenvalues corresponding to the transmitted wavefield give amplitude decay perpendicular to the stratification along with up- and downgoing phase propagation in some other direction.

Introduction

Geophysical media often exhibit anisotropic behavior due to alternating strata of material, each stratum itself being isotropic. This can occur over a wide range of

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length scales. When probed with radiation of a wavelength much larger than the width of the strata, such stratified regions exhibit mechanical properties that appear to be space-independent but direction-dependent depending only on the elevation angle with respect to the axis perpendicular to the plane of stratification. Such a medium is said to be transversely isotropic and has five independent elastic moduli. Waves in periodically stratified elastic media have been considered from two points of view, that of the long wavelength approximation for which an equivalent homogeneous transversely isotropic medium is found, and that of the exact solution valid for all wavelengths (Rytov 1956). In this paper, the problem of reflection from a periodically stratified half-space will be considered from both points of view.

In the first section, elastic moduli will be easily derived following Helbig (1958) from quasistatic considerations for a periodic medium with an arbitrary number of constituent layers per period. The field is assumed to vary slowly with respect to H, the spatial period of the layer aggregate. All moduli of the equivalent transverse isotropic medium are shown to depend only on thickness-weighted averages of various parameters of the individual layers. Because this is a quasistatic analysis, the effective density is not determined. It will be shown from the exact solution for SH-wave propagation that the correct effective density for the SH-wave is the thickness-weighted average of the layer densities. Slowness surfaces in the equivalent transverse isotropic media and plane wave reflection from equivalent half-spaces are discussed.

The second approach using an exact solution employs an adaptation of the matrix methods for layered media (Thomson 1950, Haskell 1953)—the propagator matrix method of Aki and Richards (1980). The exact solution of the problem of reflection from a periodically stratified half-space is found following the suggestions of Gilbert (1979). This is valid for all wavelengths, and it may be seen how the exact solution approaches the approximate long wavelength solution.

The particular periodic medium for which results are shown is one which is homogeneous except for periodically spaced parallel slip interfaces. Along these interfaces linear tangential slip may occur, i.e. there is a tangential displacement discontinuity across the interface that is, at each frequency, proportional to the corresponding shear stress across the interface. Such a model has but one density, ρ, and this is also the density of the equivalent medium. This may be used as a model for many types of laminates, and leads to relatively simple expressions. However, the techniques presented here are applicable to any periodic media made up of homogeneous layers.

The second section deals with plane wave slowness surfaces in the transverse isotropic medium equivalent to the medium with slip interfaces. Plane wave speeds are either decreased or unchanged but never increased due to elastic tangential linear slip.

Section 3 addresses the problem of the reflection of SH-waves from a half-space, the boundary of which is parallel to the slip interfaces. First, the long wavelength approximation is used, and then the exact solution is derived. Results are shown for the case when the half-space in which the incident and reflected waves reside is the same mechanically (same shear modulus and same density) as the half-space from which reflection occurs, except for the presence of the slip interfaces. In the long wavelength limit, the reflection coefficient in this case is independent of angle of incidence and thus the half-space appears as a reducing mirror at low frequencies. This phenomenon also occurs between two media of the same speed but with different densities.

Section 4 considers SH-wave propagation in a general stratified periodic medium. It is proved that for the SH-wave reflection coefficient of the equivalent medium to be precisely the zero frequency limit of the exact reflection coefficient it is necessary to set the density of the equivalent medium to be the thickness-weighted average density (which is the average density) of the periodic medium.

The method of solution for the reflection of P- and SV-waves is outlined in section 5. The exact solution requires evaluating the eigenvalues and eigenvectors of a 4×4 matrix, and then solving a pair of linear equations for the two unknown reflection coefficients. The condition of perfect tangential slip is also considered in that section. This is analogous to considering a periodic medium composed of alternating solid and ideal fluid layers. It is shown how in this case the 4×4 matrix reduces again to a 2×2 matrix.

1. Equivalence of Transverse Isotropy to Periodic Stratification

Consider an infinite linear elastic medium made up of plane homogeneous layers. Let x_3 be the axis perpendicular to the layering and let the layering be periodic with period H (fig. 1). Assume that one period is made up of N homogeneous layers, each with shear modulus μ_i, Poisson ratio ν_i, and thickness $h_i H$, $i = 1, \ldots, N$. Let γ_i be defined as the square of the ratio of shear speed, β_i, to compressional speed α_i, so that

$$\gamma_i \equiv \beta_i^2/\alpha_i^2 = (1/2 - \nu_i)/(1 - \nu_i). \tag{1}$$

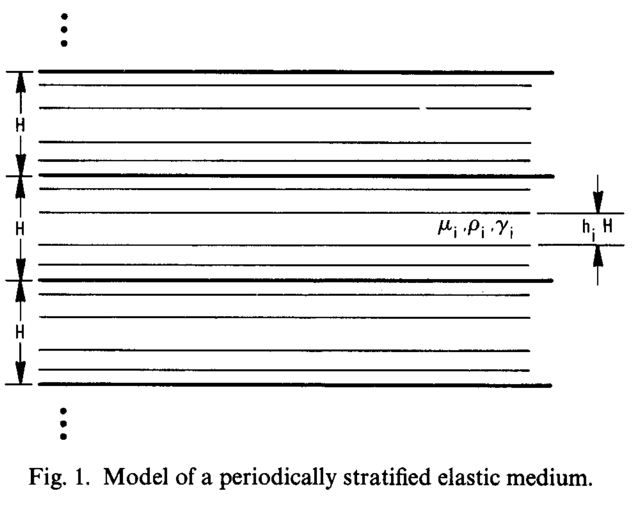

Fig. 1. Model of a periodically stratified elastic medium.

For stress and strain fields whose scale of variation (wavelength) is much greater than H, effective transverse isotropic moduli can be derived in terms of the μ_i, γ_i and h_i. There are five independent elastic constants for a transverse isotropic medium, and using condensed notation

$$\begin{aligned} \sigma_1 &= \sigma_{11} & \epsilon_1 &= \epsilon_{11} \\ \sigma_2 &= \sigma_{22} & \epsilon_2 &= \epsilon_{22} \\ \sigma_3 &= \sigma_{33} & \epsilon_3 &= \epsilon_{33} \\ \sigma_4 &= \sigma_{23} & \epsilon_4 &= \epsilon_{23} \\ \sigma_5 &= \sigma_{31} & \epsilon_5 &= \epsilon_{31} \\ \sigma_6 &= \sigma_{12} & \epsilon_6 &= \epsilon_{12}, \end{aligned} \tag{2}$$

the stress-strain relations are

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}, \tag{3}$$

where

$$C_{12} = C_{11} - 2C_{66}. \tag{4}$$

Stresses which act on a face perpendicular to the x_3-axis, i.e. σ_3, σ_4, and σ_5, are assumed constant across a set of layers of width H due to quasistatic or low-frequency equilibrium requirements. Strains that lie in a plane parallel to the layering, i.e. ϵ_1, ϵ_2 and ϵ_6, are assumed constant due to the layers being constrained to having the same in-plane motion in a medium of infinite extent in the x_1- and x_2-directions.

The other strains over a full spatial period H can be written in terms of the strains of the individual layers. For example, let displacement be indicated by u_1, u_2, and u_3. The strain ϵ_{3i} is given by u_3 at the top of the ith layer minus u_3 at the bottom of the ith layer (call this Δ_{3i}) divided by $h_i H$, the thickness of the ith layer. The average strain ϵ_3 over a full spatial period is given by

$$\begin{aligned} \epsilon_3 &= \frac{u_3(x_3 + H) - u_3(x_3)}{H} \\ &= \frac{1}{H} \sum_{i=1}^{N} \Delta_{3_i} = \frac{1}{H} \sum_{i=1}^{N} h_i H \epsilon_{3_i} = \sum h_i \epsilon_{3_i} \equiv \langle \epsilon_3 \rangle, \end{aligned} \tag{5}$$

where $\langle\ \rangle$ denotes the thickness-weighted average. Similarly, the average strains, ϵ_4 and ϵ_5, across a full period can be found, using displacements u_2 and u_1, respectively, to be given by

$$\epsilon_4 = \langle \epsilon_4 \rangle, \qquad \epsilon_5 = \langle \epsilon_5 \rangle. \tag{6}$$

The in-plane average stresses may be derived in a similar way. For example, σ_{6_i} is given by the force per unit length in the x_1-direction on an x_2-face (call it f_{12_i}) or equivalently in the x_2-direction on an x_1-face divided by $h_i H$. The average stress across the full width H is given by

$$\sigma_6 = \frac{1}{H} \sum_{i=1}^{N} f_{12_i} = \frac{1}{H} \sum h_i H \sigma_{6_i} = \langle \sigma_6 \rangle. \tag{7}$$

Similarly, using the force in the x_1-direction on the x_1-face and the force in the x_2-direction on the x_2-face gives the average stresses σ_1 and σ_2, respectively:

$$\sigma_1 = \langle \sigma_1 \rangle, \qquad \sigma_2 = \langle \sigma_2 \rangle. \tag{8}$$

Now consider the relations between shear stress and shear strain. In each layer

$$\begin{aligned} \sigma_4 &= \sigma_{4_i} = 2\mu_i \epsilon_{4_i}, \\ \sigma_5 &= \sigma_{5_i} = 2\mu_i \epsilon_{5_i}, \\ \sigma_{6_i} &= 2\mu_i \epsilon_{6_i} = 2\mu_i \epsilon_6, \end{aligned} \tag{9}$$

so we may average these equations giving

$$\begin{aligned} \epsilon_4 &= \langle \epsilon_4 \rangle = \langle \sigma_4 / 2\mu \rangle = \langle \mu^{-1} \rangle \sigma_4 / 2, \\ \epsilon_5 &= \langle \mu^{-1} \rangle \sigma_5 / 2, \\ \sigma_6 &= \langle \sigma_6 \rangle = \langle 2\mu\epsilon_6 \rangle = 2\langle \mu \rangle \epsilon_6 . \end{aligned} \tag{10}$$

A comparison of these equations with the last three equations shown in matrix form in (3) yields the effective transverse isotropic moduli C_{44} and C_{66} for the periodically layered medium as

$$C_{44} = \langle \mu^{-1} \rangle^{-1}, \qquad C_{66} = \langle \mu \rangle. \tag{11}$$

