ON THE REEB SPACES OF DEFINABLE MAPS

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Abstract. We prove that the Reeb space of a proper definable map in an arbitrary o-minimal expansion of the reals is realizable as a proper definable quotient which can be seen as the definable analog of Stein factorization of proper morphisms in algebraic geometry. We also show that the Betti numbers of the Reeb space of a map f can be arbitrarily large compared to those of X, unlike in the special case of Reeb graphs of manifolds. Nevertheless, in the special case when $f : X \to Y$ is a semi-algebraic map and X is closed and bounded, we prove a singly exponential upper bound on the Betti numbers of the Reeb space of f in terms of the number and degrees of the polynomials defining X, Y and f.

1. Introduction

Given a topological space X and a continuous function $f\colon X \to \mathbb{R}$, define an equivalence relation $\sim$ on X by setting $x \sim x'$ if and only if $f(x) = f(x')$ and x and x' are in the same connected component of $f^{-1}(f(x)) = f^{-1}(f(x'))$. The space $X/\sim$ is called the *Reeb graph* of f, denoted $\mathrm{Reeb}(f)$. The concept of the Reeb graph was introduced by Georges Reeb in [22] as a tool in Morse theory. The notion of the Reeb graph can be generalized to the notion of *Reeb space* by letting $f\colon X \to Y$, where Y is any topological space. Burlet and de Rham first introduced the Reeb space in [6] as the *Stein factorization* of a map f, but their work was limited to bivariate, generic, smooth mappings. Existence of Stein factorization for more general morphisms in algebraic geometry is proved in [14, III, Corollary 11.5], and is closely related to the well-known Zariski's Main Theorem [14, III, Corollary 11.4] (see Remarks 3 and 6 for the connection between Stein factorization in algebraic geometry and the results of the current paper). From the point of view of applied topology, Reeb spaces have been investigated from both a theoretical and practical perspective. Edelsbrunner et al. defined the Reeb space of a multivariate piecewise linear mapping on a combinatorial manifold in [9], and they proved results regarding the local and global structure of such spaces. Expanding on this work, Patel [18] produced an algorithm to construct the Reeb space of a mapping f. Mapper, introduced in [24], gives a discrete approximation of the Reeb space of a multivariate mapping; this allows for more efficient computation of the underlying data structure. Munch et al.

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[17] define the *interleaving distance* for Reeb spaces to show the convergence between the Reeb space and Mapper.

In this paper, we investigate Reeb spaces from the point of view of topological complexity. Our motivation is to understand how topologically complicated the Reeb space of a map can become in terms of the complexity of the map itself. In order to obtain meaningful results we restrict ourselves to the category of maps *definable in an o-minimal expansion of* $\mathbb{R}$ (see Section 2 for a quick overview of o-minimality), and, in particular, to *semi-algebraic* maps.

The notion of o-minimal structures has its origins in model theory but has since become a widely accepted framework for studying "tame geometry". The definable sets and maps of an o-minimal structure satisfy many uniform finiteness properties (similar to those of semi-algebraic sets) while allowing much richer families of sets and maps. We refer the reader to the survey by Wilkie [33] for the origin and motivation of this notion of tameness. The reader will also find many applications of interest.

Our first result is that the Reeb spaces of "tame" maps are themselves tame. More precisely, we prove that the quotient map corresponding to the Reeb space of a proper definable map can be realized as a proper definable map (Theorem 2 below). This implies as a special case that the Reeb spaces of proper semi-algebraic maps can be realized as semi-algebraic quotients. Theorem 2 can be viewed as the definable analog of the theorem [14, III, Corollary 11.5] on the existence of Stein factorization for proper morphisms in algebraic geometry (see Remark 3 below). Another significance of this result is that it makes it possible to ask for an algorithm to semi-algebraically describe this semi-algebraic quotient using results from the well developed area of algorithmic semi-algebraic geometry [4]. We do not pursue this question further in this paper, leaving it for future work.

It is known [10, page 141] that the sum of the Betti numbers of the *Reeb graph* of a map $f : X \to \mathbb{R}$ is bounded from above by the sum of the Betti numbers of X. We show that this is false for more general maps by exhibiting a couple of natural examples of sequences of maps $(f_n : X_n \to Y_n)_{n>0}$, such that the sum of the Betti numbers of the Reeb space of f_n is arbitrarily large compared to that of X_n. In view of these examples, it makes sense to ask whether it is still possible to bound the Betti numbers of the Reeb space of a map f in terms of some measure of the "complexity" of the map f. In particular, if the map is semi-algebraic, then one can measure the complexity of the map by the number and degrees of the polynomials defining the map. We are then led to the problem of studying the topological complexity of Reeb spaces of semi-algebraic maps.

While studying the topological complexity of Reeb spaces of semi-algebraic maps is a natural mathematical question on its own, another motivation is related to the algorithmic question mentioned earlier concerning the design of efficient algorithms for computing a semi-algebraic description of the Reeb space of a semi-algebraic map. It is a meta theorem in algorithmic semi-algebraic geometry that upper bounds on topological complexity of objects are closely related to the worst-case complexity of algorithms computing the topological invariants of such objects. Thus, a singly exponential upper bound on the Betti numbers of the Reeb space of a semi-algebraic map opens up the possibility of being able to compute the Betti numbers of the Reeb space. The singly exponential upper bound on the Betti numbers of the Reeb space of a semi-algebraic map may also hint that one could compute a semi-algebraic description of the Reeb space with an algorithm having a singly exponential complexity bound.

The problem of bounding the topological complexity (for example measured in terms of Betti numbers or the number of homotopy types of fibers) of semi-algebraic sets or maps in terms of the parameters of the formula defining them has a long history (see [2] for a survey). Bounds on these quantities which are doubly exponential in the dimension or the number of variables usually follow from the fact that semi-algebraic sets admit semi-algebraic triangulations of at most doubly exponential size. Singly exponential upper bounds are more difficult and usually involve more careful arguments involving Morse inequalities and other inequalities coming from certain spectral sequences [19, 25, 15, 3, 11, 4]. To the best of our knowledge, the problem of bounding the Betti numbers of the Reeb space of a semi-algebraic map has not been considered before. In this paper we prove a singly exponential upper bound on the Betti numbers of the Reeb space of a semi-algebraic map $f : X \to Y$, where X is a closed and bounded semi-algebraic set, in terms of the number and the degrees of the polynomials defining X, Y and f (cf. Theorem 3 below).

The rest of the paper is organized as follows: In Section 2, we recall the basic definitions related to o-minimality. In Section 3, we prove the definability of Reeb spaces of proper definable maps. In Section 4, we describe examples showing that the Betti numbers of the Reeb space of a definable map $f : X \to Y$ can be arbitrarily large compared to those of X. We also give a proof of the inequality $b_1(\mathrm{Reeb}(f)) \leq b_1(X)$ for definable proper maps $f : X \to Y$ with X connected, using a spectral sequence that plays an important role in this paper (this inequality was proved previously using alternative techniques by Dey et al. [8]). Finally, in Section 5, we prove a singly exponential upper bound on the sum of the Betti numbers of the Reeb space of a proper semi-algebraic map in terms of the number and degrees of the polynomials defining the map.

