EFFICIENT ALGORITHM FOR COMPUTING THE EULER-POINCARÉ CHARACTERISTIC OF A SEMI-ALGEBRAIC SET DEFINED BY FEW QUADRATIC INEQUALITIES

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Abstract. We present an algorithm which takes as input a closed semi-algebraic set, $S \subset \mathrm{R}^k$, defined by

$$P_1 \leq 0, \ldots, P_\ell \leq 0, P_i \in \mathrm{R}[X_1, \ldots, X_k], \deg(P_i) \leq 2,$$

and computes the Euler-Poincaré characteristic of S. The complexity of the algorithm is $k^{O(\ell)}$.
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1. Introduction

Let R be a real closed field and let $S \subset \mathrm{R}^k$ be a basic semi-algebraic set defined by $P_1 \leq 0, \ldots, P_\ell \leq 0$, with $P_i \in \mathrm{R}[X_1, \ldots, X_k], \deg(P_i) \leq 2,\ 1 \leq i \leq \ell$. It is known [2, 3] that the sum of the Betti numbers of S (and hence also its Euler-Poincaré characteristic) is bounded by $k^{O(\ell)}$. Notice that for fixed ℓ, these bounds are polynomial in k. One can also check whether S is non-empty, as well as compute a finite set of sample points meeting every connected component of S in time $k^{O(\ell)}$ [2, 10]. However, no algorithm with similar complexity is known for computing any of the individual Betti numbers of S (for instance, the number of connected components). The best known algorithm for computing all the Betti numbers of S has complexity $k^{2^{O(\ell)}}$ [5].

Here, and elsewhere in this paper the Betti number, $b_i(S)$, is the dimension of the simplicial homology group, $H_i(S, \mathbb{Q})$, in case S is closed and bounded. If S is a closed but not necessarily bounded semi-algebraic set, $b_i(S)$ is the dimension of $H_i(S \cap \overline{B_k(0,r)}, \mathbb{Q})$, for sufficiently large $r > 0$ (here and in the rest of the paper, $B_k(0,r)$ denotes the open ball of radius r in R^k centered at the origin, and $\overline{X}$ denotes the closure of a semi-algebraic set X). It is easy to see that $b_i(S)$ is well-defined and we denote by

$$\chi(S) = \sum_{i=0}^{k} (-1)^i b_i(S)$$

the Euler-Poincaré characteristic of S.

In this paper we describe an algorithm for computing the Euler-Poincaré characteristic of S, whose complexity is $k^{O(\ell)}$. Our algorithm relies on an efficient algorithm for computing the Euler-Poincaré characteristic of the realizations of all realizable sign conditions of a family of polynomials described in [6] and uses different techniques than those used in [2, 10].

The main result of this paper is the following.

Main Result: We present an algorithm (Algorithm 4.2 in Section 4) which given a set of ℓ polynomials, $\mathcal{P} = \{P_1, \dots, P_\ell\} \subset \mathrm{R}[X_1, \dots, X_k]$, with $\deg(P_i) \leq 2, 1 \leq i \leq \ell$, computes the Euler-Poincaré characteristic, $\chi(S)$, where S is the set defined by $P_1 \leq 0, \dots, P_\ell \leq 0$. The complexity of the algorithm is $k^{O(\ell)}$. If the coefficients of the polynomials in $\mathcal{P}$ are integers of bitsizes bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau k^{O(\ell)}$.

The rest of the paper is organized as follows. In Section 2, we describe some mathematical and algorithmic results we will need for our algorithm. We also include a brief introduction to spectral sequences since they play a motivating role in the design of the main algorithm described in this paper. In Section 3, we describe an algorithm for computing the Euler-Poincaré characteristic of a set defined by homogeneous quadratic inequalities. Finally, in Section 4 we describe our algorithm for the general (inhomogeneous) case.

For referring to well known results in real algebraic geometry we sometime use reference [7] as a useful source.

2. Preliminaries

In this section we describe some mathematical and algorithmic results we will require in the rest of the paper.

2.1. Definition of the Euler-Poincaré Characteristic. For the purposes of our algorithm, it will be useful to define Euler-Poincaré characteristic for locally closed semi-algebraic sets. We do this in terms of the Borel-Moore homology groups of such sets (defined below). This definition agrees with the definition of Euler-Poincaré characteristic stated earlier for closed and bounded semi-algebraic sets. They may be distinct for semi-algebraic sets which are closed but not bounded.

The simplicial homology groups of a pair of closed and bounded semi-algebraic sets $T \subset S \subset \mathrm{R}^k$ are defined as follows. Such a pair of closed, bounded semi-algebraic sets can be triangulated [7] using a pair of simplicial complexes (K, A), where A is a sub-complex of K. The p-th simplicial homology group of the pair (S,T), $H_p(S,T)$, is $H_p(K,A)$. The dimension of $H_p(S,T)$ as a $\mathbb{Q}$-vector space is called the p-th Betti number of the pair (S,T) and denoted $b_p(S,T)$. The Euler-Poincaré characteristic of the pair (S,T) is