Now consider the relation between the normal stress σ_3 and the normal strains ϵ_1, ϵ_2 and ϵ_3. In each layer

$$\sigma_3 = \sigma_{3_i} = [(1 - 2\gamma_i)(\epsilon_1 + \epsilon_2) + \epsilon_{3_i}]\mu_i/\gamma_i \tag{12}$$

and multiplying this equation by γ_i/μ_i, averaging, and then dividing by $\langle \gamma/\mu \rangle$ gives

$$\sigma_3 = (1 - 2\langle \gamma \rangle)\langle \gamma/\mu \rangle^{-1}(\epsilon_1 + \epsilon_2) + \langle \gamma/\mu \rangle^{-1}\epsilon_3 . \tag{13}$$

Comparison of this with the third equation of (3) gives the effective elastic moduli C_{33} and C_{13} as

$$C_{33} = \langle \gamma/\mu \rangle^{-1}, \qquad C_{13} = (1 - 2\langle \gamma \rangle)\langle \gamma/\mu \rangle^{-1}. \tag{14}$$

Now consider finally the relation between the normal stress σ_1, and the normal strains. In each layer,

$$\sigma_{1_i} = [\epsilon_1 + (1 - 2\gamma_i)(\epsilon_2 + \epsilon_{3_i})]\mu_i/\gamma_i \tag{15}$$

and into this equation substitute for $\mu_i \epsilon_{3_i}/\gamma_i$ the expression obtained from (12), giving

$$\begin{aligned}\sigma_{1_i} &= [\epsilon_1 + (1-2\gamma_i)\epsilon_2]\mu_i/\gamma_i + (1-2\gamma_i)[\sigma_3 - (1-2\gamma_i)(\epsilon_1+\epsilon_2)\mu_i/\gamma_i] \\ &= \{[1-(1-2\gamma_i)^2]\epsilon_1 + [1-(1-2\gamma_i)](1-2\gamma_i)\epsilon_2\}\mu_i/\gamma_i + (1-2\gamma_i)\sigma_3 \\ &= 2\mu_i\epsilon_1 + 2(1-2\gamma_i)\mu_i(\epsilon_1+\epsilon_2) + (1-2\gamma_i)\sigma_3 . \end{aligned} \tag{16}$$

Averaging and substituting for σ_3 its value given in (13) gives

$$\begin{aligned}\sigma_1 &= 2\langle\mu\rangle\epsilon_1 + [2\langle\mu\rangle - 4\langle\gamma\mu\rangle + (1-2\langle\gamma\rangle)^2\langle\gamma/\mu\rangle^{-1}] \\ &\quad\times(\epsilon_1+\epsilon_2) + (1-2\langle\gamma\rangle)\langle\gamma/\mu\rangle^{-1}\epsilon_3 . \end{aligned} \tag{17}$$

Comparison of this with the first equation of (3) gives

$$C_{11} = 4\langle\mu\rangle - 4\langle\gamma\mu\rangle + (1-2\langle\gamma\rangle)^2\langle\gamma/\mu\rangle^{-1} \tag{18}$$

with

$$C_{12} = C_{11} - 2\langle\mu\rangle = C_{11} - 2C_{66},$$

which checks with (4) and thus such a periodic medium behaves as a transverse isotropic medium with the above derived elastic moduli for fields whose characteristic scale of variation is much larger than H. These elastic constants, following Helbig (1958) were derived according to a quasistatic approximation.

A special case of some geophysical interest is when Poisson's ratio is the same for all layers so that

$$\begin{aligned}\langle\gamma\rangle &= \gamma, \\ \langle\gamma\mu\rangle &= \gamma C_{66}, \\ \langle\gamma/\mu\rangle^{-1} &= C_{44}/\gamma. \end{aligned} \tag{19}$$

This is a transverse isotropic media whose elastic modulus matrix depends on C_{44} and C_{66} as given in (11) and γ. The other moduli are given by

$$C_{33} = C_{44}/\gamma, \qquad C_{13} = (1-2\gamma)C_{44}/\gamma, \qquad C_{11} = 4(1-\gamma)C_{66} + (1-2\gamma)^2 C_{44}/\gamma. \tag{20}$$

Now suppose the layering is such that one period is made up of only two layers and let layer 2 be very thin and very soft so that $h_1 \to 1$, $h_2 \to 0$ and $\mu_2 \to 0$ such that

$$h_2 H/\mu_2 \to \eta_T, \qquad \gamma_2 h_2 H/\mu_2 \to \eta_N . \tag{21}$$

Let $\mu_1 \equiv \mu$ and $\gamma_1 \equiv \gamma$. This is equivalent to postulating a homogeneous medium with equally spaced plane linear slip interfaces, all parallel to the x_1, x_2-plane, and each with tangential and normal compliance given by η_T and η_N, respectively (Schoenberg 1980). Across such an interface, stresses are continuous but displacement discontinuities are allowed so that

$$\begin{aligned}\Delta u_1 &= \eta_T\sigma_5 , \\ \Delta u_2 &= \eta_T\sigma_4 , \\ \Delta u_3 &= \eta_N\sigma_3 . \end{aligned} \tag{22}$$

Then from (11), (14), and (18), the elastic moduli of the equivalent transverse isotropic medium are given by

$$\begin{aligned} &C_{44} = \mu/(1 + E_T), \qquad C_{66} = \mu \\ &C_{33} = \mu/(\gamma + E_N), \qquad C_{13} = (1 - 2\gamma)C_{33} \\ &C_{11} = 4(1 - \gamma)\mu + (1 - 2\gamma)^2 C_{33} = [1 - 4(1 - \gamma)E_N]C_{33}, \\ &E_T = \eta_T \mu/H, \qquad E_N = \eta_N \mu/H. \end{aligned} \tag{23}$$

There are four independent elastic parameters, the shear modulus μ, the nondimensional ratio γ, and the dimensionless compliances E_T and E_N.

A physically reasonable simplification is the case in which there is only tangential slip and the normal compliance E_N vanishes. Then all coefficients are as in the isotropic case except for C_{44}, which is given in (23).

It should be noted here that we can relax the requirement that we have equally spaced identical cracks. As the slip is manifest in the two dimensionless compliances $E_T = \mu\eta_T/H$ and $E_N = \mu\eta_N/H$, a homogeneous material with irregularly spaced parallel slip interfaces behaves identically as long as η_T/H and η_N/H are the same (where η_T is the total tangential slip of all the cracks in distance H due to unit shear stress, and η_N is the total normal slip due to unit normal stress). Thus a rock with arbitrary closely spaced parallel linear slip interfaces can be modeled as a homogeneous transverse isotropic solid as long as the average slip per distance normal to the system of cracks is independent of x_3.

2. Slowness Surfaces in a Medium with Plane Parallel Slip Interfaces

The transverse isotropic elastic moduli derived above under the quasistatic, slowly varying field approximation may be used to derive slowness surfaces and other parameters of dynamic wave propagation for wavelengths much larger than the average spacing between slip interfaces. Let ρ be the uniform density and $\beta = (\mu/\rho)^{1/2}$ be the shear speed of the material between the slip interfaces. Allow a plane wave to propagate through the medium at an angle θ to the x_3-axis. With no loss of generality, let the plane of the propagation vector be the x_1, x_3-plane, so that the displacement is not a function of x_2.

If one allows all terms with partial derivatives with respect to x_2 to vanish, the displacement equations of motion for a homogeneous transverse isotropic medium are

$$\begin{aligned} &C_{11}u_{1,11} + C_{13}u_{3,31} + C_{44}(u_{3,13} + u_{1,33}) = \rho\ddot{u}_1, \\ &C_{66}u_{2,11} + C_{44}u_{2,33} = \rho\ddot{u}_2, \\ &C_{44}(u_{3,11} + u_{1,31}) + C_{13}u_{1,13} + C_{33}u_{3,33} = \rho\ddot{u}_3. \end{aligned} \tag{24}$$

As is well known, the second equation of (24)—in terms of u_2—is not coupled to the others—in terms of u_1, u_3—so that the "antiplane strain" slowness surface, which depends only on η_T is uncoupled from the "plane strain" slowness surfaces which depend on η_N and η_T.

For "antiplane strain", let

$$u_1 = u_3 = 0, \qquad u_2 = U_2 \exp i\omega[S(x_1 \sin\theta + x_3 \cos\theta) - t], \tag{25}$$

where the slowness S is a function of θ. Substitution of (25) into (24) gives

$$S^2\left[\sin^2\theta + \frac{\cos^2\theta}{1+E_T}\right] = \beta^{-2}. \tag{26}$$

This is the polar coordinate S- θ-representation of an ellipse, a quarter of which is shown in fig. 2, for $E_T = 3.0$, labeled SH. By letting $S_1 = S\sin\theta$, $S_3 = S\cos\theta$, (26) is converted to a rectangular Cartesian (S_1, S_3) representation:

$$S_1^2 + \frac{S_3^2}{1+E_T} = \beta^{-2}. \tag{27}$$

From this it is clear that the semi-major axis is in the x_3-direction and has length $(1+E_T)^{1/2}/\beta$ and the semi-minor axis is in the x_1-direction and has length $1/\beta$. The shape of the wavefront is the polar reciprocal of the slowness surface (26), and this is given by

$$v(\theta) = S^{-1} = \left[\sin^2\theta + \frac{\cos^2\theta}{1+E_T}\right]^{1/2}\beta. \tag{28}$$

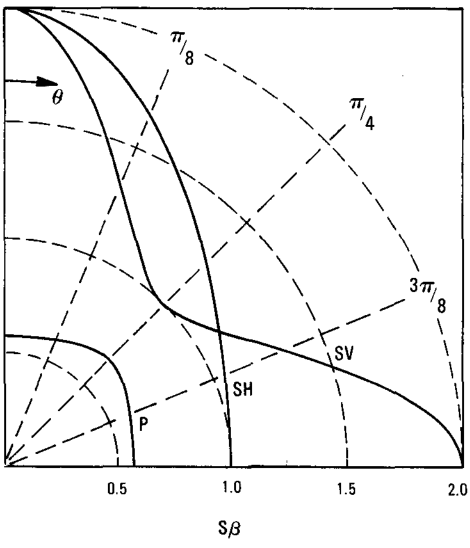

Fig. 2. Slowness surfaces, normalized by the shear speed, for the transverse isotropic medium equivalent, for the long wavelength approximation, to a homogeneous medium with equally spaced plane interfaces along which tangential elastic slip occurs. Poisson's ratio is 1/4 and the nondimensional slip compliance, E_T, is 3. The polar plot shows $S\beta$ as a function of the angle of propagation.

Clearly, the plane wave propagating normal to the layering is slowed up by an amount $(1 + E_T)^{-1/2}$ relative to a plane wave propagating parallel to the layering which propagates at the shear speed.

