ON THE REEB SPACES OF DEFINABLE MAPS

SAUGATA BASU, NATHANAEL COX, AND SARAH PERCIVAL

Abstract. We prove that the Reeb space of a proper definable map in an arbitrary o-minimal expansion of the reals is realizable as a proper definable quotient. We also show that the Betti numbers of the Reeb space of a map f can be arbitrarily large compared to those of X, unlike in the special case of Reeb graphs of manifolds. Nevertheless, in the special case when $f: X \to Y$ is a semi-algebraic map and X is closed and bounded, we prove a singly exponential upper bound on the Betti numbers of the Reeb space of f in terms of the number and degrees of the polynomials defining X, Y and f.

1. Introduction

Given a topological space X and a continuous function $f\colon X \to \mathbb{R}$, define an equivalence relation $\sim$ on X by setting $x_1 \sim x_2$ if $f(x_1) = f(x_2)$ and $f^{-1}(f(x_1))$ and $f^{-1}(f(x_2))$ are in the same connected component of X. The space $X/\sim$ is called the *Reeb graph* of f, denoted $\mathrm{Reeb}(f)$. The concept of the Reeb graph was introduced by Georges Reeb in [22] as a tool in Morse theory. The notion of the Reeb graph can be generalized to the notion of *Reeb space* by letting $f\colon X \to Y$, where Y is any topological space. Burlet and de Rham first introduced the Reeb space in [8] as the *Stein factorization* of a map f, but their work was limited to bivariate, generic, smooth mappings. Edelsbrunner et al. in [11] defined the Reeb space of a multivariate piecewise linear mapping on a combinatorial manifold, and proved results regarding its local and global structure. Expanding on this work, Patel [18] produced an algorithm to construct the Reeb space of a mapping f. Mapper, introduced in [24], gives a discrete approximation of the Reeb space of a multivariate mapping, allowing for more efficient computation of the underlying data structure. Munch et al. [17] define the *interleaving distance* for Reeb spaces to show the convergence between the Reeb space and Mapper.

In this paper we investigate Reeb spaces from the point of view of topological complexity. Our motivation is to understand how topologically complicated the Reeb space of a map can become in terms of the complexity of the map itself. In order to obtain meaningful results we restrict ourselves to the category of maps *definable in an o-minimal expansion of* $\mathbb{R}$ (see Section 2 for a quick overview of o-minimality), and in particular to *semi-algebraic* maps.

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Our first result is that the Reeb spaces of "tame" maps are themselves tame. More precisely, we prove that the quotient map corresponding to the Reeb space of a proper definable map can be realized as a proper definable map (Theorem 2 below). This implies as a special case that the Reeb spaces of proper semi-algebraic maps can be realized as semi-algebraic quotients. This gives rise to an algorithmic question of designing an algorithm to describe semi-algebraically this semi-algebraic quotient. We do not pursue this question further in this paper leaving it for future work.

We next consider the problem of studying the topological complexity of Reeb spaces of semi-algebraic maps. The problem of bounding the topological complexity (for example measured in terms of Betti numbers or the number of homotopy types of fibers) of semi-algebraic sets or maps in terms of the parameters of the formula defining them has a long history (see [4] for a survey). Bounds on these quantities which are doubly exponential in the dimension or the number of variables usually follow from the fact that semi-algebraic sets admit semi-algebraic triangulations of at most doubly exponential size. Singly exponential upper bounds are more difficult and usually involve more careful arguments involving Morse inequalities and other inequalities coming from certain spectral sequences [19, 25, 16, 2, 12, 6]. To the best of our knowledge, the problem of bounding the Betti numbers of the Reeb space of a semi-algebraic map has not been considered before. In this paper we prove a singly exponential upper bound on the Betti numbers of the Reeb space of a semi-algebraic map $f : X \to Y$, where X is a closed and bounded semi-algebraic set, in terms of the number and the degrees of the polynomials defining X, Y and f (cf. Theorem 3 below).

While studying the topological complexity of Reeb spaces of semi-algebraic maps is a natural mathematical question on its own, another motivation is related to complexity of algorithms for computing a semi-algebraic description of the Reeb spaces of a semi-algebraic continuous map. It is a meta theorem in algorithmic semi-algebraic geometry that upper bounds on topological complexity of objects are closely related to the worst-case complexity of algorithms computing the topological invariants of such objects. Thus, a singly exponential upper bound on the Betti numbers of the Reeb space of a semi-algebraic map also opens the possibility of being able to compute the Betti numbers of the Reeb space as well as computing a semi-algebraic description with an algorithm having a singly exponential complexity bound.

The rest of the paper is organized as follows. In Section 2 we recall the basic definitions related to o-minimality. In Section 3 we prove the definability of the Reeb spaces of proper definable maps. In Section 4, we describe an example showing that the Betti numbers of the Reeb space of a definable map $f : X \to Y$ could be arbitrarily large compared to those of X. Finally, in Section 5, we prove a singly exponential upper bound on the Betti numbers of the Reeb spaces of proper semi-algebraic maps in terms of the number and degrees of the polynomials defining the map. We conclude in Section 6 with some open problems.

2. Basic definitions

We first recall the important model theoretic notion of o-minimality which plays an important role in what follows.