$$\chi(S,T) = \sum_i (-1)^i b_i(S,T).$$

The p-th Borel-Moore homology group of $S \subset \mathrm{R}^k$, denoted $H_p^{BM}(S)$, is defined in terms of the homology groups of a pair of closed and bounded semi-algebraic sets as follows. For $r > 0$, let $S_r = S \cap B_k(0,r)$. Note that, for a locally closed semi-algebraic set S, both $\overline{S_r}$ and $\overline{S_r} \setminus S_r$ are closed and bounded and hence $H_p(\overline{S_r}, \overline{S_r} \setminus S_r)$ is well defined. Moreover, it is a consequence of Hardt's triviality theorem [11] that the homology group $H_p(\overline{S_r}, \overline{S_r} \setminus S_r)$ is invariant for all sufficiently large $r > 0$. We define, $H_p^{BM}(S) = H_p(\overline{S_r}, \overline{S_r} \setminus S_r)$, for $r > 0$ sufficiently large, and it follows from the remark above that it is well defined. The Borel-Moore homology groups are invariant under semi-algebraic homeomorphisms (see [8]. It also follows clearly from the definition that for a closed and bounded semi-algebraic set, the Borel-Moore homology groups coincide with the simplicial homology groups.

For a locally closed semi-algebraic set S, we define the Borel-Moore Euler-Poincaré characteristic by,

$$\chi^{BM}(S) = \sum_{i=0}^{k} b_i^{BM}(S),$$

where $b_i^{BM}(S)$ denotes the dimension of $H_i^{BM}(S, \mathbb{Q})$. Note that if S is closed and bounded, then $\chi^{BM}(S) = \chi(S)$.

The Borel-Moore Euler-Poincaré characteristic has the following additive property.

Proposition 2.1. *Let X, X_1 and X_2 be locally closed semi-algebraic sets such that*

$$X_1 \cup X_2 = X, X_1 \cap X_2 = \emptyset.$$

Then

$$\chi^{BM}(X) = \chi^{BM}(X_1) + \chi^{BM}(X_2).$$

Proof. This is classical (see for example, Proposition 2.6 in [6] for a proof). □

2.2. Sign Conditions and their realizations. A *sign condition* is an element of $\{0, 1, -1\}$. We define

$$\mathrm{sign}(x) = \begin{cases} 0 & \text{if and only if} \quad x = 0 \\ 1 & \text{if and only if} \quad x > 0 \\ -1 & \text{if and only if} \quad x < 0 \end{cases}$$

Let $Z \subset \mathrm{R}^k$ be a locally closed semi-algebraic set and let $\mathcal{P} = \{P_1, \ldots, P_s\}$ be a finite subset of $\mathrm{R}[X_1, \ldots, X_k]$. A *sign condition* on $\mathcal{P}$ is an element of $\{0, 1, -1\}^{\mathcal{P}}$.

The realization of the sign condition σ on Z is

$$\mathcal{R}(\sigma, Z) = \{x \in Z \mid \bigwedge_{P \in \mathcal{P}} \mathrm{sign}(P(x)) = \sigma(P)\},$$

and its Euler-Poincaré characteristic is denoted $\chi^{BM}(\sigma, Z)$.

We denote by $\mathrm{Sign}(\mathcal{P}, Z)$ the list of $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$ such that $\mathcal{R}(\sigma, Z)$ is non-empty. We denote by $\chi^{BM}(\mathcal{P}, Z)$ the list of Euler-Poincaré characteristics $\chi^{BM}(\sigma, Z) = \chi^{BM}(\mathcal{R}(\sigma, Z))$ for $\sigma \in \mathrm{Sign}(\mathcal{P}, Z)$.

We will use the following algorithm for computing the list $\chi^{BM}(\mathcal{P}, Z)$ described in [6]. We describe here the input, output and complexity of the algorithm.

Algorithm 2.2. Euler-Poincaré Characteristic of Sign Conditions.

Input: an algebraic set $Z = \mathrm{Z}(Q, \mathrm{R}^k) \subset \mathrm{R}^k$ and a finite list $\mathcal{P} = P_1, \ldots, P_s$ of polynomials in $\mathrm{R}[X_1, \ldots, X_k]$.

Output: the list $\chi^{BM}(\mathcal{P}, Z)$.

Complexity: Let k' be the dimension of Z, d a bound on the degree of Q and the elements of $\mathcal{P}$ and $s = \#(\mathcal{P})$). The number of arithmetic operations is

$$s^{k'+1} O(d)^k + s^{k'}((k' \log_2(s) + k \log_2(d))d)^{O(k)}.$$

The algorithm also involves the inversion matrices of size $s^{k'} O(d)^k$ with integer coefficients.

If $\mathrm{D} = \mathbb{Z}$, and the bitsizes of the coefficients of the polynomials are bounded by τ, then the bitsizes of the integers appearing in the intermediate computations and the output are bounded by $\tau((k' \log_2(s) + k \log_2(d))d)^{O(k)}$.

2.3. Infinitesimals. In our algorithms we will use infinitesimals in order to ensure that the set we are dealing with is bounded. To ensure this we will extend the ground field R to $\mathrm{R}\langle\varepsilon\rangle$, the real closed field of algebraic Puiseux series in ε with coefficients in R. The sign of a Puiseux series in $\mathrm{R}\langle\varepsilon\rangle$ agrees with the sign of the coefficient of the lowest degree term in ε. This induces a unique order on $\mathrm{R}\langle\varepsilon\rangle$ which makes ε infinitesimal: ε is positive and smaller than any positive element of R.