2. Basic definitions

We first recall the important model theoretic notion of o-minimality which plays an important role in what follows.

2.0.1. *O-minimal Structures.* O-minimal structures were invented and first studied by Pillay and Steinhorn in the pioneering papers [20, 21], motivated by the prior work of van den Dries [26]. Later, the theory was further developed through contributions of other researchers, most notably van den Dries, Wilkie, Rolin, and Speissegger, amongst others [28, 29, 30, 31, 32, 23]. We particularly recommend the book by van den Dries [27] and the notes by Coste [7] for an easy introduction to the topic as well as for the proofs of the basic results that we use in this paper.

Definition 1 (o-minimal structure)**.** An o-minimal structure over a real closed field R (or equivalently an o-minimal expansion of R) is a sequence $\mathcal{S}(\mathrm{R}) = (\mathcal{S}_n)_{n\in\mathbb{N}}$ where each $\mathcal{S}_n$ is a collection of subsets of R^n (called the *definable sets* in the structure) satisfying the following axioms (following the exposition in [7]):

(A) All algebraic subsets of R^n are in $\mathcal{S}_n$.
(B) The class $\mathcal{S}_n$ is closed under complementation and finite unions and intersections.
(C) If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$ then $A \times B \in \mathcal{S}_{m+n}$.
(D) If $\pi : \mathrm{R}^{n+1} \to \mathrm{R}^n$ is the projection map on the first n coordinates and $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$.
(E) The elements of $\mathcal{S}_1$ are finite unions of points and intervals. (Note that these are precisely the subsets of R which are definable by a first-order formula in the language of the reals with one free variable.)

A map $f : X \to Y$ between two definable sets X and Y is *definable* if its graph is a definable set. Note that for any definable map $f : X \to Y$, there exists a finite partition $(X_i)_{i\in I}$ of X into definable subsets such that f restricted to each X_i is continuous. In light of this, for rest of this paper we use the term "definable map" to mean a map that is definable and continuous.

The class of semi-algebraic sets is one obvious example of an o-minimal structure, but in fact there are much richer classes of sets which have been proven to be o-minimal. The class of *sub-analytic sets* is one such example [32].

We now consider quotients by definable equivalence relations.

Definition 2. Let $E \subset X \times X$ be a definable equivalence relation on a definable set X. A *definable quotient* of X by E is a pair (p, Y) consisting of a definable set Y and a definable surjective map $p : X \to Y$ such that

(i) $(x_1, x_2) \in E \Leftrightarrow p(x_1) = p(x_2)$ for all $x_1, x_2 \in X$;
(ii) p is definably identifying; that is, for all definable $K \subset Y$, if $p^{-1}(K)$ is closed in X, then K is closed in Y.

We say that the definable quotient (p, Y) is definably proper if p is a definably proper map, i.e. for every definable $K \subset Y$ with K closed and bounded in R^n, the ambient space of Y, $p^{-1}(K) \subset X$ is closed and bounded in R^m, the ambient space of X.

Definition 3. A definable equivalence relation $E \subset X \times X$ is said to be *definably proper* if the two maps $\mathrm{pr}_1, \mathrm{pr}_2 : E \to X$ are definably proper.

We will use the following theorem which appears in [27]:

Theorem 1. [27, page 166] *Let X be a definable set and $E \subset X \times X$ a definably proper equivalence relation on X. Then X/E exists as a definably proper quotient of X.*

3. The Reeb space of a definable map $f : X \to Y$

We now fix an o-minimal expansion of $\mathbb{R}$. Let $X \subset \mathbb{R}^n$ be a closed and bounded definable set, and $f : X \to Y$ be a definable map.

Definition 4. The Reeb space of the map f, henceforth denoted $\mathrm{Reeb}(f)$, is the topological space $X/{\sim}$, equipped with the quotient topology, where $x \sim x'$ if and only if $f(x) = f(x')$, and x, x' belong to the same connected component of $f^{-1}(f(x))$.

Remark 1. Note that a definable (resp. semi-algebraic) set $S \subset \mathbb{R}^k$ is connected if and only if S is definably (resp. semi-algebraically) path-connected, i.e. for all $x, y \in S$, there exists a definable (resp. semi-algebraic) path $\gamma : [0,1] \to S$ with $\gamma(0) = x, \gamma(1) = y$.

Our first result is that with the above assumptions:

Theorem 2. *The space* $\mathrm{Reeb}(f) \triangleq X/{\sim}$ *exists as a definably proper quotient. In other words, there exists a proper definable map* $\psi : X \to Z$ *and a homeomorphism* $\theta : \mathrm{Reeb}(f) \to Z$ *such that the following diagram commutes:*

$$\begin{array}{ccc} & X & \\ {\scriptstyle\phi}\swarrow & & \searrow{\scriptstyle\psi} \\ \mathrm{Reeb}(f) = X/{\sim} & \xrightarrow{\ \theta\ } & Z \end{array}$$

(here ϕ is the quotient map). In particular, $\mathrm{Reeb}(f)$ *is homeomorphic to a definable set.*

Remark 2. The assumption of compactness of X is needed. For example, suppose $X = \mathbb{R}^2 \setminus \mathbf{0}$ and $f : X \to \mathbb{R}$ is the projection map forgetting the second coordinate. Then, each fiber $f^{-1}(x)$ has one connected component if $x \neq 0$, and $f^{-1}(0)$ has two connected components. The Reeb space of f is homeomorphic to the real line with a doubled point, and cannot be a definable subset of any real affine space.

Remark 3. Theorem 2 can also be seen as a definable analog of Stein factorization for projective morphisms [14, III, Corollary 11.5] which states that every projective morphism $f : X \to Y$ of Noetherian schemes factors as $f = g \circ f'$, with $g : Y' \to Y$ a finite morphism, and $f' : X \to Y'$, a morphism with connected fibers. Here the scheme Y' plays the role of Reeb space of f.

Proof of Theorem 2. We first claim that the relation, "$x \sim x'$ if and only if $f(x) = f(x')$, and x, x' belong to the same connected component of $f^{-1}(f(x))$" is a definably proper equivalence relation. Using Hardt's triviality theorem for o-minimal structures [27, 7], we have that there exists a finite definable partition of Y into locally closed definable sets $(Y_\alpha)_{\alpha \in I}$, $y_\alpha \in Y_\alpha$, and definable homeomorphisms $\phi_\alpha : Y_\alpha \times f^{-1}(y_\alpha) \to f^{-1}(Y_\alpha)$ such that the following diagram commutes for each $\alpha \in I$:

$$\begin{array}{ccc} Y_\alpha \times f^{-1}(y_\alpha) & \xrightarrow{\ \phi_\alpha\ } & f^{-1}(Y_\alpha) \\ {\scriptstyle\pi_1}\searrow & & \swarrow{\scriptstyle f|_{f^{-1}(Y_\alpha)}} \\ & Y_\alpha & \end{array}$$

(here π_1 is the projection to the first factor in the direct product). For each $\alpha \in I$, let $(C_{\alpha,\beta})_{\beta \in J_\alpha}$ be the connected components of $f^{-1}(y_\alpha)$, and for each $\alpha \in I, \beta \in J_\alpha$, let $D_{\alpha,\beta} = \phi_\alpha(Y_\alpha \times C_{\alpha,\beta})$.