2.0.1. *O-minimal Structures.* O-minimal structures were invented and first studied by Pillay and Steinhorn in the pioneering papers [20, 21] motivated by prior work of van den Dries [26]. Later the theory was further developed through contributions of other researchers, most notably van den Dries, Wilkie, Rolin, Speissegger amongst others [28, 29, 30, 31, 32, 23]. We particularly recommend the book by van den Dries [27] and the notes by Coste [10] for an easy introduction to the topic as well as the proofs of the basic results that we use in this paper.

Definition 1 (o-minimal structure)**.** An o-minimal structure over a real closed field R is a sequence $\mathcal{S}(\mathrm{R}) = (\mathcal{S}_n)_{n\in\mathbb{N}}$, where each $\mathcal{S}_n$ is a collection of subsets of R^n (called the *definable sets* in the structure) satisfying the following axioms (following the exposition in [10]).

(A) All algebraic subsets of R^n are in $\mathcal{S}_n$.
(B) The class $\mathcal{S}_n$ is closed under complementation and finite unions and intersections.
(C) If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$ then $A \times B \in \mathcal{S}_{m+n}$.
(D) If $\pi : \mathrm{R}^{n+1} \to \mathrm{R}^n$ is the projection map on the first n co-ordinates and $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$.
(E) The elements of $\mathcal{S}_1$ are finite unions of points and intervals. (Note that these are precisely the subsets of R which are definable by a first-order formula in the language of the reals with one free variable.)

The class of semi-algebraic sets is one obvious example of such a structure, but in fact there are much richer classes of sets which have been proved to be o-minimal (see [10, 27]).

We now consider quotients by definable equivalence relations.

Definition 2. Let $E \subset X \times X$ be a definable equivalence relation on a definable set X. A definable quotient of X by E is a pair (p, Y) consisting of a definable set Y and a definable surjective map $p : X \to Y$ such that:

(i) $(x_1, x_2) \in E \Leftrightarrow p(x_1) = p(x_2)$, for all $x_1, x_2 \in X$;
(ii) p is definably identifying, that is, for all definable $K \subset Y$, if $p^{-1}(K)$ is closed in X, then K is closed in Y.

We say that the definable quotient (p, Y) is definably proper if p is definably proper map, i.e. for every definable $K \subset Y$, with K closed and bounded in R^n, $p^{-1}(K) \subset X$ is closed and bounded in R^m.

Definition 3. A definable equivalence relation $E \subset X \times X$ is said to be definably proper if the two maps $\mathrm{pr}_1, \mathrm{pr}_2 : E \to X$ are definably proper.

We will use the following theorem which appears in [27].

Theorem 1. [27, page 166] *Let X be a definable set and $E \subset X \times X$ a definably proper equivalence relation on X. Then X/E exists as a definably proper quotient of X.*

3. Reeb space of a definable map $f : X \to Y$

We now fix an o-minimal expansion of $\mathbb{R}$. Let $X \subset \mathbb{R}^n$ be a closed and bounded definable set, and $f : X \to Y$ be a continuous definable map.

Definition 4. The Reeb space of the map f, henceforth denoted $\mathrm{Reeb}(f)$, is the topological space (equipped with the quotient topology), $X/\sim$, where $x \sim x'$ if and only if $f(x) = f(x')$, and x, x' belong to the same connected component of $f^{-1}(f(x))$.

Remark 1. Note that a definable (resp. semi-algebraic) set $S \subset \mathbb{R}^k$ is connected if and only if S is definably (resp. semi-algebraically) path-connected i.e. for all $x, y \in S$, there exists a definable (resp. semi-algebraic) path $\gamma : [0,1] \to S$, with $\gamma(0) = x, \gamma(1) = y$.

Our first result is that with the above assumptions:

Theorem 2. *The space* $\mathrm{Reeb}(f) \triangleq X/\sim$ *exists as a definably proper quotient. In other words, there exists a proper definable map $\psi : X \to Z$, and a homeomorphism $\theta : \mathrm{Reeb}(f) \to Z$ such that the following diagram commutes:*

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ \mathrm{Reeb}(f) = X/\sim \arrow[rr, "\theta"] & & Z \end{tikzcd}$$

(here ϕ is the quotient map). In particular, $\mathrm{Reeb}(f)$ *is homeomorphic to a definable set.*

Proof of Theorem 2. We first claim that the relation, "$x \sim x'$ if and only if $f(x) = f(x')$, and x, x' belong to the same connected component of $f^{-1}(f(x))$" is a definably proper equivalence relation. Using Hardt's triviality theorem for o-minimal structures [27, 10], we have that there exists a finite definable partition of Y into locally closed definable sets $(Y_\alpha)_{\alpha \in I}$, and $y_\alpha \in Y_\alpha$, and definable homeomorphisms $\phi_\alpha : Y_\alpha \times f^{-1}(y_\alpha) \to f^{-1}(Y_\alpha)$, such that the following diagram commutes for each $\alpha \in I$:

$$\begin{tikzcd} Y_\alpha \times f^{-1}(y_\alpha) \arrow[rr, "\phi_\alpha"] \arrow[dr, "\pi_1"'] & & f^{-1}(Y_\alpha) \arrow[dl, "f|_{f^{-1}(Y_\alpha)}"] \\ & Y_\alpha & \end{tikzcd}$$

(here π_1 is the projection to the first factor in the direct product). Let for each $\alpha \in I$, $(C_{\alpha,\beta})_{\beta \in J_\alpha}$ be the connected components of $f^{-1}(y_\alpha)$, and for each α, β, let $D_{\alpha,\beta} = \phi_\alpha(Y_\alpha \times C_{\alpha,\beta})$.