If R' is a real closed field containing R, then given a semi-algebraic set S in R^k, we denote the *extension* of S to R' by $\mathrm{Ext}(S, \mathrm{R}')$. $\mathrm{Ext}(S, \mathrm{R}')$ is the semi-algebraic subset of ${\mathrm{R}'}^k$ defined by the same quantifier free formula that defines S. The set $\mathrm{Ext}(S, \mathrm{R}')$ is well defined (i.e. it only depends on the set S and not on the quantifier free formula chosen to describe it). This is an easy consequence of the Tarski-Seidenberg transfer principle (see for example Section 2.4.1 in [7]).

2.4. Spectral Sequences. For the benefit of the readers we include a brief introduction to the theory of spectral sequences pointing to [9, 12] for more details.

A *spectral sequence* is a sequence of bigraded complexes $(E_r, d_r : E_r^{p,q} \to E_r^{p+r,q-r+1})$ such that the complex E_{r+1} is obtained from E_r by taking its homology with respect to d_r (that is $E_{r+1} = H_{d_r}(E_r)$).

There are two spectral sequences, ${}'E_*^{p,q}, {}''E_*^{p,q}$, (corresponding to taking row-wise or column-wise filtrations respectively) associated with a double complex $C^{\bullet,\bullet}$, which will be important for us. Both of these converge to $H^*(\mathrm{Tot}^\bullet(C^{\bullet,\bullet}))$. This means that the homomorphisms d_r are eventually zero, and hence the spectral sequences stabilize, and

$$\bigoplus_{p+q=i} {}'E_\infty^{p,q} \cong \bigoplus_{p+q=i} {}''E_\infty^{p,q} \cong H^i(\mathrm{Tot}^\bullet(C^{\bullet,\bullet})),$$

for each $i \geq 0$.

The first terms of these are ${}'E_1 = H_\delta(C^{\bullet,\bullet}), {}'E_2 = H_d H_\delta(C^{\bullet,\bullet})$, and ${}''E_1 = H_d(C^{\bullet,\bullet}), {}''E_2 = H_\delta H_d(C^{\bullet,\bullet})$.

In particular, assuming that the complex $C^{\bullet,\bullet}$ is bounded in both directions, we have that,

Proposition 2.3.

$$\begin{aligned} \textstyle\sum_{i\geq 0}(-1)^i \dim(H^i(\mathrm{Tot}^\bullet(C^{\bullet,\bullet}))) &= \textstyle\sum_{p,q\geq 0}(-1)^{p+q} \dim({}'E_2^{p,q}) \\ &= \textstyle\sum_{p,q\geq 0}(-1)^{p+q} \dim({}''E_2^{p,q}). \end{aligned}$$

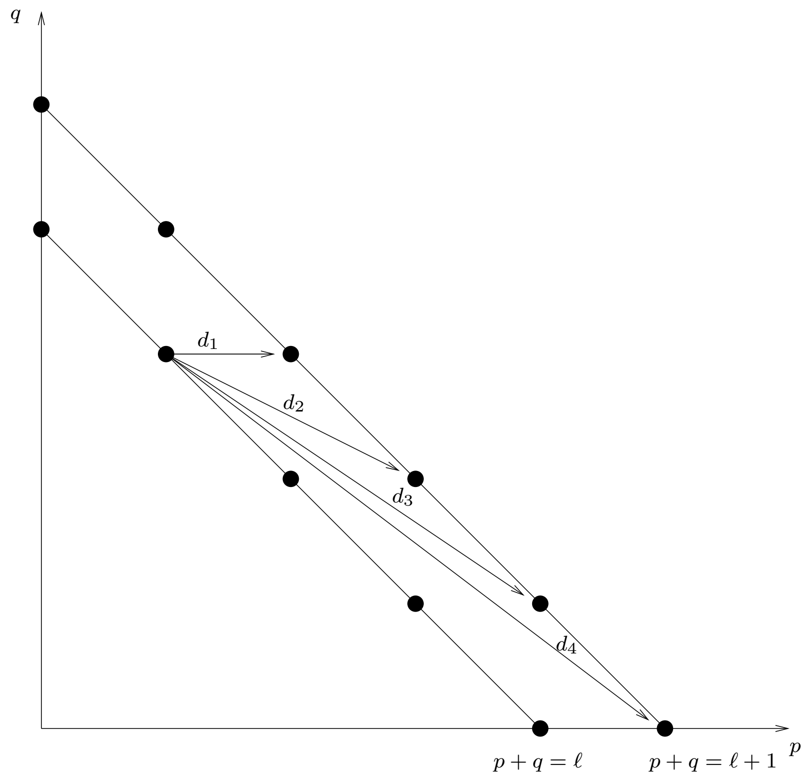


Figure 2.1: $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$

2.5. Leray Spectral Sequence of a map. The Leray spectral sequence of a proper map,

$$f : A \longrightarrow B,$$

is a classical tool in algebraic topology which relates the cohomology of the space A, with those of the space B and of the fibers of the map f. Its most common use is in the theory of sheaf cohomology [9]. We will need it in a very special situation where the sets A and B are compact semi-algebraic sets and f a continuous semi-algebraic map. In this special situation it is possible to define the Leray spectral sequence in terms of triangulations, which we do below.