Let

$$E = \bigcup_{\alpha \in I, \beta \in J_\alpha} (\phi_\alpha \times \phi_\alpha)((Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta})),$$

where $(Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta})$ is the definable subset of $(Y_\alpha \times f^{-1}(y_\alpha)) \times (Y_\alpha \times f^{-1}(y_\alpha))$ defined by

$$((y,x),(y',x')) \in (Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta}) \Leftrightarrow y = y', x, x' \in C_{\alpha,\beta}.$$

It is clear that E is a definable subset of $X \times X$, and that $x \sim x'$ if and only if $(x,x') \in E$.

Since X is assumed to be closed and bounded, if we can show that E is closed in $X \times X$, it would follow that E is a definably proper equivalence relation, and we can apply Theorem 1.

The rest of the proof is devoted to showing that E is a closed definable subset of $X \times X$. For each $\alpha \in I, \beta \in J_\alpha$, let

$$E_{\alpha,\beta} = (\phi_\alpha \times \phi_\alpha)((Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta}).$$

Since $E = \bigcup_{\alpha \in I, \beta \in J_\alpha} E_{\alpha,\beta}$, in order to prove that E is closed it suffices to prove that for each $\alpha \in I, \beta \in J_\alpha$,

$$\overline{E_{\alpha,\beta}} \subset E,$$

where $\overline{E_{\alpha,\beta}}$ is the closure of $E_{\alpha,\beta}$ in $X \times X$.

It follows from the curve selection lemma for o-minimal structures [7] that for every $z \in \overline{E_{\alpha,\beta}}$ there exists a definable curve $\gamma : [0,1] \to E_{\alpha,\beta}$ with $\gamma(0) = z$, $\gamma((0,1]) \subset E_{\alpha,\beta}$. Thus, in order to prove that $\overline{E_{\alpha,\beta}} \subset E$, it suffices to show that for each definable curve $\gamma : (0,1] \to E_{\alpha,\beta}$, $z_0 = \lim_{t \to 0} \gamma(t) \in E$.

Let $\gamma : (0,1] \to E_{\alpha,\beta}$ be a definable curve, and suppose that $\lim_{t \to 0} \gamma(0) \notin E_{\alpha,\beta}$. Otherwise, $\lim_{t \to 0} \gamma(0) \in E_{\alpha,\beta} \subset E$, and we are done.

For $t \in (0,1]$, let $y_t = f(\gamma(t))$ and let $(x_t, x'_t) \in (\phi_\alpha \times \phi_\alpha)((Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta}))$ be such that $\gamma(t) = (x_t, x'_t)$. Note that $f(x_t) = f(x'_t) = y_t$. Finally, let $z_0 = (x_0, x'_0) = \lim_{t \to 0} \gamma(t)$.

Since, $z_0 \notin E_{\alpha,\beta}$ by assumption and $\gamma((0,1]) \subset E_{\alpha,\beta}$, there exists $t_0 > 0$ such that $\lambda = f \circ \gamma|_{(0,t_0]} : (0,t_0] \to Y_\alpha$ is an injective definable map and $\lim_{t \to 0} \lambda(t) = y_0 = f(x_0) = f(x'_0) \in Y_{\alpha'}$ for some $\alpha' \in I$. We need to show that x_0 and x'_0 belong to the same connected component of $f^{-1}(y_0)$, which would imply that $(x_0, x'_0) \in E$.

Let $D_{\alpha,\beta,\gamma} = f^{-1}(\lambda((0,t_0])) \cap D_{\alpha,\beta}$ and let $g : D_{\alpha,\beta,\gamma} \to (0,t_0]$ be defined by $g(x) = \lambda^{-1}(f(x))$ (which is well defined by the injectivity of λ). Note that for each $t \in (0,t_0]$, $g^{-1}(t)$ is definably homeomorphic to $C_{\alpha,\beta}$, and hence is connected. It also follows from Hardt's triviality theorem that there exists $t'_0 \in (0,t_0]$ and a definable homeomorphism $\theta : g^{-1}(t'_0) \times (0,t'_0] \to g^{-1}((0,t'_0])$ such that the following diagram commutes:

$$\begin{array}{ccc} g^{-1}(t'_0) \times (0, t'_0] & \xrightarrow{\theta} & g^{-1}((0, t'_0]) \\ & \searrow^{\pi_2} \quad \swarrow_{g} & \\ & (0, t'_0] & \end{array}$$

Extend θ continuously to a definable map $\overline{\theta} : g^{-1}(t'_0) \times [0, t_0] \to \overline{g^{-1}((0, t'_0])}$ by setting $\overline{\theta}(x, 0) = \lim_{t \to 0} \theta(x, t)$. Finally, let $\theta' : g^{-1}(t'_0) \to f^{-1}(y_0)$ be the definable map obtained by setting $\theta'(x) = \overline{\theta}(x, 0)$.

Note that since $g^{-1}(t'_0)$ is connected, $\theta'(g^{-1}(t'_0))$ is connected as well, since it is the image of a connected set under a continuous map. Also note that for each $t \in (0, t'_0]$, we have that $x_t, x'_t \in D_{\alpha,\beta,\gamma}$ and $f(x, t) = f(x'_t) = \lambda(t)$, hence $x_t, x'_t \in g^{-1}(t)$, and thus $x_0, x'_0 \in \theta'(g^{-1}(t'_0))$. Moreover, $f(x_0) = f(x'_0) = y_0$. Therefore, since $\theta'(g^{-1}(t'_0))$ is connected, x_0 and x'_0 belong to the same connected component of $f^{-1}(y_0)$.

This shows that $(x_0, x'_0) \in E$, which in turn implies that E is closed in $X \times X$.

The fact that $\mathrm{Reeb}(f)$ exists as a definably proper quotient now follows from Theorem 1. □

Remark 4. Theorem 2 opens up an algorithmic problem of actually realizing the Reeb space as a definable quotient in the special case where the o-minimal structure is that of semi-algebraic sets and maps. More precisely, the problem is to design an algorithm that, given a proper semi-algebraic map $f : X \to Y$, will compute a description of a semi-algebraic map $g : X \to Z \cong \mathrm{Reeb}(f)$ realizing the Reeb space of f as a semi-algebraic quotient. The complexity of the algorithm will then depend on the number and degrees of the polynomials defining X. In this paper, we do not pursue this algorithmic problem any further leaving it for future work.

4. The Betti numbers of the Reeb space of $f\colon X \to Y$ can exceed that of X

Notation 1. For any topological space X and $i \geq 0$ let $b_i(X)$ denote the i-th Betti number (that is, the dimension of the i-th singular homology group of X with coefficients in $\mathbb{Q}$), and let $b(X) = \sum_i b_i(X)$.