For plane strain,

$$u_2 = 0, \quad \begin{bmatrix} u_1 \\ u_3 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_3 \end{bmatrix} \exp i\omega[S(x_1 \sin\theta + x_3 \cos\theta) - t]. \tag{29}$$

The slowness surfaces resulting from such a plane wave in a transverse isotropic medium have been discussed by several authors. We shall consider the special case for which $\eta_N = 0$, i.e. on the interfaces normal slip vanishes but tangential slip is finite. Then substitution of (29) into (24), using isotropic values for C_{ij} except for C_{44} [which is given by (23)], yields

$$\begin{gathered} \left[S^2\left(\frac{\sin^2\theta}{\gamma} + \frac{\cos^2\theta}{1+E_T}\right) - \beta^{-2}\right]U_1 + S^2\left(\frac{1-2\gamma}{\gamma} + \frac{1}{1+E_T}\right)\sin\theta\cos\theta U_3 = 0, \\ S^2\left(\frac{1-2\gamma}{\gamma} + \frac{1}{1+E_T}\right)\sin\theta\cos\theta U_1 + \left[S^2\left(\frac{\sin^2\theta}{1+E_T} + \frac{\cos^2\theta}{\gamma}\right) - \beta^{-2}\right]U_3 = 0. \end{gathered} \tag{30}$$

For a nontrivial solution to exist, the determinant of the displacement amplitudes U_1, U_3 must vanish, giving

$$(S^2\beta^2)^2 \frac{1}{\gamma(1+E_T)}[1 + (1-\gamma)E_T \sin^2 2\theta] - S^2\beta^2\left(\frac{1}{\gamma} + \frac{1}{1+E_T}\right) + 1 = 0. \tag{31}$$

Figure 2 shows, for $E_T = 3$ and $\gamma = 1/3$, the two slowness surfaces which are solutions of (31) labeled P for the compressional-type solution and SV for the shear-type solution. It may be seen that a plane wave can only be slowed by the presence of closely spaced slip interfaces, never speeded up. At the angles $\theta = 0$, $\pi/4$, and $\pi/2$, the P-wave displacement is parallel to the slowness vector and the SV-wave displacement is perpendicular to the slowness vector. The values of the slownesses squared of the P-, SV-, and SH-waves are given in table 1.

Table 1. *$\beta^2 S^2$ for plane waves in a medium with slip interfaces.*

θ	P	SV	SH
0	γ	$1 + E_T$	$1 + E_T$
$\pi/4$	$\gamma\left[1 + \frac{\gamma E_T}{1 + (1-\gamma)E_T}\right]$	1	$1 + \frac{E_T}{2 + E_T}$
$\pi/2$	γ	$1 + E_T$	1

The slip interfaces cause the P-wave to be slowed the most at an angle of $\pi/4$. At this angle, the SV-wave is not slowed at all. However, for positive Poisson's ratio ($\gamma < 1/2$), the P-wave is faster than the SV-wave for all real values of E_T. For negative Poisson's ratio ($3/4 > \gamma > 1/2$), the P-wave may be slowed enough to be slower than the SV-wave. This occurs when $E_T > (1-\gamma)/(2\gamma - 1)$. Note that results given by Brekhovskikh (1960) for speeds parallel and perpendicular to the layering agree with the slownesses obtained above with C_{11}, C_{33}, C_{44}, and C_{66} given by the general expressions of (11), (14), and (18) and the density given by $\langle\rho\rangle$.

3. Reflection of SH-Waves from an Elastic Half-Space with Plane Parallel Slip Interfaces

An elastic homogeneous medium with parameters μ_0, ρ_0, $\beta_0 = (\mu_0/\rho_0)^{1/2}$ occupies the region $x_3 < 0$. A medium with parameters μ, ρ, $\beta = (\mu/\rho)^{1/2}$ occupies the region $x_3 > 0$ and this medium is assumed to contain plane slip interfaces, parallel to the x_1, x_2-plane spaced a distance H apart with dimensionless tangential slip compliance E_T. A plane SH-wave is incident from medium 0 on the boundary between the two media $x_3 = 0$ (fig. 3). The wavefield in the region $x_3 < 0$ has the form

$$
\begin{aligned}
u_2 &= \exp(\mathrm{i}\omega S_{3_0} x_3) + R\exp(-\mathrm{i}\omega S_{3_0} x_3),\\
\sigma_4 &= \mu_0 \mathrm{i}\omega S_{3_0}[\exp(\mathrm{i}\omega S_{3_0} x_3) - R\exp(-i\omega S_{3_0} x_3)], \qquad (32)\\
S_{3_0} &= (\beta_0^{-2} - S_1^2)^{1/2},
\end{aligned}
$$

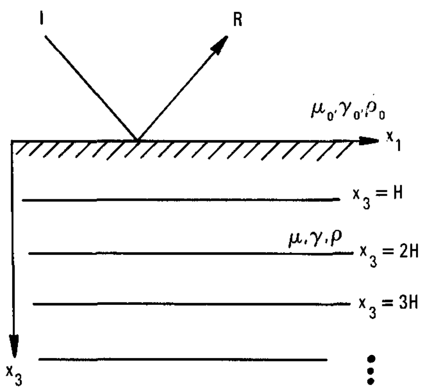

Fig. 3. Plane wave reflection from a medium with plane parallel slip interfaces.

where R is the plane SH-wave reflection coefficient. These equations can be put into a matrix form for later use:

$$
\begin{aligned}
\begin{bmatrix}\sigma_4\\ u_2\end{bmatrix} &= \mathbf{B}_0\,\mathbf{D}_0(x_3)\begin{bmatrix}1\\ R\end{bmatrix},\\
\mathbf{D}_0(x_3) &= \mathrm{diag}\,[\exp \mathrm{i}\omega S_{3_0} x_3, \quad \exp -\mathrm{i}\omega S_{3_0} x_3], \qquad (33)\\
\mathbf{B}_0 &= \begin{bmatrix}\mathrm{i}\omega\mu_0 S_{3_0} & -\mathrm{i}\omega\mu_0 S_{3_0}\\ 1 & 1\end{bmatrix}.
\end{aligned}
$$

Note that $\mathbf{D}_0(0)$ is the identity matrix.

3.1. Long wavelength, effective transverse isotropic medium application

In this case the wavelength is assumed much longer than H so the effective transverse isotropic medium theory described above is applicable. In the region $x_3 > 0$, one has from (27)

$$\begin{aligned} u_2 &= T \exp [i\omega(1 + E_T)^{1/2}(\beta^{-2} - S_1^2)^{1/2}x_3], \\ \sigma_4 &= C_{44} u_{2,3} = T\mu(1 + E_T)^{-1/2} i\omega(\beta^{-2} - S_1^2)^{1/2} \\ &\quad \times \exp [i\omega(1 + E_T)^{1/2}(\beta^{-2} - S_1^2)^{1/2}x_3]. \end{aligned} \tag{34}$$

Continuity of u_2 and σ_4 at $x_3 = 0$ gives

$$\begin{aligned} &1 + R = T, \\ &1 - R = \chi T, \\ &\chi = \frac{\mu}{\mu_0 S_{3_0}} (\beta^{-2} - S_1^2)^{1/2}(1 + E_T)^{-1/2}, \end{aligned} \tag{35}$$

which have the usual solution

$$R = \frac{1 - \chi}{1 + \chi}, \qquad T = \frac{2}{1 + \chi}. \tag{36}$$

Note that χ depends on the angle of incidence, $\theta_0 = \arcsin (\beta_0 S_1)$, and that $\beta_0 S_{3_0} = \cos \theta_0$. From the expression for χ in (35), it follows that the same function of θ_0 would be obtained for a homogeneous medium (ρ_h, β_h, $\mu_h = \rho_h \beta_h^2$) occupying region $x_3 > 0$ if it had a shear modulus $\mu_h = \mu(1 + E_T)^{-1/2}$ without changing the value of the shear speed; i.e. a homogeneous medium with shear speed $\beta_h = \beta$ and density $\rho_h = \rho(1 + E_T)^{-1/2}$ would be indistinguishable from the medium with plane parallel slip interfaces through its long wavelength reflection and transmission coefficients R and T, provided the compliance E_T is frequency-independent for small frequency. However, the transmission angle θ from (26) is given by

$$\frac{\sin \theta_0}{\beta_0} = S(\theta) \sin \theta = \frac{\sin \theta}{\beta[\sin^2 \theta + (1 + E_T)^{-1} \cos^2 \theta]^{1/2}}, \tag{37}$$

or

$$\theta = \arctan \left[\frac{\sin \theta_0}{(1 + E_T)^{1/2}(\beta_0^2/\beta^2 - \sin^2 \theta_0)^{1/2}}\right], \tag{38}$$

whereas the homogeneous medium with $\beta_h = \beta$ and $\rho_h = \rho(1 + E_T)^{1/2}$ has a transmission angle given by (38) but with $E_T = 0$; i.e. this homogeneous medium and the medium with plane parallel slip interfaces, for long wavelength, have identical reflection and transmission coefficients but different transmission angles.

When the shear speed ratio is unity, we obtain from (35) and (38)

$$\chi = \frac{\rho(1 + E_T)^{-1/2}}{\rho_0}, \qquad \theta = \arctan \frac{\tan \theta_0}{(1 + E_T)^{1/2}}, \tag{39}$$

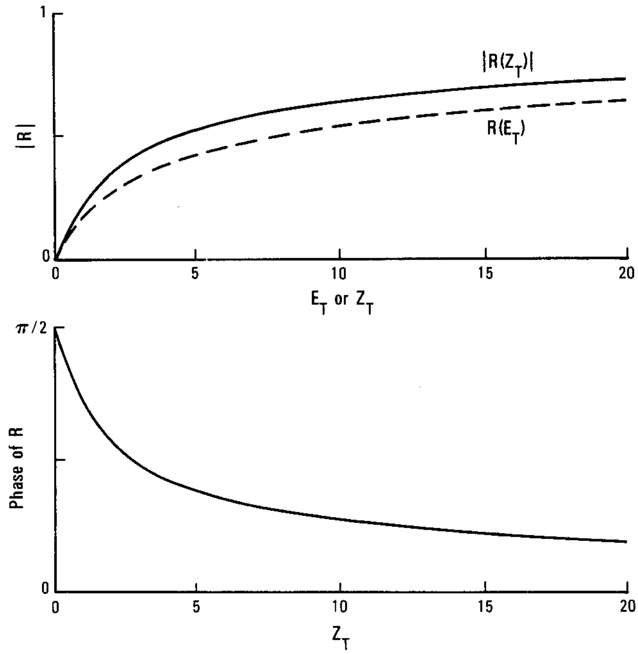

Fig. 4. Real long wavelength reflection coefficient R as a function of the elastic slip compliance E_T (- - -); amplitude and phase of R as a function of the viscous slip compliance $Z_T = \zeta_T \mu/\omega H$ for SH-waves from an elastic medium incident on a half-space of the same medium with equally spaced slip interfaces.

and now the reflection and transmission coefficients are independent of angle of incidence—in particular, when the medium occupying $x_3 > 0$ is the same medium as that occupying $x_3 < 0$ except for the slip interfaces, $\chi = (1 + E_T)^{-1/2}$ and R for this case is represented by the dashed line in fig. 4 plotted as a function of E_T. For $E_T \ll 1$, $R \sim E_T/4$ and as E_T grows, R grows very gradually to unity. The case of $E_T \to \infty$ corresponds to the surface ($x_3 = 0$) being stress-free, in which case $R = 1$.