Let

$$E = \bigcup_{\alpha \in I, \beta \in J_\alpha} (\phi_\alpha \times \phi_\alpha)((Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta})),$$

where $(Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta})$ is the definable subset of $(Y_\alpha \times f^{-1}(y_\alpha)) \times (Y_\alpha \times f^{-1}(y_\alpha))$ defined by

$$((y,x),(y',x')) \in (Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta}) \Leftrightarrow y = y', x, x' \in C_{\alpha,\beta}.$$

It is clear that E is a definable subset of $X \times X$, and that $x \sim x'$ if and only if $(x, x') \in E$.

Since X is assumed to be closed and bounded, if we can show that E is closed in $X \times X$, it would follow that E is a definably proper equivalence relation, and we can apply Theorem 1.

The rest of the proof is devoted to showing that E is a closed definable subset of $X \times X$. For each $\alpha \in I, \beta \in J_\alpha$, let

$$E_{\alpha,\beta} = (\phi_\alpha \times \phi_\alpha)((Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta}).$$

Since $E = \bigcup_{\alpha \in I, \beta \in J_\alpha} E_{\alpha,\beta}$, in order to prove that E is closed it suffices to prove that for each $\alpha \in I, \beta \in J_\alpha$,

$$\overline{E_{\alpha,\beta}} \subset E,$$

where $\overline{E_{\alpha,\beta}}$ is the closure of $E_{\alpha,\beta}$ in $X \times X$.

It follows from the curve selection lemma for o-minimal structures [10], that for every $z \in \overline{E_{\alpha,\beta}}$ there exists a definable curve $\gamma : [0,1] \to E_{\alpha,\beta}$ with $\gamma(0) = z, \gamma((0,1]) \subset E_{\alpha,\beta}$. Thus, in order to prove that $\overline{E_{\alpha,\beta}} \subset E$, it suffices to show that for each definable curve $\gamma : (0,1] \to E_{\alpha,\beta}$, $z_0 = \lim_{t\to 0} \gamma(t) \in E$.

Let $\gamma : (0,1] \to E_{\alpha,\beta}$ be a definable curve, and suppose that $\lim_{t\to 0} \gamma(0) \notin E_{\alpha,\beta}$. Otherwise, $\lim_{t\to 0} \gamma(0) \in E_{\alpha,\beta} \subset E$, and we are done.

For $t \in (0,1]$, let $y_t = f(\gamma(t))$, and $(x_t, x'_t) \in (\phi_\alpha \times \phi_\alpha)((Y_\alpha \times C_{\alpha,\beta}) \times_{\pi_1} (Y_\alpha \times C_{\alpha,\beta}))$ such that $\gamma(t) = (x_t, x'_t)$. Note that $f(x_t) = f(x'_t) = y_t$. Finally, let $z_0 = (x_0, x'_0) = \lim_{t\to 0} \gamma(t)$.

Since, $z_0 \notin E_{\alpha,\beta}$ by assumption, and $\gamma((0,1]) \subset E_{\alpha,\beta}$, there exists $t_0 > 0$ such that $\lambda = f \circ \gamma|_{(0,t_0]} : (0,t_0] \to Y_\alpha$ is an injective definable map, and $\lim_{t\to 0} \lambda(t) = y_0 = f(x_0) = f(x'_0) \in Y_{\alpha'}$ for some $\alpha' \in I$. We need to show that x_0, x'_0 belong to the same connected component of $f^{-1}(y_0)$, which would imply that $(x_0, x'_0) \in E$.

Let $D_{\alpha,\beta,\gamma} = f^{-1}(\lambda((0,t_0])) \cap D_{\alpha,\beta}$ and let $g : D_{\alpha,\beta,\gamma} \to (0,t_0]$ be defined by $g(x) = \lambda^{-1}(f(x))$ (which is well defined by the injectivity of λ). Note that for each $t \in (0,t_0]$, $g^{-1}(t)$ is definably homeomorphic to $C_{\alpha,\beta}$, and hence is connected. It also follows from Hardt's triviality theorem that there exists $t'_0 \in (0,t_0]$, and a definable homeomorphism $\theta : g^{-1}(t'_0) \times (0,t'_0] \to g^{-1}((0,t'_0])$ such that the following diagram commutes:

$$\begin{tikzcd} g^{-1}(t'_0) \times (0,t'_0] \arrow[rr, "\theta"] \arrow[dr, "\pi_2"'] & & g^{-1}((0,t'_0]) \arrow[dl, "g"] \\ & (0,t'_0] & \end{tikzcd}$$

Extend θ continuously to a definable map $\overline{\theta} : g^{-1}(t_0') \times [0, t_0] \to \overline{g^{-1}((0, t_0'])}$ by setting $\overline{\theta}(x, 0) = \lim_{t\to 0} \theta(x, t)$. Finally, let $\theta' : g^{-1}(t_0') \to f^{-1}(y_0)$, be the definable continuous map obtained by setting $\theta'(x) = \overline{\theta}(x, 0)$.