Consider a semi-algebraic continuous map, $f : A \longrightarrow B$, where A and B are compact semi-algebraic sets. Moreover, let $h : \Delta \longrightarrow B$ be a semi-algebraic triangulation of B, and let $\mathcal{H}(A)$ denote a cell-complex, such that A is the union of the cells in $\mathcal{H}(A)$, and such that for any simplex $\sigma \in \Delta$, $A_\sigma = f^{-1}(\overline{h(\sigma)})$ is a subcomplex of $\mathcal{H}(A)$ (where $\overline{X}$ denotes the topological closure of X).

Then, the Leray spectral sequence of f is isomorphic to the spectral sequence (corresponding to the column-wise filtration) associated to the double complex $C^{\bullet,\bullet}$ defined as follows:

$$C^{p,q} = \bigoplus_{\sigma \text{ a } p\text{-simplex in } \Delta} C^q(A_\sigma),$$

where $C^q(A_\sigma)$ denotes the vector space of q-co-chains of the complex A_σ. The horizontal and the vertical differentials are the obvious ones (see [9]). The spectral sequence associated to the double complex defined above converges to the co-homology of A.

3. The basic homogeneous case

Let $\mathcal{P} = \{P_1, \ldots, P_\ell\} \subset \mathrm{R}[X_0, X_1, \ldots, X_k]$ be a set of homogeneous quadratic polynomials, and let S be the basic closed semi-algebraic set defined on the unit sphere $S^k \subset \mathrm{R}^{k+1}$ by the inequalities,

$$P_1 \leq 0, \ldots, P_\ell \leq 0.$$

We denote by S_i the subset of S^k defined by $P_i \leq 0$. Then, $S = \cap_{i=1}^{\ell} S_i$. For $J \subset \{1, \ldots, \ell\}$, we denote by $S^J = \cup_{j \in J} S_j$. The following equality is a consequence of the exactness of the Mayer-Vietoris sequence.

Lemma 3.1.

$$\chi(S) = \sum_{J \subset \{1,\ldots,\ell\}} (-1)^{\#(J)+1} \chi(S^J).$$

Proof. In case $\ell = 2$, this is a direct consequence of the exactness of Mayer-Vietoris sequence (see for example, [7], Corollary 6.28). The general case follows from an easy induction. □

Thus, in order to compute $\chi(S)$ it suffices to compute $\chi(S^J)$ for each $J \subset \{1, \ldots, \ell\}$.

3.1. Topology of sets defined by quadratic constraints. We first recall some facts about topology of sets defined by quadratic inequalities [1]. Let $P_1, \ldots, P_s$ be homogeneous quadratic polynomials in $\mathrm{R}[X_0, \ldots, X_k]$.

We denote by $P = (P_1, \ldots, P_s) : \mathrm{R}^{k+1} \to \mathrm{R}^s$, the map defined by the polynomials $P_1, \ldots, P_s$. Let

$$A = \bigcup_{1 \leq i \leq s} \{x \in S^k \mid P_i(x) \leq 0\}.$$

Let

$$\Omega = \{\omega \in R^s \mid |\omega| = 1, \omega_i \leq 0, 1 \leq i \leq s\}.$$

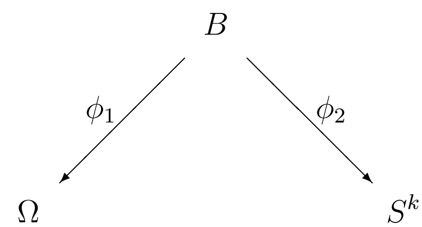
For $\omega \in \Omega$ we denote by ωP the quadratic form defined by

$$\omega P = \sum_{i=1}^{s} \omega_i P_i.$$

Let $B \subset \Omega \times S^k$ be the set defined by,

$$B = \{(\omega, x) \mid \omega \in \Omega, x \in S^k \text{ and } \omega P(x) \geq 0\}.$$

We denote by $\phi_1 : B \to \Omega$ and $\phi_2 : B \to S^k$ the two projection maps.



The following was proved by Agrachev [1]. With the notation developed above,

Proposition 3.2. *The map ϕ_2 gives a homotopy equivalence between B and $\phi_2(B) = A$.*

Proof. We first prove that $\phi_2(B) = A$. If $x \in A$, then there exists some $i, 1 \leq i \leq s$, such that $P_i(x) \leq 0$. Then for $\omega = (-\delta_{1i}, \ldots, -\delta_{si})$ (where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise), we see that $(\omega, x) \in B$. Conversely, if $x \in \phi_2(B)$, then there exists $\omega = (\omega_1, \ldots, \omega_s) \in \Omega$ such that, $\sum_{i=1}^{s} \omega_i P_i(x) \geq 0$. Since, $\omega_i \leq 0, 1 \leq i \leq s$, and not all $\omega_i = 0$, this implies that $P_i(x) \leq 0$ for some $i, 1 \leq i \leq s$. This shows that $x \in A$.