A long wavelength reflection coefficient R, independent of angle of incidence with absolute value less than unity, implies that the interface $x_3 = 0$ acts like a reducing mirror, i.e. the reflection of a point source located at $x_3 = -d$ appears to come from its image point at $x_3 = +d$ but with its strength multiplied by the factor R.

If the slip interfaces behave as a viscous fluid tangentially (so that $h_2 H$ divided by the viscosity is given by ζ_T), we have $\Delta \dot{u}_2 = \zeta_T \sigma_4$ or $\Delta u = (i\zeta_T/\omega)\sigma_4$ for both the viscosity and h_2 approaching zero (Schoenberg 1980). Thus the dimensionless tangential compliance has the form

$$E_T = i\zeta_T \mu/\omega H \equiv iZ_T. \tag{40}$$

Let the material occupying the region $x_3 > 0$ be the same as that occupying $x_3 < 0$ between the thin interfaces. Then $\chi = (1 + iZ_T)^{-1/2}$, magnitude and phase angle of R

as a function of Z_T are shown by the solid line in fig. 4. Note that Z_T increases as the fluid viscosity or the frequency decrease. For $Z_T \ll 1$, $|R| \sim Z_T/4$ and the phase of $R \sim \pi/2$. As Z_T grows, R approaches unity gradually.

3.2. *Exact theory*

In this case, the wavefield in the region $x_3 > 0$ is analyzed for all wavelengths. The slip interfaces with tangential compliance η_T are located at $x_3 = nH$, $n = 0, 1, 2, 3, \ldots$ The wavefield in the homogeneous region is given by (33), and the wavefield in the region $H > x_3 > 0$ has the form

$$\begin{bmatrix} \sigma_4 \\ u_2 \end{bmatrix}_{x_3} = \mathbf{BD}(x_3)\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \tag{41}$$

where A_1, A_2 are the coefficients of the down- and upgoing waves, respectively. **B** and **D** are as given in (33) but for medium parameters μ, β. Then

$$\begin{bmatrix} \sigma_4 \\ u_2 \end{bmatrix}_{x_3=H-} = \mathbf{BD}(H)\mathbf{B}^{-1}\begin{bmatrix} \sigma_4 \\ u_2 \end{bmatrix}_{x_3=0+}. \tag{42}$$

Due to slip at $x_3 = 0$

$$\begin{bmatrix} \sigma_4 \\ u_2 \end{bmatrix}_{x_3=0+} = \begin{bmatrix} 1 & 0 \\ \eta_T & 1 \end{bmatrix}\begin{bmatrix} \sigma_4 \\ u_2 \end{bmatrix}_{x_3=0-},$$

and thus

$$\begin{bmatrix} \sigma_4 \\ u_2 \end{bmatrix}_{x_3=H-} = \mathbf{BD}(H)\mathbf{B}^{-1}\begin{bmatrix} 1 & 0 \\ \eta_T & 1 \end{bmatrix}\begin{bmatrix} \sigma_4 \\ u_2 \end{bmatrix}_{x_3=0-} \equiv \mathbf{Q}\begin{bmatrix} \sigma_4 \\ u_2 \end{bmatrix}_{x_3=0-}, \tag{43}$$

$$\mathbf{Q} = \begin{bmatrix} \cos\Delta - E_T\,\Delta\sin\Delta & -\omega\mu S_3\sin\Delta \\ \dfrac{\sin\Delta + E_T\,\Delta\cos\Delta}{\omega\mu S_3} & \cos\Delta \end{bmatrix},$$

$$E_T = \eta_T\,\mu/H, \qquad \Delta = \omega S_3 H.$$

Q is the transfer matrix for stress and displacement across a complete cycle of width H. The determinant of **Q** is unity. Across n such cycles one has

$$\begin{bmatrix} \sigma_4 \\ u_2 \end{bmatrix}_{x_3=nH-} = \mathbf{Q}^n\begin{bmatrix} \sigma_4 \\ u_2 \end{bmatrix}_{x_3=0-} = \mathbf{Q}^n\mathbf{B}_0\begin{bmatrix} 1 \\ R \end{bmatrix}. \tag{44}$$

The matrix **Q** may be decomposed by means of the following similarity transformation:

$$\mathbf{Q} = \mathbf{V\Lambda V}^{-1}, \qquad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}, \tag{45}$$

where λ_1 and λ_2 are the eigenvalues, the first column of $\mathbf{V}$ is the eigenvector belonging to λ_1, and the second column is the eigenvector belonging to λ_2. Then

$$\mathbf{Q}^n = \mathbf{V}\mathbf{\Lambda}^n\mathbf{V}^{-1} = \frac{1}{|V|}\left\{\lambda_1^n\begin{bmatrix} v_{11} \\ v_{21}\end{bmatrix}[v_{22} \quad -v_{12}] + \lambda_2^n\begin{bmatrix} v_{12} \\ v_{22}\end{bmatrix}[-v_{21} \quad v_{11}]\right\}, \tag{46}$$

where $|V|$ is the determinant of $\mathbf{V}$. The product of the eigenvalues is equal to unity. The introduction of an infinitesimal amount of loss (Gilbert 1979) leads to the requirement that the field must vanish at infinity, i.e.

$$\lim_{n\to\infty}\begin{bmatrix}\sigma_4 \\ u_2\end{bmatrix}_{x_3=nH^-} = 0. \tag{47}$$

This enables us to solve for the reflection coefficient R. If the eigenvalues are complex conjugates, then $|\lambda_1| = |\lambda_2| = 1$, but with the introduction of a small amount of loss, one eigenvalue, say λ_1, has an absolute value less than unity; correspondingly, $|\lambda_2| > 1$. Thus, in the limit as $n \to \infty$, $\lambda_1^n \to 0$. If the eigenvalues are real, one will have absolute value less than unity, say λ_1, and one has an absolute value greater than unity, say λ_2. Equations (46) and (47), with $\lambda_1^n \to 0$ substituted into (44), yield one independent equation for R:

$$[-v_{21} \quad v_{11}]\mathbf{B}_0\begin{bmatrix}1 \\ R\end{bmatrix} = 0 \tag{48}$$

or

$$R = \frac{1-\chi}{1+\chi}, \qquad \chi = v_{11}/\mathrm{i}\omega\mu_0 S_{3_0} v_{21} = (\lambda_1 - Q_{22})/\mathrm{i}\omega\mu_0 S_{3_0} Q_{21}, \tag{49}$$

and the problem is reduced to finding the real eigenvalue with an absolute value less than unity or the complex eigenvalue which, with the introduction of a small loss, has an absolute value less than unity.

The transfer matrix $\mathbf{Q}$ given in (43) has the eigenvalues tr $\mathbf{Q}/2 \pm [(\text{tr } \mathbf{Q}/2)^2 - 1]$, where "tr" denotes the trace. These are given by

$$\lambda^\pm = \begin{cases}\cos\Delta - (E_\mathrm{T}\Delta/2)\sin\Delta \pm \mathrm{i}K^{1/2}\sin\Delta, & K > 0,\\ \cos\Delta - (E_\mathrm{T}\Delta/2)\sin\Delta \pm (-K)^{1/2}\sin\Delta, & K < 0,\end{cases} \tag{50}$$

$$K = 1 + E_\mathrm{T}\Delta\cot\Delta - E_\mathrm{T}^2\Delta^2/4.$$

For $K > 0$, the absolute values of the eigenvalues are unity. Letting $E_\mathrm{T} \to E_\mathrm{T}(1 + \mathrm{i}\epsilon)$, $0 < \epsilon \ll 1$ gives, to first order in ϵ,

$$\lambda^\pm\Big|_\epsilon = \lambda^\pm\left(1 + \frac{\mathrm{i}\epsilon E_\mathrm{T}}{\lambda^\pm}\frac{\partial\lambda^\pm}{\partial E_\mathrm{T}}\right) = \lambda^\pm(1 \mp \epsilon E_\mathrm{T}\Delta/2K^{1/2}), \tag{51}$$

which shows that λ_1 is λ^+. This follows from the fact that with $E_\mathrm{T} \to 0$ one has $\lambda^+ \to \exp(\mathrm{i}\Delta)$ corresponding to the wave propagating in the $+x_3$-direction, while $\lambda^- \to \exp(-\mathrm{i}\Delta)$ corresponds to the wave propagating in the $-x_3$-direction. In the presence of loss, the first wave decays as $x_3 \to \infty$ while the second wave grows. The requirement that the coefficient of the growing wave must vanish identically gives (48) on R. The eigenvalues (50) are real when $K < 0$. In this case, λ_1 is λ^+ when $\cot\Delta > E_T\Delta/2$ and λ_1 is λ^- when $\cot\Delta < E_T\Delta/2$. Again, the coefficient of the growing wave must vanish giving (48) on R. Thus, for any frequency and angle of incidence

$$\begin{aligned}\chi &= \frac{\mu S_3}{\mu_0 S_{3_0}} \frac{K^{1/2} + \mathrm{i}E_T\Delta/2}{1 + E_T\Delta\cot\Delta}, \qquad K > 0.\\ \chi &= \mathrm{i}\,\frac{\mu S_3}{\mu_0 S_{3_0}} \frac{\operatorname{sgn}(E_T\Delta/2 - \cot\Delta)(-K)^{1/2} + E_T\Delta/2}{1 + E_T\Delta\cot\Delta}, \qquad K < 0,\end{aligned} \tag{52}$$

and R is given by substituting (52) into (49). Note that for $K < 0$, $|R| = 1$ and for $K > 0$, $|R| < 1$.

For purely viscous interfaces, with E_T given by (40), $E_T\Delta = \mathrm{i}\zeta_T\mu S_3$ is independent of frequency, and the only dependence of χ on frequency is through $\cot\Delta$. Thus, for any angle of incidence, χ and thus R are periodic functions of frequency. Again one must determine which eigenvalue corresponds to the wave that vanishes at infinity.