In [10, page 141] it is noted that the inequality $b(\mathrm{Reeb}(f)) \leq b(X)$ holds for arbitrary maps $f : X \to \mathbb{R}$.

We first show that the same is not true for Reeb spaces of more general maps by giving several examples.

Example 1. Consider the closed n-dimensional disk $\mathbf{D}^n$ with $n \geq 1$, and let $\sim$ be the equivalence relation identifying all points on the boundary of $\mathbf{D}^n$. Then $\mathbf{D}^n/\sim\, \cong \mathbf{S}^n$, where $\mathbf{S}^n$ is the n-dimensional sphere. Let f_n denote the quotient map $f_n : \mathbf{D}^n \to \mathbf{S}^n$. The fibers of f_n consist of either one point or the boundary $\mathbf{S}^{n-1}$ of $\mathbf{D}^n$, and hence $\mathrm{Reeb}(f_n) \cong \mathbf{S}^n$ for all $n > 1$. Note that $b_0(\mathbf{D}^n) = 1$ and $b_i(\mathbf{D}^n) = 0$

for all $i > 0$. Moreover, $b_0(\mathbf{S}^n) = 1$, $b_n(\mathbf{S}^n) = 1$, and $b_i(\mathbf{S}^n) = 0, i \neq 0, n$. Thus, we have for $n > 1$,

$$\begin{aligned} b(\mathbf{D}^n) &= 1, \\ b(\mathrm{Reeb}(f_n)) &= 2. \end{aligned}$$

More generally, for $k \geq 0$, let

$$f_{n,k} = \underbrace{f \times \cdots \times f}_{k \text{ times}} : \underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}} \longrightarrow \underbrace{\mathbf{S}^n \times \cdots \times \mathbf{S}^n}_{k \text{ times}}.$$

Using the same argument as before, for $n > 1$ and $k > 0$,

$$\mathrm{Reeb}(f_{n,k}) \cong \underbrace{\mathbf{S}^n \times \cdots \times \mathbf{S}^n}_{k \text{ times}}.$$

Thus,

$$\begin{aligned} b_0(\underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}}) &= 1, \\ b_i((\underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}}) &= 0, \ i > 0, \end{aligned}$$

and hence

$$b(\underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}}) = 1.$$

Moreover, for $n > 1$,

$$\begin{aligned} b_i(\mathrm{Reeb}(f_{n,k})) &= 0 \text{ if } n \not| i \text{ or if } i > nk, \\ b_i(\mathrm{Reeb}(f_{n,k})) &= \binom{k}{i/n} \text{ otherwise}, \end{aligned}$$

and hence for $n > 1$,

$$b(\mathrm{Reeb}(f_{n,k})) = 2^k.$$

This example shows that even for definably proper maps $f : X \to Y$, the individual as well as the total Betti numbers of $\mathrm{Reeb}(f)$ can be arbitrarily large compared to those of X.

Our second example comes from the topology of compact Lie groups, in particular the complex unitary group:

Example 2. For $n > 0$, let $U(n)$ denote the group of $n \times n$ complex unitary matrices, and let $T^n \subset U(n)$ denote the maximal torus. (Note that T^n is the group of $n \times n$ unitary diagonal matrices $\mathrm{diag}(z_1, \ldots, z_n)$ with $|z_i| = 1, 1 \leq i \leq n$, and is thus homeomorphic to the product of n circles.) Denote the quotient map by $\pi_n : U(n) \to U(n)/T^n$. We have that:

$$b(U(n)/T^n) = n! \text{ (see [16, Theorem 4.6])}, \quad (4.1)$$

$$b(U(n)) = 2^n \text{ (see [16, Corollary 3.11])}. \quad (4.2)$$

Observing that the fibers of π_n are all connected, one has that $\mathrm{Reeb}(\pi_n) \cong U(n)/T^n$, and it follows from (4.1) and (4.2) that for all $n \geq 4$,

$$b(\mathrm{Reeb}(\pi_n)) = n! \geq 2^n = b(U(n)).$$

Remark 5. We note that recently Dey et al. [8] have shown that

$$b_1(\mathrm{Reeb}(f)) \leq b_1(X) \tag{4.3}$$

if $f : X \to Y$ is a proper map and X is connected. Notice that the examples given above do not violate this bound since the stated inequalities involve only the sum rather than the individual Betti numbers.

We sketch below an alternative proof of the inequality (4.3) for a proper definable map $f : X \to Y$, with X connected, using an inequality coming from a spectral sequence associated to the quotient map $\phi : X \to \mathrm{Reeb}(f)$. This spectral sequence also plays a key role in the proof of the main result in this paper.

More precisely, for a proper definable surjective map $g : A \to B$, Gabrielov, Vorobjov and Zell [12] proved that there exists a spectral sequence (which we write as a cohomological spectral sequence for convenience) which converges to $\mathrm{H}^*(B)$. This spectral sequence is referred to as the *descent spectral sequence* of g below and it's E_1-term is given by

$$E_1^{p,q} = \mathrm{H}^q(\underbrace{A \times_g \cdots \times_g A}_{p+1}).$$

Returning to the case of a proper definable map $f : X \to Y$, we first note that if X is connected, then so is $\underbrace{X \times_\phi \cdots \times_\phi X}_{p+1}$, and $\dim(E_1^{p,0}) = 1$ for all $p \geq 0$. Moreover, the differential $d_1^{p,0} : E_1^{p,0} \to E_1^{p+1,0}$ has rank 0 or 1 depending on whether p is even or odd, respectively. This implies that $E_2^{p,0} = 0$ for all $p > 0$ in the descent spectral sequence of the quotient map $\phi : X \to \mathrm{Reeb}(f)$. Moreover, notice that $E_1^{0,1} \cong \mathrm{H}^1(X)$, and hence

$$\dim(E_1^{0,1}) = b_1(X).$$

Since the spectral sequence converges to $\mathrm{H}^{p+q}(\mathrm{Reeb}(f))$, the following inequality holds for each $n \geq 0$ and $r \geq 1$:

$$\mathrm{H}^n(\mathrm{Reeb}(f)) \quad \leq \quad \sum_{p+q=n} \dim(E_r^{p,q}). \tag{4.4}$$

Moreover, for $r \geq r'$ and for any p, q,

$$\dim(E_r^{p,q}) \quad \leq \quad \dim(E_{r'}^{p,q}), \tag{4.5}$$

since $E_r^{p,q}$ is a sub-quotient of $E_{r'}^{p,q}$.

It follows from the inequalities (4.4) and (4.5) with $n = 1$, $r' = 1$, and $r = 2$, that

$$\begin{aligned} b_1(\mathrm{Reeb}(f)) &\leq \dim(E_2^{0,1}) + \dim(E_2^{1,0}) \\ &\leq \dim(E_1^{0,1}) + \dim(E_2^{1,0}) \\ &= b_1(X) + 0 \\ &= b_1(X). \end{aligned}$$

We note here that an inequality (cf. inequality (5.4)) coming from the consideration of the E_1-term of the spectral sequence of the map ϕ plays a key role in the proof of Theorem 3, which is the main result of this paper.