For $x \in \phi_2(B)$, the fibre

$$\phi_2^{-1}(x) = \{(\omega, x) \mid \omega \in \Omega \text{ such that } \omega P(x) \geq 0\},$$

can be identified with a non-empty subset of Ω defined by a single linear inequality. From convexity considerations, all such fibres can clearly be retracted to their center of mass continuously, proving the first half of the proposition. □

For any quadratic form Q, we will denote by $\mathrm{index}(Q)$, the number of negative eigenvalues of the symmetric matrix of the corresponding bilinear form, that is of the matrix M such that, $Q(x) = \langle Mx, x\rangle$ for all $x \in R^{k+1}$. We will also denote by $\lambda_i(Q), 0 \le i \le k$, the eigenvalues of Q, in non-decreasing order, i.e.

$$\lambda_0(Q) \le \lambda_1(Q) \le \cdots \le \lambda_k(Q).$$

Given a quadratic map $P = (P_1, \ldots, P_s) : \mathrm{R}^{k+1} \to \mathrm{R}^s$, and $0 \le j \le k$, we denote by

$$\Omega_j = \{\omega \in \Omega \mid \lambda_j(\omega P) \ge 0\}.$$

For notational convenience, Ω_{-1} will denote the empty set and Ω_{k+1} the whole space Ω.

It is clear that the Ω_j's induce a filtration of the space Ω, i.e., $\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_{k+1}$.

Agrachev [1] showed that the Leray spectral sequence of the map ϕ_1 (converging to the cohomology $H^*(B) \cong H^*(A)$), has as its E_2 terms,

$$E_2^{pq} = H^p(\Omega_{k-q}, \Omega_{k-q-1}). \tag{3.3}$$

This follows from the fact that the fibre of the map ϕ_1 over a point $\omega \in \Omega_j \setminus \Omega_{j-1}$ has the homotopy type of a sphere of dimension $k-j$. To see this notice that for $\omega \in \Omega_j \setminus \Omega_{j-1}$, $\lambda_0(\omega P), \ldots, \lambda_{j-1}(\omega P) < 0$. Moreover, letting $Y_0(\omega P), \ldots, Y_k(\omega P)$ be an orthonormal basis consisting of the eigenvectors of ωP, we have that $\phi_1^{-1}(\omega)$ is the subset of S^k defined by,

$$\sum_{i=0}^{k} \lambda_i(\omega P) Y_i(\omega P)^2 \ge 0,$$

$$\sum_{i=0}^{k} Y_i(\omega P)^2 = 1.$$

Since, $\lambda_i(\omega P) < 0, 0 \le i < j$, it follows that for $\omega \in \Omega_j \setminus \Omega_{j-1}$, $\phi_1^{-1}(\omega)$ is homotopy equivalent to the $(k-j)$-dimensional sphere defined by setting $Y_0(\omega P) = \cdots = Y_{j-1}(\omega P) = 0$ on the sphere defined by $\sum_{i=0}^{k} Y_i(\omega P)^2 = 1$.

The following proposition relates the Euler-Poincaré characteristic of the set A with those of $\Omega_j \setminus \Omega_{j-1}$, $0 \le j \le k+1$.

Proposition 3.4.

$$\chi(A) = \chi^{BM}(A) = \sum_{j=0}^{k+1} \chi^{BM}(\Omega_j \setminus \Omega_{j-1})(1 + (-1)^{(k-j)}).$$

Proof. Notice that the sets $\Omega_j \setminus \Omega_{j-1}$ are locally closed, and the fibre over a point $\omega \in \Omega_j \setminus \Omega_{j-1}$ is compact and has the homotopy type of a $(k-j)$-dimensional sphere. Now consider a sufficiently fine triangulation of Ω, which respects the filtration $\Omega_0 \subset \cdots \subset \Omega_{k+1}$, and such that over each simplex σ of the triangulation lying in $\Omega_j \setminus \Omega_{j-1}$, $\phi_1^{-1}(\sigma)$ is homotopy equivalent to $\sigma \times S^{k-j}$. The Euler-Poincaré characteristic of a $(k-j)$-dimensional sphere, S^{k-j}, is equal to $1+(-1)^{(k-j)}$. The proposition now follows from the additivity property of the Borel-Moore Euler-Poincaré characteristic and Proposition 3.2.

□

Since Proposition 3.4 is central to the algorithm presented in this paper, we include a different proof below which uses the spectral sequence (3.3). First note that by Proposition 2.1,

$$\chi^{BM}(\Omega_j \setminus \Omega_{j-1}) = \chi(\Omega_j) - \chi(\Omega_{j-1}).$$

It follows from the convergence of the spectral sequence in (3.3) and Proposition 2.3 that,

$$\begin{aligned}
\chi(A) &= \textstyle\sum_{p+q=i}(-1)^i \dim(E_2^{p,q}) \\
&= \textstyle\sum_{p+q=i}(-1)^i \dim H^p(\Omega_{k-q}, \Omega_{k-q-1}) \\
&= \textstyle\sum_{0\le q\le k+1} \sum_{0\le p\le k}(-1)^{p+q} \dim(H^p(\Omega_{k-q}, \Omega_{k-q-1})) \\
&= \textstyle\sum_{0\le q\le k+1}(-1)^q \sum_{0\le p\le k}(-1)^p \dim(H^p(\Omega_{k-q}, \Omega_{k-q-1}))
\end{aligned}$$