To show the behavior of the transmitted waves in the periodic medium, note that substitution of (46) into (44) along with the condition that the coefficients of λ_2^n vanish (see (48)) gives

$$\begin{bmatrix}\sigma_4\\ u_2\end{bmatrix}_{x_3=nH-} = \frac{\lambda_1^n}{|V|}\begin{bmatrix}v_{11}\\ v_{21}\end{bmatrix}[v_{22} \;\; -v_{12}]\mathbf{B}_0\begin{bmatrix}1\\ R\end{bmatrix} = \lambda_1^n\,\mathbf{B}_0\begin{bmatrix}1\\ R\end{bmatrix}. \tag{53}$$

Using the values for $\mathbf{B}_0$ and R gives

$$\begin{bmatrix}\sigma_4\\ u_2\end{bmatrix}_{x_3=nH-} = \frac{2\lambda_1^n}{1+\chi}\begin{bmatrix}v_{11}/v_{21}\\ 1\end{bmatrix}. \tag{54}$$

The displacement field in the nth layer may be written with (41) and (54) as

$$\begin{aligned}u_2 &= A_1^{(n)}\exp \mathrm{i}\omega S_3(x_3 - (n-1)H) + A_2^{(n)}\exp -\mathrm{i}\omega S_3(x_3 - (n-1)H)\\ &= [1 \quad 1]\mathbf{D}(x_3 - nH)\mathbf{D}(H)\begin{bmatrix}A_1^{(n)}\\ A_2^{(n)}\end{bmatrix} = [1 \quad 1]\mathbf{D}(x_3 - nH)\mathbf{B}^{-1}\begin{bmatrix}\sigma_4\\ u_2\end{bmatrix}_{x_3=nH-}\\ &= \frac{2\lambda_1^n}{1+\chi}[1 \quad 1]\mathbf{D}(x_3 - nH)\mathbf{B}^{-1}\begin{bmatrix}v_{11}/v_{21}\\ 1\end{bmatrix}\\ &= \frac{\lambda_1^n}{1+\chi}[1 \quad 1]\mathbf{D}(x_3 - nH)\begin{bmatrix}1 + (\mu_0 S_{3_0}/\mu S_3)\chi\\ 1 - (\mu_0 S_{3_0}/\mu S_3)\chi\end{bmatrix}.\end{aligned} \tag{55}$$

Thus we see that the coefficient ratio of the upgoing to downgoing wave in each layer is just the reflection coefficient for the half-space when the medium occupying $x_3 < 0$ is identical with that occupying $x_3 > 0$ between the slip interfaces. From layer to layer the only change is that the field is multiplied by λ_1. With no loss, when the eigenvalues are a complex conjugate pair, $K > 0$, then $|\lambda_1| = 1$ which corresponds to propagation into the periodic medium with just a phase change from layer to layer. When the eigenvalues are real, $K < 0$, then λ_1 is real and $|\lambda_1| < 1$. The sign of λ_1 is the same as the sign of tr $\mathbf{Q}$. When λ_1 is positive the waves undergo an exponential decay but with no phase change from layer to layer. When λ_1 is negative in addition to decay there is π phase shift from layer to layer. Thus, real eigenvalues correspond to the stop bands of propagation in periodic media, for which even with no loss, only decaying and growing waves are possible.

The limit of long wavelength compared to H is approached as $\Delta \to 0$. From (52) and the fact that as $\Delta \to 0$, K is positive and $\Delta \cot \Delta \to 1$, we have

$$\lim_{\Delta \to 0} \chi = \frac{\mu S_3}{\mu_0 S_{3_0}} (1 + E_T)^{-1/2}, \tag{56}$$

which agrees with the approximate result obtained in (35) for reflection from the equivalent transverse isotropic medium. As K is positive, there is no stop band in the low-frequency limit. In this exact case, when the shear speeds β_0 and β are equal, $\mu S_3/\mu_0 S_{3_0}$ becomes ρ/ρ_0, but χ is still dependent on the angle of incidence through its dependence on $\Delta = \omega S_3 H$.

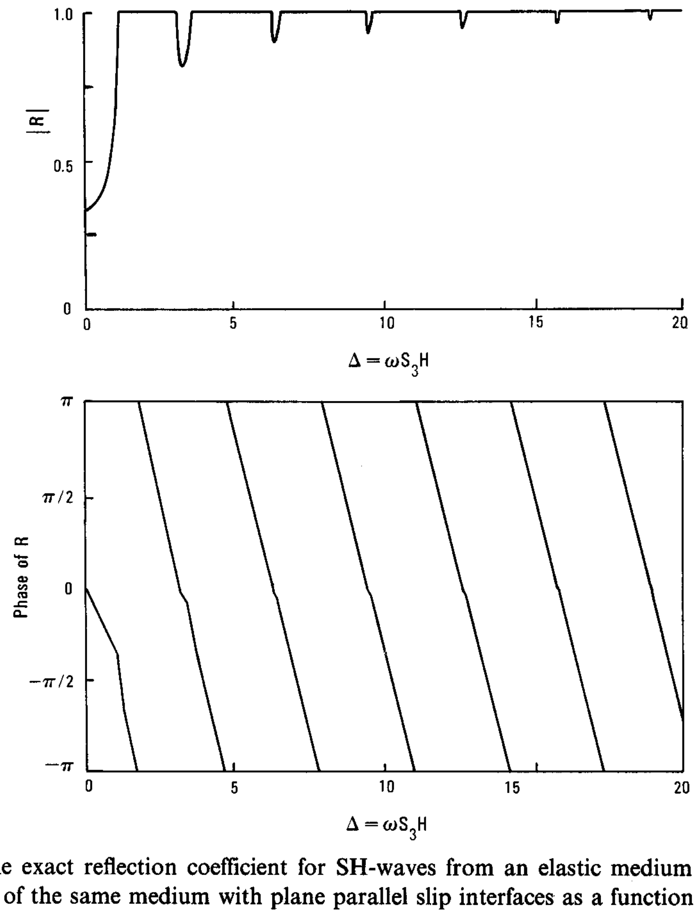

Fig. 5. The exact reflection coefficient for SH-waves from an elastic medium incident on a half-space of the same medium with plane parallel slip interfaces as a function of $\Delta = \omega S_3 H$ with the nondimensional elastic slip compliance, $E_T = \eta_T \mu/H$, equal to 3.

The high-frequency limit is approached as $\Delta \to \infty$. In this case, for S_3 real, K is negative, and $|R| = 1$ except when $\Delta = k\pi + \epsilon^2$, $k \gg 1$, $\epsilon^2 < 4/\pi k E_T$ so that the size of the regions for which there is propagation into the periodic medium shrinks with increasing Δ. Figure 5 shows the amplitude and phase of R as a function of the nondimensional parameter Δ for $E_T = 3$ when the media occupying $x_3 > 0$ and $x_3 < 0$ are identical so that $\mu S_3/\mu_0 S_{3_0} = 1$. Note that for large Δ the pass bands become vanishingly small and R is approximately equal to $\exp(-2i\Delta)$, independent of E_T. This is the reflection coefficient found if the surface $x_3 = -H$ were stress-free.

When $\beta > \beta_0$ and the angle of incidence is greater than critical, S_3 and Δ are imaginary. Letting

$$S_3 = i\tilde{S}_3 = i(S_1^2 - \beta^{-2}), \qquad \Delta = i\omega\tilde{S}_3 H = i\tilde{\Delta} \tag{57}$$

gives positive K and

$$\chi = \frac{i\mu\tilde{S}_3}{\mu_0 S_{3_0}} \frac{(1 + E_T\tilde{\Delta}\coth\tilde{\Delta} + E_T^2\tilde{\Delta}^2/4)^{1/2} - E_T\tilde{\Delta}/2}{1 + E_T\tilde{\Delta}\coth\tilde{\Delta}} \tag{58}$$

so that $|R|$ is equal to unity for all frequencies, as expected. As $\tilde{\Delta} \to \infty$, $\chi \to 0$ and $R \to 1$.

4. Effective Density in the Long Wavelength Approximation

The question of the density of the effective medium does not arise in the case of periodically spaced slip interfaces because the density of the medium is essentially uniform. However, in the general case in which one period is made up of N layers, each with its own elastic parameters and its own density, the question that arises is: What is the effective density to be used in a model whose effective elastic parameters were derived using a quasistatic approximation? The analysis of the previous section for the exact solution can be carried out for the periodic medium with N layers, retaining only terms of second order in ω, and then we can examine the reflection coefficient in the limit as frequency approaches zero. In this case the matrix $\mathbf{Q}$, analogous to that given in (43), may be written

$$\mathbf{Q} = \prod_{i=N}^{1} \begin{bmatrix} \cos\Delta_i & -\omega\mu_i S_{3_i}\sin\Delta_i \\ \dfrac{\sin\Delta_i}{\omega\mu_i S_{3_i}} & \cos\Delta_i \end{bmatrix} \tag{59}$$

$$= \begin{bmatrix} 1 - \omega^2 H^2 \sum_{i=1}^{N} S_{3_i}^2 h_i\left(\dfrac{h_i}{2} + \mu_i \sum_{j=1}^{i-1} h_j \mu_j^{-1}\right) + O(\omega^4) & \omega \sum_{1}^{N} \mu_i S_{3_i}\Delta_i + O(\omega^4) \\ H\sum_{i=1}^{N} \dfrac{h_i}{\mu_i} + O(\omega_2) & 1 - \omega^2 h^2 \sum_{i=1}^{N} S_{3_i}^2 h_i\left(\dfrac{h_i}{2} + \mu_i \sum_{j=i+1}^{N} h_j\mu_j^{-1}\right) + O(\omega^4) \end{bmatrix},$$

noting that $\Delta_i = \omega S_{3_i} h_i H$. This expansion is used because, although the exact coefficients for any value N may easily be found, the author knows of no exact analytical expression for arbitrary N. The expression in (61) may be proved to be valid for all N by induction. The eigenvalues of $\mathbf{Q}$ are found by noting that, from $\mathbf{Q}$ as given in (59), det $\mathbf{Q} = 1$ to order ω^4 and

$$\begin{aligned}\frac{\operatorname{tr}\mathbf{Q}}{2} &= 1 - \frac{\omega^2 H^2}{2}\left(\sum_{i=1}^{N} S_{3_i}^2 h_i \mu_i\right)\left(\sum_{j=1}^{N} h_j \mu_j^{-1}\right) + O(\omega^4)\\ &= 1 - \frac{\omega^2 H^2}{2}\langle S_3^2 \mu\rangle\langle \mu^{-1}\rangle + O(\omega^4),\\ &\quad \langle S_3^2 \mu\rangle = \langle \rho\rangle - S_1^2\langle \mu\rangle.\end{aligned} \tag{60}$$