Note that since $g^{-1}(t_0')$ is connected, so is $\theta'(g^{-1}(t_0'))$ being the image of a connected set under a continuous map. Also note that for each $t \in (0, t_0']$, $x_t, x_t' \in D_{\alpha,\beta,\gamma}$, $f(x, t) = f(x_t') = \lambda(t)$, and hence $x_t, x_t' \in g^{-1}(t)$, and thus $x_0, x_0' \in \theta'(g^{-1}(t_0'))$. Moreover, $f(x_0) = f(x_0') = y_0$

Since $\theta'(g^{-1}(t_0'))$ is connected, x_0, x_0' belong to the same connected component of $f^{-1}(y_0)$.

This shows that $(x_0, x_0') \in E$, which in turn implies that E is closed in $X \times X$.

The fact that Reeb(f) exists as a definably proper quotient now follows from Theorem 1. □

Remark 2. Theorem 2 opens up an algorithmic problem of actually realizing the Reeb space as a definable quotient in the special case where the o-minimal structure is that of semi-algebraic sets and maps. More precisely, the problem is to design an algorithm that, given a proper semi-algebraic map $f : X \to Y$, will compute a description of a semi-algebraic map $g : X \to Z \cong \mathrm{Reeb}(f)$, realizing the Reeb space of f as a semi-algebraic quotient. The complexity of the algorithm will then depend on the number and degrees of the polynomials defining X. In this paper, we do not pursue this algorithmic problem any further, leaving it for future work.

4. The Betti numbers of the Reeb space of $f: X \to Y$ can exceed that of X

Notation 1. For any topological space X, and $i \geq 0$, we will denote by $b_i(X)$ the i-th Betti number (that is the rank of the i-th singular homology group of X), and we will denote by $b(X) = \sum_i b_i(X)$.

In [9], Cole-McLaughlin et al. prove that for a manifold M and a Morse function $f: M \to \mathbb{R}$, $b_1(\mathrm{Reeb}(f)) \leq b_1(M)$, from which it follows that for such a function,

$$b(\mathrm{Reeb}(f)) \leq b(M).$$

We first show that the same is not true for Reeb spaces of more general maps.

Example 1. Consider the closed n-dimensional disk $\mathbf{D}^n$ with $n \geq 1$, and let $\sim$ be the equivalence relation identifying all points on the boundary of $\mathbf{D}^n$. Then $\mathbf{D}^n/\sim \,\cong \mathbf{S}^n$, where $\mathbf{S}^n$ is the n-dimensional sphere. Let f denote the quotient map $f : \mathbf{D}^n \to \mathbf{S}^n$. The fibers of f consist of either one point or the boundary $\mathbf{S}^{n-1}$ of $\mathbf{D}^n$, so they are connected, and hence $\mathrm{Reeb}(f) \cong \mathbf{S}^n$. Note that $b_0(\mathbf{D}^n) = 1$ and $b_i(\mathbf{D}^n) = 0$ for all $i > 0$. Moreover, $b_0(\mathbf{S}^n) = 1$, $b_n(\mathbf{S}^n) = 1$, and $b_i(\mathbf{S}^n) = 0, i \neq 0, n$. Thus, in this case

$$\begin{aligned} b(\mathbf{D}^n) &= 1, \\ b(\mathrm{Reeb}(f)) &= 2. \end{aligned}$$

More generally, for any $k \geq 0$, let

$$f_k = \underbrace{f \times \cdots \times f}_{k \text{ times}} : \underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}} \longrightarrow \underbrace{\mathbf{S}^n \times \cdots \times \mathbf{S}^n}_{k \text{ times}}.$$

Using the same argument as before $\mathrm{Reeb}(f_k) \cong \underbrace{\mathbf{S}^n \times \cdots \times \mathbf{S}^n}_{k \text{ times}}$. Thus,

$$\begin{aligned} b_0(\underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}}) &= 1, \\ b_i((\underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}}) &= 0, i > 0, \end{aligned}$$

and hence

$$b(\underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}}) = 1.$$

Moreover,

$$\begin{aligned} b_i(\mathrm{Reeb}(f_k)) &= 0, \text{ if } n \not| i \text{ or if } i > nk, \\ b_i(\mathrm{Reeb}(f_k)) &= \binom{k}{i/n}, \text{ otherwise,} \end{aligned}$$

and hence

$$b(\mathrm{Reeb}(f_k)) = 2^k.$$

This example shows that even for definably proper maps $f : X \to Y$, the individual as well as the total Betti numbers of $\mathrm{Reeb}(f)$ can be arbitrarily large compared to those of X.

5. Quantitative Bounds

We now consider the problem of bounding effectively from above the Betti numbers of the Reeb spaces of definable continuous maps. We have seen from Example 1 that given a continuous semi-algebraic map $f : X \to Y$, $b(\mathrm{Reeb}(f))$ can be arbitrarily large compared to $b(X)$ (unlike in the case of Reeb graphs i.e. when $\dim(Y) \leq 1$). In this section, we prove an upper bound on $b(\mathrm{Reeb}(f))$ in terms of the "semi-algebraic" complexity of the map f.

We first introduce some more notation.