Now, from the exact sequence of the pair $(\Omega_{k-q}, \Omega_{k-q-1})$, namely,

$$\cdots \to H^{i-1}(\Omega_{k-q}) \to H^{i-1}(\Omega_{k-q-1}) \to H^i(\Omega_{k-q}, \Omega_{k-q-1}) \to H^i(\Omega_{k-q}) \to \cdots,$$

we get that,

$$\sum_{i\ge 0}(-1)^i(\dim(H^i(\Omega_{k-q-1})) - \dim(H^i(\Omega_{k-q})) + \dim(H^i(\Omega_{k-q}, \Omega_{k-q-1})) = 0,$$

which yields

$$\sum_{0\le p\le k}(-1)^p \dim(H^p(\Omega_{k-q}, \Omega_{k-q-1})) = \chi(\Omega_{k-q}) - \chi(\Omega_{k-q-1}).$$

Thus, the previous sum

$$\begin{array}{ll}
= & \sum_{0 \le q \le k+1} (-1)^q (\chi(\Omega_{k-q}) - \chi(\Omega_{k-q-1})) \\
= & \sum_{0 \le j \le k+1} (-1)^{k+1-j} (\chi(\Omega_{j-1}) - \chi(\Omega_{j-2})) \\
= & \sum_{0 \le j \le k+1} (-1)^{k+1-j} \chi(\Omega_{j-1}) - \sum_{0 \le j \le k+1} (-1)^{k+1-j} \chi(\Omega_{j-2}) \\
= & \sum_{0 \le j \le k} (-1)^{k-j} \chi(\Omega_j) - \sum_{0 \le j \le k-1} (-1)^{k+1-j} \chi(\Omega_j) \\
= & \sum_{0 \le j \le k} (-1)^{k-j} \chi(\Omega_j) + \sum_{0 \le j \le k-1} (-1)^{k-j} \chi(\Omega_j) \\
= & \chi(\Omega_k) - 2\chi(\Omega_{k-1}) + 2\chi(\Omega_{k-2}) + \cdots + (-1)^k 2\chi(\Omega_0) \\
= & \sum_{0 \le j \le k+1} (\chi(\Omega_j) - \chi(\Omega_{j-1}))(1 + (-1)^{k-j}) \\
= & \sum_{0 \le j \le k+1} \chi^{BM}(\Omega_j \setminus \Omega_{j-1})(1 + (-1)^{k-j}).
\end{array}$$

□

Let $Z = (Z_1, \ldots, Z_s)$ be variables and let $M(Z)$ be the symmetric matrix corresponding to the quadratic form $Z \cdot P = Z_1 P_1 + \cdots + Z_s P_s$. The entries of $M(Z)$ depend linearly on Z. Let

$$F(Z, T) = \det(M(Z) + T \cdot I_{k+1}) = T^{k+1} + C_k T^k + \cdots + C_0,$$

where each $C_i \in \mathrm{R}[Z_1, \ldots, Z_s]$ is a polynomial of degree at most $k+1$.

It follows from the well known Descarte's rule of signs (see for example, Remark 2.42 in [7]) that for any $z \in \Omega$, $\mathrm{index}(zP)$ is equal to the number of sign variations in the sequence $C_0(z), \ldots, C_k(z), +1$. Thus, the signs of the polynomials $C_0, \ldots, C_k$ determine the index of zP. For $\sigma \in \{0, +1, -1\}^{\mathcal{C}}$ a sign condition on the family $\mathcal{C} = \{C_0, \ldots, \mathrm{C}_k\}$, let $n(\sigma)$ denote the number of sign variations in the sequence, $\sigma(C_0), \ldots, \sigma(C_k), +1$. Let $\mathrm{Sign}(\mathcal{C}, \Omega)$ be the set of sign conditions realized by the family $\mathcal{C}$ on Ω. The following proposition is an immediate consequence of Proposition 3.4 and the additivity of the Euler-Poincaré characteristic.

Proposition 3.5.

$$\chi(A) = \chi^{BM}(A) = \sum_{\sigma \in \mathrm{Sign}(\mathcal{C}, \Omega)} \chi^{BM}(\mathcal{R}(\sigma, \Omega)) \cdot (1 + (-1)^{(k - n(\sigma))}).$$

Before proceeding further we discuss a small example.

Example 3.6. Let $\ell = 2, k = 2$ and $P = (P_1, P_2) : \mathrm{R}^3 \to \mathrm{R}^2$ be the quadratic map with,

$$\begin{aligned} P_1 &= X_0^2 + X_1^2 - X_2^2, \\ P_2 &= X_0^2 - X_1^2 - X_2^2. \end{aligned}$$

In this example,

$$\Omega = \{(\omega_1, \omega_2) \;\mid\; \omega_1^2 + \omega_2^2 = 1, \omega_1, \omega_2 \leq 0\}$$

consists of the arc of the unit circle in the third quadrant.