Thus, the eigenvalues are given explicitly by

$$\begin{aligned}\lambda^{\pm} &= 1 - \frac{\omega^2 H^2}{2}(\langle \rho\rangle - S_1^2\langle \mu\rangle)\langle \mu^{-1}\rangle\\ &\quad \pm i\omega H[(\langle \rho\rangle - S_1^2\langle \mu\rangle)\langle \mu^{-1}\rangle]^{1/2} + O(\omega^3).\end{aligned} \tag{61}$$

For less than critical incidence, i.e. S_1^2 less than $\langle \rho\rangle/\langle \mu\rangle$, $\lambda^{\pm}$ are a complex conjugate pair and there is propagation into the periodic medium. Again, a small amount of loss in any of the μ_i implies that $\lambda_1 = \lambda^+$. Then, from (49), χ is given by

$$\begin{aligned}\chi &= \frac{[(\langle \rho\rangle - S_1^2\langle \mu\rangle)\langle \mu^{-1}\rangle]^{1/2} - i\omega H \sum_{i=1}^{N} S_{3_i}^2 \mu_i h_i\left(\sum_{j=i+1}^{N} h_j \mu_j^{-1} - \sum_{j=1}^{i-1} h_j \mu_j^{-1}\right) + O(\omega^2)}{\mu_0 S_{3_0}\langle \mu^{-1}\rangle + O(\omega^2)}\\ &= \frac{1}{\mu_0 S_{3_0}}\left[\frac{\langle \rho\rangle - S_1^2\langle \mu\rangle}{\langle \mu^{-1}\rangle}\right]^{1/2} - \frac{i\omega H}{\mu_0 S_{3_0}\langle \mu^{-1}\rangle}\sum_{i=1}^{N} S_{3_i}^2 \mu_i h_i\\ &\quad \times\left(\sum_{j=i+1}^{N} h_j \mu_j^{-1} - \sum_{j=1}^{L-1} h_j \mu_j^{-1}\right) O(\omega^2).\end{aligned} \tag{62}$$

To find the reflection coefficient using the effective transverse isotropic medium in the region $x_3 > 0$ with C_{44} and C_{66} as given in (11), the second of (24) must be used. That equation requires a given effective value of the density. Then, analogous to (34), for $x_3 > 0$,

$$\begin{aligned}u_2 &= T \exp\,[i\omega\langle \mu^{-1}\rangle^{1/2}(\rho_{\text{eff}} - \langle \mu\rangle S_1^2)^{1/2} x_3],\\ \sigma_4 &= Ti\omega\langle \mu^{-1}\rangle^{-1/2}(\rho_{\text{eff}} - \langle \mu\rangle S_1^2)^{1/2}\\ &\quad \times \exp\,[i\omega\langle \mu^{-1}\rangle^{1/2}(\rho_{\text{eff}} - \langle \mu\rangle S_1^2)^{1/2} x_3],\end{aligned} \tag{63}$$

and continuity of u_2 and σ_4 at $x_3 = 0$ [where u_2 and σ_4 for $x_3 < 0$ are given by (32)] yields the solution (36) with

$$\chi = \frac{1}{\mu_0 S_{3_0}} \left[\frac{\rho_{\text{eff}} - \langle\mu\rangle S_1^2}{\langle\mu^{-1}\rangle} \right]^{1/2}. \tag{64}$$

This agrees with the leading term for χ in (62) if, and only if, ρ_{eff} for the effective medium is the thickness-weighted average density $\langle\rho\rangle$. From (64) it can also be seen that for long wavelength χ—and thus the reflection coefficient—is independent of the angle of incidence if, and only if, β_0 is equal to $(\langle\mu\rangle/\langle\rho\rangle)^{1/2}$.

5. Reflection of P- and SV-Waves from a Medium with Plane Parallel Slip Interfaces

The geometry is assumed the same as in section 3 but now the plane strain case is considered. The compressional speed in the region $x_3 < 0$ is $\alpha_0 = \beta_0 \gamma_0^{-1/2}$ and the wavefield in this region, denoted with subscript 0, may be written as

$$\mathbf{Y}(x_3) \equiv \begin{bmatrix} \sigma_3 \\ \sigma_5 \\ u_1 \\ u_3 \end{bmatrix}_{x_3} = \mathbf{B}_0 \mathbf{D}_0(x_3) \begin{bmatrix} J \\ 1-J \\ R_{\text{P}} \\ R_{\text{S}} \end{bmatrix}, \tag{65}$$

where $J = 1$ for an incident P-wave of unit amplitude, $J = 0$ for an incident SV-wave of unit amplitude and R_{P} and R_{S} are the reflection coefficients for the reflected P- and SV-waves in either case, respectively. The matrices $\mathbf{B}_0$ and $\mathbf{D}_0$ are given by

$$\mathbf{D}_0(x_3) = \text{diag}\,[\exp i\omega S'_{3_0} x_3,\ \exp i\omega S_{3_0} x_3,\ \exp(-i\omega S'_{3_0} x_3),\ \exp(-i\omega S_{3_0} x_3)],$$

$$\mathbf{B}_0 = \begin{bmatrix} i\omega\mu_0 \gamma_0^{-1/2}\Gamma_0/\beta_0 & 2i\omega\mu_0 \beta_0 S_1 S_{3_0} & B_{0_{11}} & -B_{0_{12}} \\ 2i\omega\mu_0 \gamma_0^{-1/2}\beta_0 S_1 S'_{3_0} & -\gamma_0^{1/2} B_{0_{11}} & -B_{0_{21}} & B_{0_{22}} \\ \gamma_0^{-1/2}\beta_0 S_1 & -\beta_0 S_{3_0} & B_{0_{31}} & -B_{0_{32}} \\ \gamma_0^{-1/2}\beta_0 S'_{3_0} & \beta_0 S_1 & -B_{0_{41}} & B_{0_{42}} \end{bmatrix}, \tag{66}$$

$$S_{3_0} = (\beta_0^{-2} - S_1^2)^{1/2}, \qquad S'_{3_0} = (\gamma_0 \beta_0^{-2} - S_1^2)^{1/2}, \qquad \Gamma_0 = 1 - 2\beta_0^2 S_1^2.$$

Note that $\mathbf{D}_0(0)$ is the identity matrix $\mathbf{I}$, and that $\Gamma_0 = \cos(2 \sin^{-1} \beta_0 S_1)$.

5.1. Long wavelength, effective transverse isotropic approximation

For an incident P-wave at an angle of incidence θ_0, $S_1 = \gamma_0^{1/2}\beta_0^{-1} \sin\theta_0$ is given. For an incident SV-wave, $S_1 = \beta_0^{-1} \sin\theta_0$. The determinant of the displacement amplitudes, U_1, U_3 of (30) must vanish, but now $S_1 = S \sin\theta$ is the given quantity and $S_3 = S \cos\theta$ is the unknown slowness in the z-direction. This gives rise to the following quadratic equation on S_3^2.

$$\begin{vmatrix} \dfrac{S_1^2}{\gamma}+\dfrac{S_3^2}{1+E_T}-\beta^{-2} & S_1S_3\left(\dfrac{1-2\gamma}{\gamma}+\dfrac{1}{1+E_T}\right) \\ S_1S_3\left(\dfrac{1-2\gamma}{\gamma}+\dfrac{1}{1+E_T}\right) & \dfrac{S_1^2}{1+E_T}+\dfrac{S_3^2}{\gamma}-\beta^{-2} \end{vmatrix} = 0, \tag{67}$$

which always has a positive discriminant. For $S_1 < \sqrt{\gamma}$ there are two positive roots, corresponding to the transmitted P- and SV-waves. For $\sqrt{(\gamma)} < S_1 < \sqrt{(1+E_T)}$ there will be one positive and one negative root corresponding to a transmitted evanescent P-wave and a transmitted propagating SV-wave. For $S_1 > \sqrt{(1+E_T)}$ there will be two negative roots corresponding to the transmitted evanescent waves. The positive real or positive imaginary square roots must always be chosen for propagation or decay in the x_3-direction.

For each wavenumber S_3, a displacement vector is now found from (30) to within an arbitrary complex constant, the transmission coefficient, so that the displacement field has the form

$$\begin{bmatrix} u_1 \\ u_3 \end{bmatrix} = \begin{bmatrix} U_{P_1} & U_{S_1} \\ U_{P_3} & U_{S_3} \end{bmatrix} \begin{bmatrix} \exp i\omega S_3' x_3 & 0 \\ 0 & \exp i\omega S_3 x_3 \end{bmatrix} \begin{bmatrix} T_P \\ T_S \end{bmatrix}, \tag{68}$$

where $S_3'^2$, S_3^2 are the roots of (67) and $S_3'^2 < S_3^2$ for positive Poisson's ratio. The stresses are obtained from (3), and in this case of tangentially slipping interfaces,

$$\begin{aligned} \sigma_3 &= \frac{\mu}{\gamma}[(1-2\gamma)i\omega S_1 u_1 + u_{3,3}], \\ \sigma_5 &= \frac{\mu}{1+E_T}(i\omega S_1 u_3 + u_{1,3}). \end{aligned} \tag{69}$$

Thus, at $x_3 > 0$, from (68) and (69)

$$Y(x_3) \equiv \begin{bmatrix} \sigma_3 \\ \sigma_5 \\ u_1 \\ u_3 \end{bmatrix}_{x_3} \equiv \mathbf{B}^+ \begin{bmatrix} \exp i\omega S_3' x_3 & 0 \\ 0 & \exp i\omega S_3 x_3 \end{bmatrix} \begin{bmatrix} T_P \\ T_S \end{bmatrix},$$

$$\mathbf{B}^+ = \begin{vmatrix} \dfrac{i\omega\mu}{\gamma}[(1-2\gamma)S_1 U_{P_1} + S_3' U_{P_3}] & \dfrac{i\omega\mu}{\gamma}[(1-2\gamma)S_1 U_{S_1} + S_3 U_{S_3}] \\ \dfrac{i\omega\mu}{1+E_T}[S_1 U_{P_3} + S_3' U_{P_1}] & \dfrac{i\omega\mu}{1+E_T}[S_1 U_{S_3} + S_3 U_{S_1}] \\ U_{P_1} & U_{S_1} \\ U_{P_3} & U_{S_3} \end{vmatrix}. \tag{70}$$

The boundary conditions at $x_3 = 0$ are that $Y(0^-) = Y(0^+)$ giving four equations on R_P, R_S, T_P, T_S:

$$\mathbf{B}_0 \begin{bmatrix} J \\ 1-J \\ R_P \\ R_S \end{bmatrix} = \mathbf{B}^+ \begin{bmatrix} T_P \\ T_S \end{bmatrix}, \tag{71}$$

which may be solved identically for an incident P- or an incident SV-wave.