5. Quantitative Bounds

We now consider the problem of bounding effectively from above the Betti numbers of the Reeb space of a definable continuous map. We have seen from Example 1 that, given a continuous semi-algebraic map $f : X \to Y$, $b(\mathrm{Reeb}(f))$ can be arbitrarily large compared to $b(X)$, unlike in the case of Reeb graphs (i.e. when $\dim(Y) \leq 1$). In this section, we prove an upper bound on $b(\mathrm{Reeb}(f))$ in terms of the "semi-algebraic" complexity of the map f.

We first introduce some more notation.

Notation 2. For any finite family of polynomials $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]$, we call an element $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$ a *sign condition* on $\mathcal{P}$. For any semi-algebraic set $Z \subset \mathbb{R}^k$ and sign condition $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$, we denote by $\mathcal{R}(\sigma, Z)$ the semi-algebraic set defined by

$$\{\mathbf{x} \in Z \mid \mathbf{sign}(P(\mathbf{x})) = \sigma(P), P \in \mathcal{P}\},$$

and call it the *realization* of σ on Z. More generally, we call any Boolean formula Φ with atoms $P\{=, >, <\}0, P \in \mathcal{P}$, a *$\mathcal{P}$-formula*. We call the realization of Φ, namely the semi-algebraic set

$$\mathcal{R}(\Phi, \mathbb{R}^k) \quad = \quad \{\mathbf{x} \in \mathbb{R}^k \mid \Phi(\mathbf{x})\},$$

a *$\mathcal{P}$-semi-algebraic set*. Finally, we call a Boolean formula without negations and with atoms $P\{\geq, \leq\}0$, $P \in \mathcal{P}$, a *$\mathcal{P}$-closed formula*, and we call the realization, $\mathcal{R}(\Phi, \mathbb{R}^k)$, a *$\mathcal{P}$-closed semi-algebraic set*.

We will denote by $\mathrm{SIGN}(\mathcal{P})$ the set of *realizable sign conditions of* $\mathcal{P}$, i.e.

$$\mathrm{SIGN}(\mathcal{P}) = \{\sigma \in \{0, 1, -1\}^{\mathcal{P}} \mid \mathcal{R}(\sigma, \mathbb{R}^k) \neq \emptyset\}.$$

Finally, for any semi-algebraic set S, we will denote the set of its connected components by $\mathrm{Cc}(S)$.

We prove the following theorem.

Theorem 3. *Let $S \subset \mathbb{R}^n$ be a bounded $\mathcal{P}$-closed semi-algebraic set, and $f = (f_1, \ldots, f_m) : S \to \mathbb{R}^m$ be a polynomial map. Suppose that $s = \mathrm{card}(\mathcal{P})$ and the maximum of the degrees of the polynomials in $\mathcal{P}$ and $f_1, \ldots, f_m$ are bounded by d. Then,*

$$b(\mathrm{Reeb}(f)) \leq (sd)^{(n+m)^{O(1)}}.$$

The rest of the paper is devoted to the proof of Theorem 3. We first outline the main idea behind the proof.

5.1. **Outline of the proof of Theorem 3.** We first replace the map $f : S \to \mathbb{R}^m$, by a new map $\tilde{f} : \tilde{S} \to \mathbb{R}^m$, where $\tilde{S} \subset \mathbb{R}^n \times \mathbb{R}^m$ and $\tilde{f}$ is the restriction to $\tilde{S}$ of the projection map to $\mathbb{R}^m$. From the definitions it is evident that $\mathrm{Reeb}(f)$ and $\mathrm{Reeb}(\tilde{f})$ are homeomorphic. We next prove that there exists a semi-algebraic partition of $\mathbb{R}^m$ of controlled complexity (more precisely given by the connected components of the realizable sign conditions of a family of polynomials of singly exponentially bounded degrees and cardinality) into connected semi-algebraic sets C, such that the connected components of the fibers $\tilde{f}^{-1}(z)$ are in 1-1 correspondence with each other as z varies over C. Moreover each of these connected components C is described by a quantifier-free first order formula and the complexity of these formulas (i.e. the number of polynomials appearing in the formula and their respective degrees) is bounded singly exponentially (see Theorem 4 below for the precise formulation of this statement).

The proof of this result (Theorem 4) uses a certain sheaf-theoretic generalization of effective real quantifier elimination proved in [5] and recalled below (Theorem 6). The fact that the connected components of a semi-algebraic set can be described efficiently (with singly exponential complexity) is a consequence of a result in [4] (Theorem 5 below).

Next, we use the fact that the canonical surjection $\phi : \tilde{S} \to \mathrm{Reeb}(\tilde{f})$ is a proper semi-algebraic map. We then use an inequality proved in [12] (see Theorem 7 below) to obtain an upper bound on the Betti numbers of the image of a proper semi-algebraic map $F : X \to Y$ in terms of the sum of the Betti numbers of various fiber products $X \times_F \cdots \times_F X$ of the same map. Recall that for $p \geq 0$, the $(p+1)$-fold fiber product is given by

$$\underbrace{X \times_F \cdots \times_F X}_{(p+1)\text{-times}} \triangleq \{(x^{(0)}, \ldots, x^{(p)}) \in X^{p+1} \ \mid \ F(x^{(0)}) = \cdots = F(x^{(p)})\}.$$

Theorem 4 provides us with a well controlled description (i.e. by quantifier-free first order formulas involving singly exponentially any polynomials of singly exponentially bounded degrees) of the fibered products $\tilde{S} \times_{\tilde{f}} \cdots \times_{\tilde{f}} \tilde{S}$. Finally, using these descriptions and results on bounding the Betti numbers of general semi-algebraic sets in terms of the number and degrees of polynomials defining them (cf. Theorem 9 below) we obtain the claimed bound on $\mathrm{Reeb}(f)$.

In order to make the above summary precise we first need to state some preliminary results.

5.2. **Parametrized description of connected components.** The following theorem, which states that given any finite family of polynomials

$$\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell],$$

there exists a semi-algebraic partition of $\mathbb{R}^\ell$ of controlled complexity which has good properties with respect to $\mathcal{P}$, will play a crucial role in the proof of Theorem 3.

Theorem 4. *Let $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell]$ be a finite set of polynomials of degrees bounded by d, with* $\mathrm{card}(\mathcal{P}) = s$. *Let $S \subset \mathbb{R}^k \times \mathbb{R}^\ell$ be a $\mathcal{P}$-semi-algebraic set. Then there exists a finite set of polynomials $\mathcal{Q} \subset \mathbb{R}[Y_1, \ldots, Y_\ell]$ such that* $\mathrm{card}(\mathcal{Q})$ *and the degrees of polynomials in $\mathcal{Q}$ are bounded by $(sd)^{(k+\ell)^{O(1)}}$, and $\mathcal{Q}$ has the following additional property.*

For each $\sigma \in \mathrm{SIGN}(\mathcal{Q}) \subset \{0, 1, -1\}^{\mathcal{Q}}$ and $C \in \mathrm{Cc}(\mathcal{R}(\sigma, \mathbb{R}^\ell))$, there exists

(i) an index set $I_{\sigma,C}$,
(ii) a finite family of polynomials $\mathcal{P}_{\sigma,C} \subset \mathbb{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell]$, and
(iii) $\mathcal{P}_{\sigma,C}$-formulas, $(\Theta_\alpha(\overline{X}, \overline{Y}))_{\alpha \in I_{\sigma,C}}$,

such that

(A) $\Theta_\alpha(x, y) \Rightarrow y \in C$;

(B) for each $y \in C$ and each connected component D of $\pi_Y^{-1}(C) \cap S$, there exists a unique $\alpha \in I_{\sigma,C}$ such that $\mathcal{R}(\Theta_\alpha(\cdot, y)) = \pi_Y^{-1}(y) \cap D$ and $\pi_Y^{-1}(y) \cap D$ is a connected component of $\pi_Y^{-1}(y) \cap S$.