Notation 2. For any finite family of polynomials $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]$, we call an element $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$, a *sign condition* on $\mathcal{P}$. For any semi-algebraic set $Z \subset \mathbb{R}^k$, and a sign condition $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$, we denote by $\mathcal{R}(\sigma, Z)$ the semi-algebraic set defined by

$$\{\mathbf{x} \in Z \mid \mathbf{sign}(P(\mathbf{x})) = \sigma(P), P \in \mathcal{P}\},$$

and call it the *realization* of σ on Z. More generally, we call any Boolean formula Φ with atoms, $P\{=,>,<\}0, P \in \mathcal{P}$, a *$\mathcal{P}$-formula*. We call the realization of Φ, namely the semi-algebraic set

$$\mathcal{R}(\Phi, \mathbb{R}^k) \quad = \quad \{\mathbf{x} \in \mathbb{R}^k \mid \Phi(\mathbf{x})\}$$

a *$\mathcal{P}$-semi-algebraic set*. Finally, we call a Boolean formula without negations, and with atoms $P\{\geq, \leq\}0$, $P \in \mathcal{P}$, to be a *$\mathcal{P}$-closed formula*, and we call the realization, $\mathcal{R}(\Phi, \mathbb{R}^k)$, a *$\mathcal{P}$-closed semi-algebraic set*.

We will denote by $\mathrm{SIGN}(\mathcal{P})$ the set of *realizable sign conditions of* $\mathcal{P}$, i.e.

$$\mathrm{SIGN}(\mathcal{P}) = \{\sigma \in \{0, 1, -1\}^{\mathcal{P}} \mid \mathcal{R}(\sigma, \mathbb{R}^k) \neq \emptyset\}.$$

Finally, for any semi-algebraic set S, we will denote by $\mathrm{Cc}(S)$, the set of its connected components.

We prove the following theorem.

Theorem 3. *Let $S \subset \mathbb{R}^n$ be a bounded $\mathcal{P}$-closed semi-algebraic set, and $f = (f_1, \ldots, f_m) : S \to \mathbb{R}^m$ be a polynomial map. Suppose that $s = \mathrm{card}(\mathcal{P})$, and the maximum of the degrees of the polynomials in $\mathcal{P}$ and $f_1, \ldots, f_m$ are bounded by d. Then,*

$$b(\mathrm{Reeb}(f)) \leq (sd)^{(n+m)^{O(1)}}.$$

The rest of the paper is devoted to the proof of Theorem 3. We first outline the main idea behind the proof.

5.1. **Outline of the proof of Theorem 3.** We first replace the map $f : S \to \mathbb{R}^m$, by a new map $\tilde{f} : \tilde{S} \to \mathbb{R}^m$, where $\tilde{S} \subset \mathbb{R}^n \times \mathbb{R}^m$, and $\tilde{f}$ is the restriction to $\tilde{S}$ of the projection map to $\mathbb{R}^m$. From their definitions is evident that $\mathrm{Reeb}(f)$ and $\mathrm{Reeb}(\tilde{f})$ are homeomorphic. We next prove that there exists a semi-algebraic partition of $\mathbb{R}^m$ of controlled complexity (more precisely given by the connected components of the realizable sign conditions of a family of polynomials of singly exponentially bounded degrees and cardinality), into connected semi-algebraic sets C, such that the connected components of the fibers $\tilde{f}^{-1}(z)$ are in 1-1 correspondence with each other as z varies over C, and moreover there exists quantifier-free first order formulas describing each of these connected components and the complexites of these formulas (i.e. the number of polynomials appearing in the formula and their respective degrees) are bounded singly exponentially (see Theorem 4 below for the precise formulation of this statement).

The proof of this result (Theorem 4) uses a certain sheaf-theoretic generalization of effective real quantifier elimination proved in [7] and recalled below (Theorem 6), and also the fact that the connected components of a semi-algebraic set can be described efficiently (with singly exponential complexity) which is a consequence of a result in [6] (Theorem 5 below).

Next we use the fact that the canonical surjection $\phi : \tilde{S} \to \mathrm{Reeb}(\tilde{f})$ is a proper semi-algebraic map. We now use an inequality proved in [13] (see Theorem 7 below), which gives an upper bound on the Betti numbers of the image of a proper semi-algebraic map $F : X \to Y$ in terms of the sum of the Betti numbers of various fiber products, $X \times_F \cdots \times_F X$ of the same map. Recall that, for $p \geq 0$, the $(p+1)$-fold fiber product

$$\underbrace{X \times_F \cdots \times_F X}_{(p+1)\text{-times}} \triangleq \{(x^{(0)}, \ldots, x^{(p)}) \in X^{p+1} \mid F(x^{(0)}) = \cdots = F(x^{(p)})\}.$$

Theorem 4 provides us with a well controlled descriptions (i.e. by quantifier-free first order formulas involving singly exponentially, any polynomials of singly exponentially bounded degrees) of the fibered products, $\tilde{S} \times_{\tilde{f}} \cdots \times_{\tilde{f}} \tilde{S}$. Finally, using these descriptions and results on bounding the Betti numbers of general semi-algebraic sets in terms of the number and degrees of polynomials defining them (cf. Theorem 9 below) we obtain the claimed bound on $\mathrm{Reeb}(f)$.

In order to make the above summary precise we first need to state some preliminary results.

5.2. **Parametrized description of connected components.** The following theorem which states that given any finite family of polynomials

$$\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell],$$

there exists a semi-algebraic partition of $\mathbb{R}^\ell$ of controlled complexity and which additionally has good properties with respect to $\mathcal{P}$, will play a crucial role in the proof of Theorem 3.