5.2. Exact theory

The method of solution for the exact reflection coefficient is described in this subsection. The wavefield in each layer between slip interfaces is composed of downgoing P- and SV-waves and upgoing P- and SV-waves. Thus, for each layer there are four unknown constants A_i, $i = 1, \ldots, 4$, which can be thought of as the components of a constant vector **A**. The stress and displacements on a constant x_3-plane in the first layer occupying the region $H > x_3 > 0$ can be written in matrix form

$$Y(x_3) = \mathbf{BD}(x_3)\mathbf{A}, \tag{72}$$

where **B** and **D** are as given in (66) but with the material parameters μ, β, γ, and hence S'_3 and S_3 of the layering medium rather than the parameters of the medium occupying $x_3 < 0$ (subscript 0). From (72)

$$Y(H^-) = \mathbf{BD}(H)\mathbf{B}^{-1}Y(0^+) = \mathbf{P}Y(0^+) \tag{73}$$

and the transfer matrix, **P** depends only on the width H of the layer and the material elastic properties $\mu = \rho\beta^2$, β, γ. Thus, this transfer matrix is the same for any of the layers occupying the region $x_3 > 0$. It is written explicitly as

$$\mathbf{P} = \begin{bmatrix} 2\beta^2 S_1^2 c_S + \Gamma c_P & -i\left(2\beta^2 S_1 S_3 s_S - \Gamma \dfrac{S_1}{S'_3} s_P\right) & -2i\omega\mu S_1 \Gamma (c_S - c_P) & -\omega\mu S_1 \left(4\beta^2 S_1 S_3 s_S + \dfrac{\Gamma^2}{\beta^2 S_1 S'_3} s_P\right) \\ i\left(2\beta^2 S_1 S'_3 s_P - \Gamma \dfrac{S_1}{S_3} s_S\right) & 2\beta^2 S_1^2 c_P + \Gamma c_S & -\omega\mu S_1 \left(4\beta^2 S_1 S'_3 s_P + \dfrac{\Gamma^2}{\beta^2 S_1 S_3} s_S\right) & P_{13} \\ \dfrac{iS_1}{\omega\rho}(c_S - c_P) & \dfrac{S_1}{\rho\omega}\left(\dfrac{S_1}{S'_3} s_P + \dfrac{S_3}{S_1} s_S\right) & P_{22} & P_{12} \\ \dfrac{S_1}{\omega\rho}\left(\dfrac{S'_3}{S_1} s_P + \dfrac{S_1}{S_3} s_S\right) & P_{31} & P_{21} & P_{11} \end{bmatrix}$$

$$\begin{aligned} \Gamma &= 1 - 2\beta^2 S_1^2, \quad & c_S &= \cos \omega S_3 H, \quad & c_P &= \cos \omega S'_3 H, \\ & & s_S &= \sin \omega S_3 H, \quad & s_P &= \sin \omega S'_3 H. \end{aligned} \tag{74}$$

Again, assuming no normal slip but nonzero tangential slip with compliance η_T,

$$Y(0^+) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \eta_T & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} Y(0^-) \equiv (\mathbf{I} + \eta_T \mathbf{J}_{32}) Y(0^-), \tag{75}$$

and thus across one period of this periodically layered medium

$$Y(H^-) = \mathbf{P}(\mathbf{I} + \eta_T \mathbf{J}_{32}) Y(0^-) \equiv \mathbf{Q} Y(0^-). \tag{76}$$

Note that $\mathbf{Q}$ is the same as $\mathbf{P}$ except that the second column of $\mathbf{Q}$ is given by the second column plus η_T times the third column of $\mathbf{P}$. For n such periods

$$Y(nH^-) = \mathbf{Q}^n Y(0^-) = \mathbf{Q}^n \begin{bmatrix} J \\ 1 - J \\ R_P \\ R_S \end{bmatrix}. \tag{77}$$

Again, $\mathbf{Q}$ may be expressed as a similarity transformation on the diagonal eigenvalue matrix

$$\mathbf{Q} = \mathbf{V} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \mathbf{V}^{-1} = \sum_{j=1}^{4} \lambda_j \begin{bmatrix} v_{1j} \\ v_{2j} \\ v_{3j} \\ v_{4j} \end{bmatrix} [v_{j1}^{-1} v_{j2}^{-1} v_{j3}^{-2} v_{j4}^{-1}], \tag{78}$$

where the jth column of $\mathbf{V}$ is the eigenvector of $\mathbf{Q}$ belonging to λ_j. When η_T is real, the quartic equation for the eigenvalues has real coefficients. The four eigenvalues are

$$\lambda = \delta^+ \pm [(\delta^+)^2 - 1]^{1/2}, \qquad \delta^- \pm [(\delta^-)^2 - 1]^{1/2},$$

$$\delta^\pm = \frac{c_S + c_P}{2} + \frac{\eta_T P_{23}}{4} \pm \left[\left(\frac{c_S + c_P}{2} + \frac{\eta_T P_{23}}{4} \right)^2 - c_S c_P - \frac{\eta_T}{2} (P_{11} P_{23} - P_{21} P_{13}) \right]^{1/2}. \tag{79}$$

Not only does the product of the four eigenvalues equal unity, but the four may be broken into two pairs, each pair having a product of unity. The necessary and sufficient condition for this to hold is that the quartic equation on the eigenvalues have the coefficient of λ^3 equal to the coefficient of λ, i.e. tr $\mathbf{Q}$ equal to the third invariant of $\mathbf{Q}$. This condition can be shown explicitly to be true for the elastic slip problem and it has been proven to be true for n layered propagation matrices $\mathbf{Q}$ of the form of a product of n homogeneous layer propagation matrices, $\mathbf{P}_n \cdot \mathbf{P}_{n-1} \ldots \cdot P_2 \cdot P_1$, by noting that (1) for any 4×4 matrix $\mathbf{Q}$ of determinant unity, tr $\mathbf{Q}^{-1}$ equals the third invariant of $\mathbf{Q}$, and (2) that tr $\mathbf{Q}$ = tr $(\mathbf{P}_n \ldots \cdot \mathbf{P}_1)$ is invariant when H is replaced by $-H$, and from the properties of propagator matrices, replacing H by $-H$ inverts the matrix, so that tr $\mathbf{Q}$ = tr $\mathbf{Q}^{-1}$. [The proof outlined here was developed in a series of communications in 1981 between I. Kaplansky of the University of Chicago and the author.]

The eigenvalues of absolute value less than unity tend to zero when raised to the nth power. The eigenvalues of absolute value unity are considered by introducing a small amount of loss into any of the parameters. Those that then have an absolute value less than unity are dropped when raised to the nth power. Let λ_3 and λ_4 be the two eigenvalues that are retained. Then, substituting (78) into (77) and taking the limit as $n \to \infty$ gives

$$\lim_{n\to\infty} Y(nH^-) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{V} \lim_{n\to\infty} \begin{bmatrix} \lambda_1^n & 0 & 0 & 0 \\ 0 & \lambda_2^n & 0 & 0 \\ 0 & 0 & \lambda_3^n & 0 \\ 0 & 0 & 0 & \lambda_4^n \end{bmatrix} \mathbf{V}^{-1} Y(0^-)$$

$$= \lim_{n\to\infty} \left\{ \lambda_3^n \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \\ v_{43} \end{bmatrix} [v_{31}^{-1} v_{32}^{-1} v_{33}^{-1} v_{34}^{-1}] + \lambda_4^n \begin{bmatrix} v_{14} \\ v_{24} \\ v_{34} \\ v_{44} \end{bmatrix} [v_{41}^{-1} v_{42}^{-1} v_{43}^{-1} v_{44}^{-1}] \right\} \mathbf{B}_0 \begin{bmatrix} J \\ 1-J \\ R_P \\ R_S \end{bmatrix}. \tag{80}$$

The independence of the eigenvectors belonging to λ_3 and λ_4, with (80), gives two equations on the reflection coefficients, R_P and R_S:

$$0 = [v_{31}^{-1} v_{32}^{-1} v_{33}^{-1} v_{34}^{-1}] \mathbf{B}_0 \begin{bmatrix} J \\ 1-J \\ R_P \\ R_S \end{bmatrix}$$

and

$$0 = [v_{41}^{-1} v_{42}^{-1} v_{43}^{-1} v_{44}^{-1}] \mathbf{B}_0 \begin{bmatrix} J \\ 1-J \\ R_P \\ R_S \end{bmatrix}, \tag{81}$$

where $J = 1$ for an incident P-wave and $J = 0$ for an incident SV-wave in the homogeneous medium. Figure 6 shows the amplitude and phase of the reflection coefficient as a function of angle for an incident SV-wave at several frequencies. The incident medium occupying the region $x_3 < 0$ is identical to the elastic medium between the slip interfaces. The nondimensional slip compliance is set equal to 3 and γ is set equal to 1/3 (Poisson's ratio of 1/4). The case of an incident SV-wave is chosen because it contains the interesting feature at the critical angle for P-waves

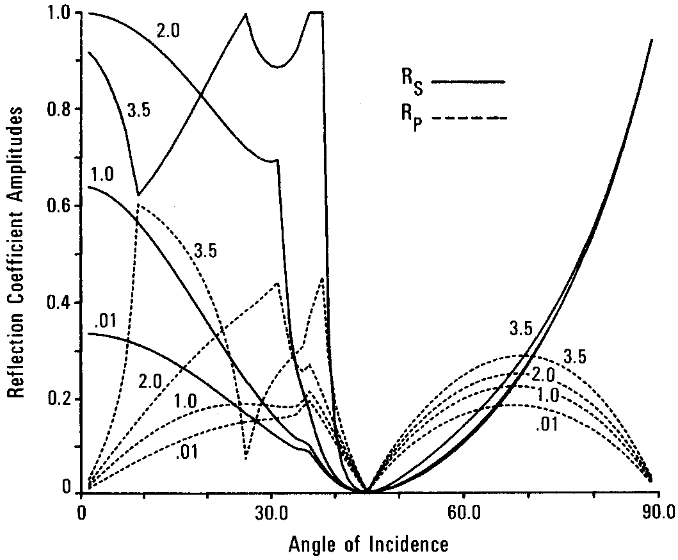

Fig. 6. The long wavelength, zero frequency reflection coefficients and the exact reflection coefficients for incident SV-waves at $\omega H/\beta = 0.01$ (which overlays the $\omega = 0$ curve), 1, 2, and 3.5 given as a function of angle of incidence. The incident medium and the medium with the slip interfaces have the same elastic properties. Poisson's ratio is 1/4 and E_T is 3.

which is arcsin $(1/\sqrt{3})$ and at $\pi/4$, whereas the case of an incident P-wave with these parameters is fairly uninteresting. The calculation of the reflection coefficient in the long wavelength limit using the effective medium theory agrees perfectly with the calculation of the reflection coefficient using the exact theory with $\omega = 0.01$ or less. At the incident angle of $\pi/4$ for which $\Gamma = 0$ and $S_3 = S_1$, R_S and R_P are both zero at any frequency with any value of γ and E_T. For about 3 or larger, there are angles for which $\delta^{\pm}$ becomes complex leading to four complex roots of the form $\exp i\omega(\pm a \pm ib)H$. These correspond to a downward-propagating and decaying wave $(+, +)$, a downward-propagating and growing wave $(+, -)$, an upward-propagating and growing wave (which decays downward) $(-, +)$, and an upward-propagating and decaying wave (which grows downward) $(-, -)$. These waves completely couple shear and longitudinal parameters and in no sense can any of them be thought of as quasishear or quasilongitudinal. λ_3 and λ_4 must be taken as those waves which grow in the $+x_3$-direction, namely the $(+, -)$ and the $(-, -)$ roots.