The proof of Theorem 4 will use the following result on efficient descriptions of the connected components of semi-algebraic sets which can easily be deduced from [4, Theorem 16.3] and which we state without proof.

Theorem 5. *Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbb{R}[X_1, \ldots, X_k]$ with $\deg(P_i) \leq d$ for $1 \leq i \leq s$ and let a semi-algebraic set S be defined by a $\mathcal{P}$ quantifier-free formula. Then there exists an algorithm that outputs quantifier-free semi-algebraic descriptions of all the connected components of S. The number of polynomials that appear in the output is bounded by $s^{k+1} d^{O(k^4)}$, while the degrees of the polynomials are bounded by $d^{O(k^3)}$.*

In order to prove Theorem 4 we will also need the following theorem, which is a consequence of a more general result on the complexity of constructible sheafs proved in [5].

Theorem 6. *Let $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell]$ be a finite set of polynomials with degrees bounded by d and with* $\mathrm{card}(\mathcal{P}) = s$, *and let $S \subset \mathbb{R}^k \times \mathbb{R}^\ell$ be a bounded $\mathcal{P}$-semi-algebraic set. Then there exists a finite set of polynomials, $\mathcal{Q} \subset \mathbb{R}[Y_1, \ldots, Y_\ell]$, with degrees and cardinality bounded by $(sd)^{(k+\ell)^{O(1)}}$, and for each connected component C of each realizable sign condition $\sigma \in \mathrm{SIGN}(\mathcal{Q}) \subset \{0, 1, -1\}^{\mathcal{Q}}$, each $y \in C$, and for each connected component of D of $\pi_Y^{-1}(C) \cap S$, $D_y = \pi_Y^{-1}(y) \cap D$ is a connected component of $S_y = \pi_Y^{-1}(y) \cap S$.*

Proof. The theorem is a consequence of a somewhat more general theorem [5, Theorem 4.21] in the special case, when $\mathcal{F}$ is the constant sheaf $\mathbb{Q}_S$ supported on S. Using Theorem 4.21 in [5] we obtain a family of polynomials $\mathcal{Q} \subset \mathbb{R}[Y_1, \ldots, Y_\ell]$ with degrees and cardinality bounded by $(sd)^{(k+\ell)^{O(1)}}$ such that the sheaf $R^0\pi_{Y,*}\mathcal{F}$ is constant on the realization of each realizable sign condition σ on $\mathcal{Q}$. This implies that for each connected component C of each realizable sign condition $\sigma \in \mathrm{SIGN}(\mathcal{Q}) \subset$

$\{0,1,-1\}^{\mathcal{Q}}$, each $y \in C$, and for each connected component of D of $\pi_Y^{-1}(C) \cap S$, $D_y = \pi_Y^{-1}(y) \cap D$ is a connected component of $S_y = \pi_Y^{-1}(y) \cap S$. □

We are now in a position to prove Theorem 4.

Proof of Theorem 4. Let $\Phi(\overline{X}, \overline{Y})$ be the $\mathcal{P}$-closed formula describing S.

First apply Theorem 6 to obtain a set of polynomials $\mathcal{Q} \subset \mathbb{R}[Y_1, \ldots, Y_\ell]$ with degrees and cardinality bounded by $(sd)^{(k+\ell)^{O(1)}}$, and for each connected component C of each realizable sign condition $\sigma \in \mathrm{SIGN}(\mathcal{Q}) \subset \{0,1,-1\}^{\mathcal{Q}}$, each $y \in C$, and for each connected component of D of $\pi_Y^{-1}(C) \cap S$, $D_y = \pi_Y^{-1}(y) \cap D$ is a connected component of $S_y = \pi_Y^{-1}(y) \cap S$.

Next using Theorem 5 obtain for each realizable sign condition σ of $\mathcal{Q}$, and for each connected component of C of $\mathcal{R}(\sigma, \mathbb{R}^\ell)$, a quantifier-free fomula $\Phi_{\sigma,C}(\overline{Y})$ describing C.

Now using Theorem 5 one more time, obtain for each σ, C, and each connected component D_α of the semi-algebraic set defined by $\Phi_{\sigma,C}(\overline{Y}) \wedge \Phi(\overline{X}, \overline{Y})$, a quantifier-free formula $\Theta_\alpha(\overline{X}, \overline{Y})$ describing D_α. □

5.3. **Bounding the topology of the image of a polynomial map.** The following theorem proved in [12] allows one to bound the Betti numbers of the image of a closed and bounded definable set X under a definable map F in terms of the Betti numbers of the iterated fibered product of X over F. More precisely:

Theorem 7. [12] *Let $F : X \to Y$ be a definable continuous map, and X a closed and bounded definable set. Then, for for all $p \geq 0$,*

$$b_p(F(X)) \leq \sum_{\substack{i,j \geq 0 \\ i+j=p}} b_i(\underbrace{X \times_F \cdots \times_F X}_{(j+1)}).$$

5.4. **Bounds on the Betti numbers of semi-algebraic sets.** Finally, in order to prove Theorem 3, we will need singly exponential upper bounds on the Betti numbers of semi-algebraic sets in terms of the number and degrees of the polynomials appearing in any quantifier-free formula defining the set. We give a brief overview of these results. The key result that we will need in the proof of Theorem 3 is Theorem 9.

5.4.1. *General Bounds.* The first results on bounding the Betti numbers of real varieties were proved by Oleĭnik and Petrovskiĭ [19], Thom [25], and Milnor [15]. Using a Morse-theoretic argument and Bezout's theorem they proved the following theorem which appears in [3] and makes more precise an earlier result which appeared in [1] :

Theorem 8. [3] *If $S \subset \mathbb{R}^k$ is a $\mathcal{P}$-closed semi-algebraic set, then*

$$b(S) \leq \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{s+1}{j} 6^j d(2d-1)^{k-1}, \tag{5.1}$$

where $s = \mathrm{card}(\mathcal{P}) > 0$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

Using an additional ingredient (namely, a technique to replace an arbitrary semi-algebraic set by a locally closed one with a very controlled increase in the number of polynomials used to describe the given set), Gabrielov and Vorobjov [11] extended Theorem 8 to arbitrary $\mathcal{P}$-semi-algebraic sets with only a small increase in the bound. Their result in conjunction with Theorem 8 gives the following theorem.