Theorem 4. *Let $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell]$ be a finite set of polynomials of degrees bounded by d, with $\mathrm{card}(\mathcal{P}) = s$, and let $S \subset \mathbb{R}^k \times \mathbb{R}^\ell$ be a $\mathcal{P}$-semi-algebraic set. Then, there exists a finite set of polynomials, $\mathcal{Q} \subset \mathbb{R}[Y_1, \ldots, Y_\ell]$, with degrees and cardinality bounded by $(sd)^{(k+\ell)^{O(1)}}$, and for each realizable sign condition $\sigma \in \mathrm{SIGN}(\mathcal{Q}) \subset \{0, 1, -1\}^{\mathcal{Q}}$ and a connected component C of $\mathcal{R}(\sigma, \mathbb{R}^\ell)$, an index set $I_{\sigma,C}$, a finite family of polynomials $\mathcal{P}_{\sigma,C} \subset \mathbb{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell]$, and $\mathcal{P}_{\sigma,C}$-formulas, $(\Theta_\alpha(\overline{X}, \overline{Y}))_{\alpha \in I_{\sigma,C}}$, such that for each $y \in C$, for each connected component of D of $\pi_Y^{-1}(C) \cap S$, there exists a unique $\alpha \in I_{\sigma,C}$, such that $\mathcal{R}(\Theta_\alpha(\cdot, y)) = \pi_Y^{-1}(y) \cap D$ and $\pi_Y^{-1}(y) \cap D$ is a connected component of $\pi_Y^{-1}(y) \cap S$.*

The proof of Theorem 4 will use the following result on efficient description of the connected components of semi-algebraic sets which can be easily deduced from [6, Theorem 16.3] and which we state without proof.

Theorem 5. *Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathbb{R}[X_1, \ldots, X_k]$ with $\deg(P_i) \leq d$ for $1 \leq i \leq s$ and let a semi-algebraic set S be defined by a $\mathcal{P}$ quantifier-free formula. There exists an algorithm that outputs quantifier-free semi-algebraic descriptions of all the connected components of S. The number of polynomials that appear in the output is bounded by $s^{k+1} d^{O(k^4)}$, while the degrees of the polynomials are bounded by $d^{O(k^3)}$.*

In order to prove Theorem 4 we will also need the following theorem which is a consequence of a more general result on the complexity of constructible sheafs proved in [7].

Theorem 6. *Let $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_\ell]$ be a finite set of polynomials of degrees bounded by d, with $\mathrm{card}(\mathcal{P}) = s$, and let $S \subset \mathbb{R}^k \times \mathbb{R}^\ell$ be a bounded $\mathcal{P}$-semi-algebraic set. Then there exists a finite set of polynomials, $\mathcal{Q} \subset \mathbb{R}[Y_1, \ldots, Y_\ell]$, with degrees and cardinality bounded by $(sd)^{(k+\ell)^{O(1)}}$, and for each connected component C of each realizable sign condition $\sigma \in \mathrm{SIGN}(\mathcal{Q}) \subset \{0, 1, -1\}^{\mathcal{Q}}$, each $y \in C$, and for each connected component of D of $\pi_Y^{-1}(C) \cap S$, $D_y = \pi_Y^{-1}(y) \cap D$ is a connected component of $S_y = \pi_Y^{-1}(y) \cap S$.*

Proof. The theorem is a consequence of a somewhat more general theorem [7, Theorem 4.21] in the special case, when $\mathcal{F}$ is the constant sheaf $\mathbb{Q}_S$ supported on S. Using Theorem 4.21 in [7] we obtain a family of polynomials $\mathcal{Q} \subset \mathbb{R}[Y_1, \ldots, Y_\ell]$ with degrees and cardinality bounded by $(sd)^{(k+\ell)^{O(1)}}$ such that the sheaf $R\pi_{Y,*}\mathcal{F}$ is locally constant on the realization of each realizable sign condition σ on $\mathcal{Q}$. It follows from the property of $\mathcal{Q}$ mentioned above that $\mathcal{Q}$ also has the property that for each connected component C of each realizable sign condition $\sigma \in \mathrm{SIGN}(\mathcal{Q}) \subset \{0, 1, -1\}^{\mathcal{Q}}$, each $y \in C$, and for each connected component of D of $\pi_Y^{-1}(C) \cap S$, $D_y = \pi_Y^{-1}(y) \cap D$ is a connected component of $S_y = \pi_Y^{-1}(y) \cap S$. $\square$

We are now in a position to prove Theorem 4.

Proof of Theorem 4. Let $\Phi(\overline{X}, \overline{Y})$ be the $\mathcal{P}$-closed formula describing S.

First apply Theorem 6 to obtain a set of polynomials $\mathcal{Q} \subset \mathbb{R}[Y_1, \ldots, Y_\ell]$ with degrees and cardinality bounded by $(sd)^{(k+\ell)^{O(1)}}$, and for each connected component C of each realizable sign condition $\sigma \in \mathrm{SIGN}(\mathcal{Q}) \subset \{0, 1, -1\}^{\mathcal{Q}}$, each $y \in C$, and for each connected component of D of $\pi_Y^{-1}(C) \cap S$, $D_y = \pi_Y^{-1}(y) \cap D$ is a connected component of $S_y = \pi_Y^{-1}(y) \cap S$.

Next using Theorem 5 obtain for each realizable sign condition σ of $\mathcal{Q}$, and for each connected component of C of $\mathcal{R}(\sigma, \mathbb{R}^\ell)$, a quantifier-free fomula $\Phi_{\sigma,C}$ describing C.