5.3. *Exact theory for the case of perfect tangential slip,* $\eta_T \to \infty$

In this case, at each interface σ_5, the shear stress is set identically equal to zero and there is no condition at all on the value of the tangential displacement u_1. This models the case where the interfaces are assumed to be filled with an ideal fluid. For each layer, (73) still holds but now $Y_2 \equiv \sigma_5$ vanishes at both $x_3 = 0$ and $x_3 = H$, so we may write

$$0 = P_{21} Y_1(0) + P_{23} Y_3(0) + P_{24} Y_4(0), \tag{82}$$

which is a condition on the value of $Y_3(0) = u_1(0)$. Substituting the value of $Y_3(0)$ obtained from (82) into the first and fourth of the equations of (73), gives—as Y_1 and Y_4 must be continuous across each interface—

$$y(H) = \begin{bmatrix} Y_1 \\ Y_4 \end{bmatrix}_H = \begin{bmatrix} P_{11} - \dfrac{P_{21}P_{13}}{P_{23}} & P_{14} - \dfrac{P_{24}P_{13}}{P_{23}} \\ P_{41} - \dfrac{P_{21}P_{43}}{P_{23}} & P_{44} - \dfrac{P_{24}P_{43}}{P_{23}} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_4 \end{bmatrix}_0 \equiv \mathfrak{q} y(0). \tag{83}$$

The procedure is then identical to that in the 2×2 antiplane strain case discussed above, giving

$$\begin{aligned} \lim_{n\to\infty} y(nH) = \mathbf{0} &= \left[\lim_{n\to\infty} \lambda_2^n\right] \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} [v_{21}^{-1} v_{22}^{-1}] y(0) \\ &= \frac{1}{|V|} \left(\lim_{n\to\infty} \lambda_2^n\right) \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} [-v_{21} v_{11}] y(0) \end{aligned} \tag{84}$$

or

$$[-v_{21} v_{11}] y(0) = 0. \tag{85}$$

The first and fourth parts of (65) substituted into (85) give one equation in R_P and R_S, and the condition that $Y_2(0)$ vanishes gives the second. These two equations are

$$\begin{aligned} &\left(B_{0_{11}} + \frac{v_{11}}{v_{21}} B_{0_{41}}\right) R_P - \left(B_{0_{12}} + \frac{v_{11}}{B_{21}} B_{0_{42}}\right) R_S \\ &\quad = \left(-B_{0_{11}} + \frac{v_{11}}{v_{21}} B_{0_{41}}\right) J + \left(-B_{0_{12}} + \frac{v_{11}}{v_{21}} B_{0_{42}}\right)(1 - J), \\ &\quad -B_{0_{21}} R_P + B_{0_{22}} R_S = -B_{0_{21}} J - B_{0_{22}}(1 - J). \end{aligned} \tag{86}$$

Letting $J = 1$ gives $R_P^{(P)}$ and $R_S^{(P)}$. Thus, we find that

$$\begin{aligned} &R_P^{(P)} = 1 - 2\Gamma_0^2/D, \qquad R_S^{(P)} = 4\Gamma_0 \alpha_0 \beta_0 S_1 S'_{3_0}/D, \\ &D = \Gamma_0^2 + 4\beta_0^4 S_1^2 S_{3_0} S'_{3_0} - \mathrm{i} \frac{S'_{3_0}}{\rho_0 \omega} \frac{v_{11}}{v_{21}}. \end{aligned} \tag{87}$$

Letting $J = 0$ gives $R_S^{(S)}$ and $R_P^{(S)}$. We find

$$R_S^{(S)} = -1 + 8\beta_0^4 S_1^2 S_{3_0} S'_{3_0}/D, \qquad R_P^{(S)} = -4\Gamma_0 \beta_0^3 S_1 S_{3_0}/\alpha_0 D. \tag{88}$$

Thus, for this reflection problem everything is determined except v_{11}/v_{21} which is found as in the case of SH-waves.

Specifically, from the matrix **P**

$$q_{11} = q_{22} = P_{11} - P_{21}P_{13}/P_{23} = [4\beta^2 S_1 S_3' c_S s_P + (\Gamma^2/\beta^2 S_1 S_3) c_P s_S]/K,$$
$$q_{12} = P_{14} - P_{13}^2/P_{23}$$
$$= -\omega\mu S_1[8\Gamma^2(1 - c_S c_P) + (16\beta^4 S_1^2 S_3 S_3' + \Gamma^4/\beta^4 S_1^2 S_3 S_3') s_S s_P]/K,$$
$$q_{21} = P_{41} - P_{21}^2/P_{23} = (S_3'/S_3) s_S s_P/\omega\mu S_1 K,$$
$$K = 4\beta^2 S_1 S_3' s_P + (\Gamma^2/\beta^2 S_1 S_3) s_S. \tag{89}$$

The determinant of **q** is unity. The eigenvalues are $q_{11} \pm \sqrt{(q_{11}^2 - 1)}$. For $|q_{11}| > 1$, the eigenvalues are real and

$$\lambda_1 = q_{11}[1 - \sqrt{(1 - q_{11}^{-2})}],$$
$$v_{11}/v_{21} = -q_{11}\sqrt{(1 - q_{11}^{-2})}/q_{21}. \tag{90}$$

This is a value of (ω, S_1) that is in the stop band of this periodic medium. For $|q_{11}| < 1$, (ω, S_1) is a point in the pass band, and the eigenvalues are a complex conjugate pair with

$$\lambda^{\pm} = q_{11} \pm \mathrm{i}\sqrt{(1 - q_{11}^2)}. \tag{91}$$

Upon including a small amount of loss, one root, say λ_1, will have $|\lambda_1| < 1$ and the other, say λ_2, will have $|\lambda_2| > 1$. So for $\lambda_1 = \lambda^{\pm}$, $v_{11}/v_{21} = \pm\mathrm{i}\sqrt{(1 - q_{11}^2)}/q_{21}$. Loss can be most easily introduced by holding β^{-2} real and letting $\alpha^{-2} = \beta^{-2}\gamma \to \beta^{-2}\gamma(1 + \mathrm{i}\epsilon)$.

For $\omega \to 0$, $\lambda_1 = \lambda^+$, and

$$\frac{v_{11}}{v_{21}} = \mathrm{i}\omega \frac{\rho}{S_3'} \sqrt{[1 - 4(1 - \gamma)\beta^2 S_1^2]} + \theta(\omega^3). \tag{92}$$

Note that $4(1 - \gamma)/\beta^2$ is the velocity of extensional waves in a plate.

Substitution of this expression into (87) gives

$$D = \Gamma_0^2 + 4\beta_0^4 S_1^2 S_{3_0} S_{3_0}' + \frac{\rho S_{3_0}'}{\rho_0 S_3'} \sqrt{[1 - 4(1 - \gamma)\beta^2 S_1^2]}, \tag{93}$$

thus giving the long wavelength reflection coefficients.

6. Conclusions

An elastic medium, homogeneous but for equally spaced parallel slip interfaces, is a special case of a periodically stratified medium. For the purposes of long wavelength elastic wave propagation, such a medium may be modeled by an equivalent transverse isotropic medium. Thus the theory of wave propagation in homogeneous anisotropic media can be applied to find slowness surfaces and reflection and transmission coefficients.

An interesting result occurs when plane equally spaced slip interfaces are embedded in an otherwise infinite homogeneous medium in a region $x_3 > 0$. Then the plane wave reflection coefficient for SH-waves incident in the region $x_3 < 0$ will be independent of angle of incidence and the surface $x_3 = 0$ will appear, to SH-radiation, as a reducing mirror.

The exact solution of the problem of reflection and transmission of SH-waves from a periodically stratified half-space was found for all frequencies and angles of incidence. In the long wavelength limit, the exact reflection coefficient approaches that of the equivalent anisotropic half-space. The density of the equivalent half-space must be taken as the thickness-weighted average density of the periodic half-space. In the high-frequency limit, the exact solution showed that the amplitude of the SH-plane-wave reflection coefficient of the half-space with equally spaced parallel elastic slip interfaces becomes unity except for narrow frequency bands centered slightly above the frequencies that correspond to H, the period of the medium, being an integral number of half-wavelengths in the x_3-direction. These narrow frequency bands correspond to the pass band of the periodic medium and they shrink as frequency increases. Energy at all other frequencies is totally reflected. The phase of the SH reflection coefficient for high frequency becomes periodic. When the material between the interfaces is the same as that occupying $x_3 < 0$, the phase approaches -2Δ. With parallel viscous slip interfaces the reflection coefficient is a purely periodic function of frequency.

For the plane strain case of incident P- and SV-waves, the approximate long wavelength reflection coefficient and the exact reflection coefficient valid for any frequency were derived formally. Numerical computations have shown that as the frequency tends to zero, the exact reflection coefficient approaches the long wavelength reflection coefficient for the incident medium the same as the layered medium. The usual pure propagating and pure decaying modes were found but, in addition, for higher frequencies, waves that decay as they propagate are found and these cannot be identified either with shear or longitudinal speeds but instead with a single speed and a single attenuation parameter. Explicit expressions for the reflection coefficients for the case when the tangential slip compliance goes to infinity (perfect tangential slip or zero shear stress on the interfaces) are derived. This is also a special case of a periodically layered medium of alternating solid and fluid layers where the fluid layer thickness is much smaller than the solid layer thickness.

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