Theorem 9. [13, 4] *If $S \subset \mathbb{R}^k$ is a $\mathcal{P}$-semi-algebraic set, then*

$$b(S) \leq \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{2ks+1}{j} 6^j d(2d-1)^{k-1}, \tag{5.2}$$

where $s = \mathrm{card}(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

We will also use the following bound on the number of connected components of the realizations of all realizable sign conditions of a family of polynomials proved in [3].

Theorem 10. *Let $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]_{\leq d}$ and let $s = \mathrm{card}(\mathcal{P})$. Then*

$$\mathrm{card}\left(\bigcup_{\sigma \in \mathrm{SIGN}(\mathcal{P})} \mathrm{Cc}(\mathcal{R}(\sigma, \mathbb{R}^k)) \right) \leq \sum_{1 \leq j \leq k} \binom{s}{j} 4^j d(2d-1)^{k-1}.$$

We now have all the ingredients needed to prove Theorem 3.

5.5. Proof of Theorem 3.

Proof of Theorem 3. Let Φ be the $\mathcal{P}$-closed formula defining S. Introducing new variables $Z_1, \ldots, Z_m$, let $\tilde{S} \subset \mathbb{R}^n \times \mathbb{R}^m$ be the $\tilde{\mathcal{P}}$-formula

$$\Phi \wedge \bigwedge_{1 \leq i \leq m} (Z_i - f_i = 0).$$

Let $\tilde{f} : \tilde{S} \to \mathbb{R}^m$ denote the restriction to $\tilde{S}$ of the projection map $\pi_Z : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ to the Z-coordinates. Then clearly S is semi-algebraically homeomorphic to $\tilde{S}$, $f(S) = \tilde{f}(\tilde{S})$, and $\mathrm{Reeb}(f)$ is semi-algebraically homeomorphic to $\mathrm{Reeb}(\tilde{f})$. We have the following commutative square where the horizontal arrows are homeomorphisms and the vertical arrows are the quotient maps.

$$\begin{array}{ccc} S & \xrightarrow{\cong} & \tilde{S} \\ \big\downarrow \phi & & \big\downarrow \tilde{\phi} \\ \mathrm{Reeb}(f) & \xrightarrow{\cong} & \mathrm{Reeb}(\tilde{f}) \end{array}$$

Now it follows from Theorem 4 that there exists a finite set of polynomials $\mathcal{Q} \subset \mathbb{R}[Z_1, \ldots, Z_m]$, with

$$\mathrm{card}(\mathcal{Q}), \max_{Q \in \mathcal{Q}} \deg(Q) \quad \leq \quad (sd)^{(n+m)^{O(1)}} \tag{5.3}$$

having the following property: for each $\sigma \in \mathrm{SIGN}(\mathcal{Q})$ and each $C \in \mathrm{Cc}(\mathcal{R}(\sigma, \mathbb{R}^m))$, there exists an index set $I_{\sigma,C}$, a finite family of polynomials

$$\mathcal{P}_{\sigma,C} \subset \mathbb{R}[X_1, \ldots, X_n, Z_1, \ldots, Z_m],$$

and $\mathcal{P}_{\sigma,C}$ formulas $(\Theta_\alpha(\overline{X}, \overline{Z}))_{\alpha \in I_{\sigma,C}}$ such that $\Theta_\alpha(x, z) \Rightarrow z \in C$, and for each $z \in C$, and each connected component D of $\pi_Z^{-1}(C) \cap \tilde{S}$, there exists a unique $\alpha \in I_{\sigma,C}$ (which does not depend on z) with $\mathcal{R}(\Theta_\alpha(\cdot, z)) = \pi_Z^{-1}(z) \cap D$.

Moreover, the cardinalities of $I_{\sigma,C}$ and $\mathcal{P}_{\sigma,C}$ and the degrees of the polynomials in $\mathcal{P}_{\sigma,C}$ are all bounded by $(sd)^{(n+m)^{O(1)}}$.

Let ϕ (resp. $\tilde{\phi}$) be the canonical surjection $\phi : S \to \mathrm{Reeb}(f) \cong S/\sim$ (resp. $\tilde{\phi} : \tilde{S} \to \mathrm{Reeb}(\tilde{f}) \cong \tilde{S}/\sim$). From Theorem 2 it follows that we can assume that ϕ is a proper semi-algebraic map. For each $i \geq 0$, we have the inequality (cf. Theorem 7)

$$b_i(\mathrm{Reeb}(f)) \leq \sum_{p+q=i} b_q(\underbrace{S \times_\phi \cdots \times_\phi S}_{(p+1)\text{ times}}). \tag{5.4}$$

Now observe that $\underbrace{\tilde{S} \times_{\tilde{\phi}} \cdots \times_{\tilde{\phi}} \tilde{S}}_{(p+1)\text{ times}}$ (and hence $\underbrace{S \times_\phi \cdots \times_\phi S}_{(p+1)\text{ times}}$) is semi-algebraically homeomorphic to the semi-algebraic set defined by the formula

$$\Theta(\overline{X}^{(0)}, \ldots, \overline{X}^{(p)}, \overline{Z}) = \bigvee_{\substack{\sigma \in \mathrm{SIGN}(\mathcal{Q}) \\ C \in \mathrm{Cc}(\mathcal{R}(\sigma, \mathbb{R}^m)) \\ \alpha \in I_{\sigma,C}}} \bigwedge_{0 \leq j \leq p} \Theta_\alpha(\overline{X}^{(j)}, \overline{Z}). \tag{5.5}$$

To see this observe that

$$((x^{(0)}, z^{(0)}), \ldots, (x^{(p)}, z^{(p)})) \in \underbrace{\tilde{S} \times_{\tilde{\phi}} \cdots \times_{\tilde{\phi}} \tilde{S}}_{(p+1)\text{ times}}$$

if and only if

$$z^{(0)} = \cdots = z^{(p)} = z,$$

for some z, and $x^{(0)}, \ldots, x^{(p)}$ belong to the same connected component of $\tilde{f}^{-1}(z)$.

It is easy to verify this last equivalence using the properties of the decomposition given by Theorem 4.

We now claim that each of the formulas

$$\Theta(\overline{X}^0, \ldots, \overline{X}^{(p)}, \overline{Z}),\ 0 \leq p \leq m,$$

is a $\tilde{\mathcal{P}}_p$-formula for some finite set $\tilde{\mathcal{P}}_p \subset \mathbb{R}[\overline{X}^0, \ldots, \overline{X}^{(p)}, \overline{Z}]$ with $\mathrm{card}(\tilde{\mathcal{P}}_p)$ and the degrees of the polynomials in $\tilde{\mathcal{P}}_p$ being bounded singly exponentially.

In order to prove the claim first observe that the cardinality of the set

$$\bigcup_{\sigma \in \mathrm{SIGN}(\mathcal{Q})} \mathrm{Cc}(\mathcal{R}(\sigma, \mathbb{R}^m))$$

is bounded singly exponentially, once the number of polynomials in $\mathcal{Q}$, and their degrees are bounded singly exponentially (using Theorem 10). The fact that the number of polynomials in $\mathcal{Q}$ and their degrees are bounded singly exponentially follows from (5.3). Moreover, for similar reasons the cardinalities of the index sets $I_{\sigma,C}$ are also bounded singly exponentially. The claim now follows from Eqn. (5.5).