Now using Theorem 5 one more time, obtain for each σ, C, and each connected component D_α of the semi-algebraic set defined by $\Phi_{\sigma,C} \wedge \Phi$, a quantifier-free formula $\Theta_\alpha(\overline{X}, \overline{Y})$ describing D_α.

$\square$

5.3. **Bounding the topology of the image of a polynomial map.** The following theorem proved in [13] allows one to bound the Betti numbers of the image of a closed and bounded definable set X under a definable map F, in terms of the Betti numbers of the iterated fibered product of X over F. More precisely:

Theorem 7. [13] *Let $F : X \to Y$ be a definable map, and X a closed and bounded definable set. Then, for for all $p \geq 0$,*

$$b_p(F(X)) \leq \sum_{\substack{i,j \geq 0 \\ i+j=p}} b_i(\underbrace{X \times_F \cdots \times_F X}_{(j+1)}).$$

5.4. **Bounds on the Betti numbers of semi-algebraic sets.** Finally, in order to prove Theorem 3, we will need singly exponential upper bounds on the Betti numbers of semi-algebraic sets in terms of the number and degrees of the polynomials appearing in any quantifier-free formula defining the set. We give a brief overview of these results. The key result that we will need in the proof of Theorem 3 is Theorem 9.

5.4.1. *General Bounds.* The first results on bounding the Betti numbers of real varieties were proved by Oleĭnik and Petrovskiĭ [19], Thom [25] and Milnor [15]. Using a Morse-theoretic argument and Bezout's theorem they proved the following theorem (which makes more precise an earlier result appearing in [1]) which appears in [5]:

Theorem 8. [5] *If $S \subset \mathbb{R}^k$ is a $\mathcal{P}$-closed semi-algebraic set, then*

$$b(S) \leq \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{s+1}{j} 6^j d(2d-1)^{k-1}, \tag{5.1}$$

where $s = \mathrm{card}(\mathcal{P}) > 0$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

Using an additional ingredient (namely, a technique to replace an arbitrary semi-algebraic set by a locally closed one with a very controlled increase in the number of polynomials used to describe the given set), Gabrielov and Vorobjov [12] extended Theorem 8 to arbitrary $\mathcal{P}$-semi-algebraic sets with only a small increase in the bound. Their result in conjunction with Theorem 8 gives the following theorem.

Theorem 9. [14, 6] *If $S \subset \mathbb{R}^k$ is a $\mathcal{P}$-semi-algebraic set, then*

$$b(S) \leq \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{2ks+1}{j} 6^j d(2d-1)^{k-1}, \tag{5.2}$$

where $s = \mathrm{card}(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

We will also use the following bound on the number of connected components of the realizations of all realizable sign conditions of a family of polynomials proved in [3].

Theorem 10. *Let $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]_{\leq d}$ and let $s = \mathrm{card}\mathcal{P}$. Then,*

$$\mathrm{card}\left(\bigcup_{\sigma \in \mathrm{SIGN}(\mathcal{P})} \mathrm{Cc}(\mathcal{R}(\sigma, \mathbb{R}^k)) \right) \leq \sum_{1 \leq j \leq k} \binom{s}{j} 4^j d(2d-1)^{k-1}.$$

We now have all the ingredients needed to prove Theorem 3.

5.5. **Proof of Theorem 3.**

Proof of Theorem 3. Let Φ be the $\mathcal{P}$-closed formula defining S. Introducing new variables, $Z_1, \ldots, Z_m$, let $\tilde{S} \subset \mathbb{R}^n \times \mathbb{R}^m$ be the $\tilde{\mathcal{P}}$-formula,

$$\Phi \wedge \bigwedge_{1 \leq i \leq m} (Z_i - f_i = 0).$$

Let $\tilde{f} : \tilde{S} \to \mathbb{R}^m$ denote the restriction to $\tilde{S}$ of the projection map, $\pi_Z : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$, to the Z-coordinates. Then clearly S is semi-algebraically homeomorphic to $\tilde{S}$, $f(S) = \tilde{f}(\tilde{S})$, and $\mathrm{Reeb}(f)$ is semi-algebraically homeomorphic to $\mathrm{Reeb}(\tilde{f})$. We have the following commutative square where the horizontal arrows are homeomorphisms, and the vertical arrows being the quotient maps.

$$\begin{tikzcd} S \arrow[r, "\cong"] \arrow[d, "\phi"] & \tilde{S} \arrow[d, "\tilde{\phi}"] \\ \mathrm{Reeb}(f) \arrow[r, "\cong"] & \mathrm{Reeb}(\tilde{f}) \end{tikzcd}$$

Now it follows from Theorem 4 that there exists a finite set of polynomials, $\mathcal{Q} \subset \mathbb{R}[Z_1, \ldots, Z_m]$, with

$$\mathrm{card}(\mathcal{Q}), \max_{Q \in \mathcal{Q}} \deg(Q) \quad \leq \quad (sd)^{(n+m)^{O(1)}}, \tag{5.3}$$

and for each connected component C of each realizable sign condition $\sigma \in \mathrm{SIGN}(\mathcal{Q}) \subset \{0, 1, -1\}^{\mathcal{Q}}$, an index set $I_{\sigma,C}$, a finite family of polynomials

$$\mathcal{P}_{\sigma,C} \subset \mathbb{R}[X_1, \ldots, X_n, Z_1, \ldots, Z_m],$$

and $\mathcal{P}_{\sigma,C}$ formulas, $(\Theta_\alpha(\overline{X}, \overline{Z}))_{\alpha \in I_{\sigma,C}}$, such that for each $z \in C$, for each connected component of D of $\pi_Z^{-1}(C) \cap \tilde{S}$, there exists a unique $\alpha \in I_{\sigma,C}$ (which does not depend on z) with $\mathcal{R}(\Theta_\alpha(\cdot, z)) = \pi_Z^{-1}(z) \cap D$.