Also,

$$\begin{aligned} A &= \{x \in S^2 \mid P_1(x) \leq 0 \vee P_2(x) \leq 0\} \\ &= \{(x_0, x_1, x_2) \in \mathrm{R}^3 \mid x_0^2 + x_1^2 + x_2^2 = 1, x_1^2 + x_2^2 \geq 1/2\}. \end{aligned}$$

The set A in this example clearly has the homotopy type of a circle, and hence,

$$\chi(A) = 0. \tag{3.7}$$

Now, for $\omega = (\omega_1, \omega_2) \in \Omega$,

$$\begin{aligned} \omega P &= \omega_1 P_1 + \omega_2 P_2 \\ &= (\omega_1 + \omega_2) X_0^2 + (\omega_1 - \omega_2) X_1^2 - (\omega_1 + \omega_2) X_2^2. \end{aligned}$$

Following notations introduced above,

$$\begin{aligned} &F(Z_1, Z_2, T) = (Z_1 + Z_2 + T)(Z_1 - Z_2 + T)(-Z_1 - Z_2 + T) \\ &= T^3 + (Z_1 - Z_2)T^2 - (Z_1 + Z_2)^2 T + (Z_1 + Z_2)^2 (Z_2 - Z_1). \end{aligned}$$

Thus, in this example,

$$\begin{aligned} C_0(Z_1, Z_2) &= (Z_1 + Z_2)^2 (Z_2 - Z_1), \\ C_1(Z_1, Z_2) &= -(Z_1 + Z_2)^2, \\ C_2(Z_1, Z_2) &= Z_1 - Z_2. \end{aligned}$$

There are three realizable sign conditions on $\mathcal{C} = \{C_0, C_1, C_2, +1\}$ on Ω. They are,

$$\begin{aligned} \sigma_1 &= (-, -, +, +), \\ \sigma_2 &= (0, -, 0, +), \\ \sigma_3 &= (+, -, -, +). \end{aligned}$$

We have,

$$\begin{aligned} n(\sigma_1) &= 1, \\ n(\sigma_2) &= 1, \\ n(\sigma_3) &= 2. \end{aligned}$$

The realizations $\mathcal{R}(\sigma_1, \Omega)$ and $\mathcal{R}(\sigma_3, \Omega)$ are each homeomorphic to $[0, 1)$ while $\mathcal{R}(\sigma_2, \Omega)$ is a point. Thus,

$$\chi^{BM}(\sigma_1, \Omega) = \chi^{BM}(\sigma_3, \Omega) = 0,$$

while

$$\chi^{BM}(\sigma_2, \Omega) = 1.$$

Finally, we have

$$\begin{aligned} \textstyle\sum_{j=1}^{3} \chi^{BM}(\sigma_j, \Omega)(1 - (-1)^{2-n(\sigma_j)}) &= 0(1 + (-1)^1) + 1(1 + (-1)^1) \\ &\quad + 0(1 + (-1)^2) \\ &= 0, \end{aligned}$$

which agrees with (3.7). ♢

We are now in a position to describe an algorithm for computing the Euler-Poincaré characteristic of a union of sets, each defined by a homogeneous quadratic inequality. In the algorithm we will use the following notation. Given two finite families of polynomials, $\mathcal{P} \subset \mathcal{P}'$, and $\sigma \in \{0, +1, -1\}^{\mathcal{P}}, \sigma' \in \{0, +1, -1\}^{\mathcal{P}'}$, we say that $\sigma \prec \sigma'$ iff for all $P \in \mathcal{P}$, $\sigma(P) = \sigma'(P)$.

Algorithm 3.8. Euler-Poincaré characteristic of a union.

Input: A set of quadratic forms $\{P_1, \ldots, P_s\} \subset \mathrm{R}[X_0, \ldots, X_k]$.

Output: $\chi(A)$, where A is the set defined on the unit sphere $S^k \subset \mathrm{R}^{k+1}$ by the formula

$$P_1 \leq 0 \vee \cdots \vee P_\ell \leq 0.$$

1. Let $P = (P_1, \ldots, P_s)$. Let $Z = (Z_1, \ldots, Z_s)$ be variables and let $M(Z)$ be the symmetric matrix corresponding to the quadratic form $Z \cdot P$. Compute the polynomials $C_i \in \mathrm{R}[Z_1, \ldots, Z_s]$ by computing the following determinant.
$$\det(M(Z) + T \cdot I_k) = T^{k+1} + C_k T^k + \cdots + C_0.$$
2. Compute $\chi^{BM}(\mathcal{C}, \Omega)$ as follows. Call Algorithm 2.2 with input $\mathcal{C}' = \mathcal{C} \cup \{Z_1, \ldots, Z_s\}$ and $Q = Z_1^2 + \cdots + Z_s^2 - 1$. The output is the list
$$\chi^{BM}(\mathcal{C}', \mathrm{Z}(Q, \mathrm{R}^k)).$$
For each $\sigma \in \{0, +1, -1\}^{\mathcal{C}}$, such that there exists $\sigma' \in \mathrm{Sign}(\mathcal{C}', \mathrm{Z}(Q, \mathrm{R}^k))$ with $\sigma \prec \sigma'$ and $\sigma'(Z_j) \in \{0, -1\}$ for $1 \leq j \leq s$, compute
$$\chi^{BM}(\sigma, \Omega) = \sum_{\sigma', \sigma \prec \sigma', \sigma'(Z_j) \in \{0,-1\}, 1 \leq j \leq s} \chi^{BM}(\sigma', \mathrm{Z}(Q, \mathrm{R}^k)).$$

3. Output

$$\chi(A) = \sum_{\sigma \in \mathrm{Sign}(\mathcal{C},\Omega)} \chi^{BM}(\mathcal{R}(\sigma,\Omega)) \cdot (1 + (-1)^{(k-n(\sigma))}).$$

Proof of Correctness: The correctness of the algorithm is a consequence of Proposition 3.5 and the correctness of Algorithm 2.2. □

Complexity Analysis: The complexity of the algorithm is $k^{O(s)}$ using the complexity of Algorithm 2.2. □

We are now in a position to describe the algorithm for computing the Euler-Poincaré characteristic in the basic, homogeneous case.