Finally, to prove the theorem we first apply inequality (5.4) and then apply Theorem 9 to bound the right hand side of the inequality (5.4). □

Remark 6. Given the analogy between Reeb spaces and Stein factorization (cf. Remark 3) it could be interesting to investigate (in the context of algebraic geometry) Stein factorization for projective morphisms from the point of view of complexity in analogy with Theorem 3. To the best of our knowledge this has not yet been investigated.

6. Conclusion

In this paper we have proved the realizability of the Reeb space of proper definable maps in an o-minimal structure as a proper definable quotient. We have exhibited examples where the Reeb spaces of maps can have arbitrarily complicated topology compared to that of the domains of the maps, a sharp contrast with the behavior of Reeb graphs. Nevertheless, we have proved singly exponential upper bounds on the Betti numbers of the Reeb spaces of proper semi-algebraic maps.

References

[1] S. Basu. On bounding the Betti numbers and computing the Euler characteristic of semi-algebraic sets. *Discrete Comput. Geom.*, 22(1):1–18, 1999. 13

[2] S. Basu, R. Pollack, and M.-F. Roy. Betti number bounds, applications and algorithms. In *Current Trends in Combinatorial and Computational Geometry: Papers from the Special Program at MSRI*, volume 52 of *MSRI Publications*, pages 87–97. Cambridge University Press, 2005. 3

[3] S. Basu, R. Pollack, and M.-F. Roy. On the Betti numbers of sign conditions. *Proc. Amer. Math. Soc.*, 133(4):965–974 (electronic), 2005. 3, 13, 14

[4] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 2006 (second edition). 2, 3, 11, 12, 14

[5] Saugata Basu. A Complexity Theory of Constructible Functions and Sheaves. *Found. Comput. Math.*, 15(1):199–279, 2015. 11, 12

[6] O Burlet and G de Rham. Sur certaines applications génériques d'une variété sur certaines applications génériques d'une variété close à 2 dimensions dans le plan. *L'Ensiegnement Mathématique*, 20:275–292, 1974. 1

[7] Michel Coste. *An introduction to o-minimal geometry*. Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000. Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica. 3, 4, 5, 6

[8] T. K. Dey, F. Memoli, and Y. Wang. Topological Analysis of Nerves, Reeb Spaces, Mappers, and Multiscale Mappers. *ArXiv e-prints*, March 2017. 3, 9

[9] Herbert Edelsbrunner, John Harer, and Amit K. Patel. Reeb spaces of piecewise linear mappings. In *Proceedings of the Twenty-fourth Annual Symposium on Computational Geometry*, SCG '08, pages 242–250, New York, NY, USA, 2008. ACM. 1

[10] Herbert Edelsbrunner and John L. Harer. *Computational topology*. American Mathematical Society, Providence, RI, 2010. An introduction. 2, 7

[11] A. Gabrielov and N. Vorobjov. Betti numbers of semialgebraic sets defined by quantifier-free formulae. *Discrete Comput. Geom.*, 33(3):395–401, 2005. 3, 14

[12] A. Gabrielov, N. Vorobjov, and T. Zell. Betti numbers of semialgebraic and sub-Pfaffian sets. *J. London Math. Soc. (2)*, 69(1):27–43, 2004. 9, 11, 13

[13] Andrei Gabrielov and Nicolai Vorobjov. Approximation of definable sets by compact families, and upper bounds on homotopy and homology. *J. Lond. Math. Soc. (2)*, 80(1):35–54, 2009. 14

[14] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. 1, 2, 5

[15] J. Milnor. On the Betti numbers of real varieties. *Proc. Amer. Math. Soc.*, 15:275–280, 1964. 3, 13

[16] Mamoru Mimura and Hirosi Toda. *Topology of Lie groups. I, II*, volume 91 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1991. Translated from the 1978 Japanese edition by the authors. 8

[17] E. Munch and B. Wang. Convergence between Categorical Representations of Reeb Space and Mapper. *ArXiv e-prints*, December 2015. 2

[18] Amit Patel. *Reeb Spaces and the Robustness of Preimages*. PhD thesis, Duke University, 2010. 1

[19] I. G. Petrovskiĭ and O. A. Oleĭnik. On the topology of real algebraic surfaces. *Izvestiya Akad. Nauk SSSR. Ser. Mat.*, 13:389–402, 1949. 3, 13

[20] A. Pillay and C. Steinhorn. Definable sets in ordered structures. I. *Trans. Amer. Math. Soc.*, 295(2):565–592, 1986. 3

[21] A. Pillay and C. Steinhorn. Definable sets in ordered structures. III. *Trans. Amer. Math. Soc.*, 309(2):469–576, 1988. 3

[22] Georges Reeb. Sur les points singuliers d'une forme de pfaff complètement intégrable ou d'une fonction numérique. *Comptes Rendus de l'Académie des Sciences*, 222:847–849, 1946. 1

[23] J.-P. Rolin, P. Speissegger, and A. J. Wilkie. Quasianalytic Denjoy-Carleman classes and o-minimality. *J. Amer. Math. Soc.*, 16(4):751–777 (electronic), 2003. 3

[24] Gurjeet Singh, Facundo Mémoli, and Gunnar Carlsson. Topological methods for the analysis of high dimensional data sets and 3d object recognition. *Eurographics Symposium of Point-Based Graphics*, 2007. 1

[25] R. Thom. Sur l'homologie des variétés algébriques réelles. In *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, pages 255–265. Princeton Univ. Press, Princeton, N.J., 1965. 3, 13

[26] L. van den Dries. Remarks on Tarski's problem concerning $(\mathbf{R}, +, \cdot, \exp)$. In *Logic colloquium '82 (Florence, 1982)*, volume 112 of *Stud. Logic Found. Math.*, pages 97–121. North-Holland, Amsterdam, 1984. 3

[27] L. van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998. 3, 4, 5

[28] L. van den Dries and C. Miller. Geometric categories and o-minimal structures. *Duke Math. J.*, 84(2):497–540, 1996. 3

[29] L. van den Dries and P. Speissegger. The real field with convergent generalized power series. *Trans. Amer. Math. Soc.*, 350(11):4377–4421, 1998. 3

[30] L. van den Dries and P. Speissegger. The field of reals with multisummable series and the exponential function. *Proc. London Math. Soc. (3)*, 81(3):513–565, 2000. 3

[31] A. J. Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. *J. Amer. Math. Soc.*, 9(4):1051–1094, 1996. 3

[32] A. J. Wilkie. A theorem of the complement and some new o-minimal structures. *Selecta Math. (N.S.)*, 5(4):397–421, 1999. 3, 4

[33] Alex J. Wilkie. o-minimal structures. *Astérisque*, (326):Exp. No. 985, vii, 131–142 (2010), 2009. Séminaire Bourbaki. Vol. 2007/2008. 2

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