Moreover, the cardinalities of $I_{\sigma,C}, \mathcal{P}_{\sigma,C}$ and the degrees of the polynomials in $\mathcal{P}_{\sigma,C}$ are all bounded by $(sd)^{(n+m)^{O(1)}}$.

Let ϕ (resp. $\tilde{\phi}$) be the canonical surjection $\phi : S \to \mathrm{Reeb}(f) \cong S/\sim$ (resp. $\tilde{\phi} : \tilde{S} \to \mathrm{Reeb}(\tilde{f}) \cong \tilde{S}/\sim$). From Theorem 2 it follows that we can take ϕ to be a proper semi-algebraic map. For each $i \geq 0$, we have the inequality (cf. Theorem 7)

$$b_i(\mathrm{Reeb}(f)) \quad \leq \quad \sum_{p+q=i} b_q(\underbrace{S \times_\phi \cdots \times_\phi S}_{(p+1) \text{ times}}). \tag{5.4}$$

Now observe that $\underbrace{\tilde{S} \times_{\tilde{\phi}} \cdots \times_{\tilde{\phi}} \tilde{S}}_{(p+1) \text{ times}}$ (and hence $\underbrace{S \times_\phi \cdots \times_\phi S}_{(p+1) \text{ times}}$) is semi-algebraically homeomorphic to the semi-algebraic set defined by the formula

$$\Theta(\overline{X}^0, \ldots, \overline{X}^{(p)}, \overline{Z}) \quad = \quad \bigvee_{\substack{\sigma \in \mathrm{SIGN}(\mathcal{Q}) \\ C \in \mathrm{Cc}(\mathcal{R}(\sigma, \mathbb{R}^m)) \\ \alpha \in I_{\sigma,C}}} \bigwedge_{0 \leq j \leq p} \Theta_\alpha(\overline{X}^{(j)}, \overline{Z}) \tag{5.5}$$

To see this observe that

$$((x^{(0)}, z^{(0)}), \ldots, (x^{(p)}, z^{(p)})) \in \underbrace{\tilde{S} \times_{\tilde{\phi}} \cdots \times_{\tilde{\phi}} \tilde{S}}_{(p+1) \text{ times}}$$

if and only if

$$z^{(0)} = \cdots = z^{(p)} = z(\text{ say}),$$

and $x^{(0)}, \ldots, x^{(p)}$ belong to the same connected component of $\tilde{f}^{-1}(z)$.

It is easy to verify this last equivalence using the properties of the decomposition given by Theorem 4.

It follows from Eqn. (5.5) and the fact that the cardinalities of the set

$$\bigcup_{\sigma \in \mathrm{SIGN}(\mathcal{Q})} \mathrm{Cc}(\mathcal{R}(\sigma, \mathbb{R}^m))$$

is bounded singly exponentially, once the number of polynomial and their degrees in $\mathcal{Q}$ are bounded singly exponentially (using Theorem 10), and the fact that the number of polynomial and their degrees are bounded singly exponentially follows from (5.3). Moreover, for similar reasons the cardinalities of the index set $I_{\sigma,C}$ is also bounded singly exponentially.

Thus, each of the formulas,

$$\Theta(\overline{X}^0, \ldots, \overline{X}^{(p)}, \overline{Z}), 0 \leq p \leq m,$$

is a $\tilde{\mathcal{P}}_p$-formula for some finite set $\tilde{\mathcal{P}}_p \subset \mathbb{R}[\overline{X}^0, \ldots, \overline{X}^{(p)}, \overline{Z}]$ with cardinality and degrees bounded singly exponentially.

The theorem now follows after applying inequality (5.4) and applying Theorem 9 to bound the right hand side of the inequality (5.4). $\square$

6. Conclusion and Open Problems

In this paper we have proved the realizability of the Reeb space of proper definable maps in an o-minimal structure as a proper definable quotient. We have exhibited examples where the Reeb spaces of maps can have arbitrarily complicated topology compared to that of the domains of the maps, a sharp contrast with the behavior of Reeb graphs. Nevertheless, we have proved singly exponential upper bounds on the Betti numbers of the Reeb spaces of proper semi-algebraic maps.

There are some problems which have been left open. One open problem is to remove the assumption of properness from our results, perhaps using the approximation techniques developed by Gabrielov and Vorobjov [14]. Another open problem is to develop an algorithm (preferably with at most singly exponential complexity bound) for computing a semi-algebraic description of the Reeb space of a semi-algebraic map.

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Department of Mathematics, Purdue University, West Lafayette, IN 47906, U.S.A.
E-mail address: sbasu@math.purdue.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47906, U.S.A.
E-mail address: cox175@math.purdue.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47906, U.S.A.
E-mail address: sperciva@math.purdue.edu