Algorithm 3.9. The basic homogeneous case.

Input: A set of quadratic forms $\{P_1,\ldots,P_\ell\} \subset \mathrm{R}[X_0,\ldots,X_k]$.

Output: $\chi(S)$, where S is the set defined on the unit sphere $S^k \subset \mathrm{R}^{k+1}$ by the inequalities,

$$P_1 \le 0, \ldots, P_\ell \le 0.$$

1. For each subset $J \subset \{1,\ldots,\ell\}$ do the following.
2. Compute using Algorithm 3.8 $\chi(S^J)$.
3. Output

$$\chi(S) = \sum_{J \subset \{1,\ldots,\ell\}} (-1)^{\#(J)+1} \chi(S^J).$$

Proof of Correctness: The correctness of the algorithm is a consequence of Lemma Lemma 3.1 and the correctness of Algorithm 3.8. □

Complexity Analysis: There are 2^ℓ calls to Algorithm 3.8. Using the complexity analysis of Algorithm 3.8, the complexity of the algorithm is bounded by $k^{O(\ell)}$. □

4. The General Case

Let $\mathcal{P} = \{P_1,\ldots,P_\ell\} \subset \mathrm{R}[X_1,\ldots,X_k]$ with $\deg(P_i) \le 2, 1 \le i \le \ell$, and let $S \subset \mathrm{R}^k$ be the basic semi-algebraic set defined by $P_1 \le 0,\ldots,P_\ell \le 0$. Let $0 < \varepsilon$ be an infinitesimal, and let

$$P_{\ell+1} = \varepsilon \sum_{j=1}^{k} X_j^2 - 1.$$

Let $S' \subset \mathrm{R}\langle\varepsilon\rangle^k$ be the set defined by $P_1 \leq 0, \ldots, P_{\ell+1} \leq 0$.

Denoting by P_i^h the homogenization of P_i, and $S^h \subset S^k$ the set defined by, $P_1^h \leq 0, \ldots, P_\ell^h \leq 0, P_{\ell+1}^h \leq 0$, on the unit sphere in $\mathrm{R}\langle\varepsilon\rangle^{k+1}$ we have,

Proposition 4.1. *For* $0 \leq i \leq k$, $\chi(S) = \chi(S') = \frac{1}{2}\chi(S^h)$.

Proof. Using the well known conic structure at infinity of semi-algebraic sets (see for example Proposition 5.50, [7]) we have that for all sufficiently large $r > 0$, $S \cap \overline{B_k(0,r)}$ is a semi-algebraic deformation retract of S. Since ε is an infinitesimal, it follows that $S' = \mathrm{Ext}(S, \mathrm{R}\langle\varepsilon\rangle) \cap \overline{B_k(0, \frac{1}{\varepsilon})}$ is a semi-algebraic deformation retract of $\mathrm{Ext}(S, \mathrm{R}\langle\varepsilon\rangle)$. This implies that $\chi(S) = \chi(S')$.

To prove the second equality, first observe that S' is bounded, and S^h is the projection from the origin of the set $1 \times S' \subset 1 \times \mathrm{R}\langle\varepsilon\rangle^k$ onto the unit sphere in R^{k+1}. Since, S' is bounded, the projection does not intersect the equator and consists of two disjoint copies in the upper and lower hemispheres, and each copy is homeomorphic to S'. □

Algorithm 4.2. The general case.

Input: A family of polynomials $\mathcal{P} = \{P_1, \ldots, P_\ell\} \subset \mathrm{R}[X_1 \ldots, X_k]$, with $\deg(P_i) \leq 2$.

Output: $\chi(S)$, where S is the set defined by

$$S = \bigcap_{P \in \mathcal{P}} \{x \in \mathrm{R}^k \mid P(x) \leq 0\}.$$

1. Replace the family $\mathcal{P}$ by the family, $\mathcal{P}^h = \{P_1^h, \ldots, P_\ell^h, P_{\ell+1}^h\}$.
2. Using Algorithm 3.9 compute $\chi(S^h)$.
3. Output $\chi(S) = \frac{1}{2}\chi(S^h)$.

Proof of Correctness: The correctness of Algorithm 4.2 is a consequence of Proposition 4.1 and the correctness of Algorithm 3.9. □

Complexity Analysis: The complexity of the algorithm is clearly $k^{O(\ell)}$ from complexity analysis of Algorithm 3.9. □

Remark 4.3. In this paper we have described an algorithm for computing the Euler-Poincaré characteristic of a basic closed semi-algebraic set defined by a constant number of quadratic inequalities $P \leq 0, P \in \mathcal{P}$. It is straightforward to extend the algorithm to the case of semi-algebraic sets defined by Boolean formulas without negations, whose atoms are of the form $P \geq 0$ or $P \leq 0$ for

$P \in \mathcal{P}$, using the technique used in [4] for reducing the problem of computing Euler-Poincaré characteristic of such sets, to the basic closed case. This reduction works perfectly well even in the quadratic situation and does not worsen the complexity.

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