A COMPLEX ANALOGUE OF TODA'S THEOREM

SAUGATA BASU

Abstract. Toda [28] proved in 1989 that the (discrete) polynomial time hierarchy, **PH**, is contained in the class $\mathbf{P}^{\#\mathbf{P}}$, namely the class of languages that can be decided by a Turing machine in polynomial time given access to an oracle with the power to compute a function in the counting complexity class $\#\mathbf{P}$. This result, which illustrates the power of counting is considered to be a seminal result in computational complexity theory. An analogous result (with a compactness hypothesis) in the complexity theory over the reals (in the sense of Blum-Shub-Smale real machines [5]) was proved in [2]. Unlike Toda's proof in the discrete case, which relied on sophisticated combinatorial arguments, the proof in [2] is topological in nature in which the properties of the topological join is used in a fundamental way. However, the constructions used in [2] were semi-algebraic – they used real inequalities in an essential way and as such do not extend to the complex case. In this paper, we extend the techniques developed in [2] to the complex projective case. A key role is played by the complex join of quasi-projective complex varieties. As a consequence we obtain a complex analogue of Toda's theorem. The results contained in this paper, taken together with those contained in [2], illustrate the central role of the Poincaré polynomial in algorithmic algebraic geometry, as well as, in computational complexity theory over the complex and real numbers – namely, the ability to compute it efficiently enables one to decide in polynomial time all languages in the (compact) polynomial hierarchy over the appropriate field.

1. Introduction and Main Results

1.1. **History and Background.** The primary motivation for this paper comes from classical (i.e. discrete) computational complexity theory. In classical complexity theory, there is a seminal result due to Toda [28] linking the complexity of counting with that of deciding sentences with a fixed number of quantifier alternations.

More precisely, Toda's theorem gives the following inclusion (see Section 1.3.1 below or refer to [21] for precise definitions of the complexity classes appearing in the theorem).

Theorem 1.1 (Toda [28])**.**

$$\mathbf{PH} \subset \mathbf{P}^{\#\mathbf{P}}.$$

In other words, any language in the (discrete) polynomial hierarchy can be decided by a Turing machine in polynomial time, given access to an oracle with the power to compute a function in $\#\mathbf{P}$.

Date: **September 20, 2011**.

1991 *Mathematics Subject Classification.* Primary 14F25, 14Q15; Secondary 68Q15.

Key words and phrases. Polynomial hierarchy, Betti numbers, constructible sets, Toda's theorem.

The author was supported in part by NSF grants CCF-0634907 and CCF-0915954.

Remark 1.2. The proof of Theorem 1.1 in [28] is quite non-trivial. While it is obvious that the classes $\mathbf{P}, \mathbf{NP}, \mathbf{coNP}$ are contained in $\mathbf{P}^{\#\mathbf{P}}$, the proof for the higher levels of the polynomial hierarchy is quite intricate and proceeds in two steps: first proving that the $\mathbf{PH} \subset \mathbf{BP} \cdot \oplus \cdot \mathbf{P}$ (using previous results of Schöning [23], and Valiant and Vazirani [29]), and then showing that $\mathbf{BP} \cdot \oplus \cdot \mathbf{P} \subset \mathbf{P}^{\#\mathbf{P}}$. Aside from the obvious question about what should be a proper analogue of the complexity class $\#\mathbf{P}$ over the reals or complex numbers, because of the presence of complexity classes such as $\mathbf{BP}$ in the proof, there seems to be no direct way of extending such a proof to real or complex complexity classes in the sense of Blum-Shub-Smale model of computation [5, 24]. This is not entirely surprising, since complexity results in the Blum-Shub-Smale over different fields, while superficially similar, often require completely different proof techniques. For example, the fact that the polynomial hierarchy, $\mathbf{PH}$ is contained in the class $\mathbf{EXPTIME}$ is obvious over finite fields, but is non-trivial to prove over real closed or algebraically closed fields (where it is a consequence of efficient quantifier elimination algorithms).

The proof of the main theorem (Theorem 2.1) of this paper, which can be seen as a complex analogue of Theorem 1.1, proceeds along completely different lines from the classical (that is over finite fields) case, and is mainly topological in nature.

In the late eighties Blum, Shub, and Smale [5, 24] introduced the notion of Turing machines over more general fields, thereby generalizing the classical problems of computational complexity theory such as $\mathbf{P}$ vs. $\mathbf{NP}$ to corresponding problems over arbitrary fields (such as the real, complex, p-adic numbers etc.) If one considers languages accepted by a Blum-Shub-Smale machine over a finite field, one recovers the classical notions of discrete complexity theory. Over the last two decades there has been a lot of research activity towards proving real as well as complex analogues of well known theorems in discrete complexity theory. The first steps in this direction were taken by the authors Blum, Shub, and Smale (henceforth B-S-S) themselves, when they proved the $\mathbf{NP}_{\mathbb{C}}$-completeness of the problem of deciding whether a systems of polynomial equations has a solution (in affine space) (this is the complex analogue of Cook-Levin's theorem that the satisfiability problem is $\mathbf{NP}$-complete in the discrete case), and subsequently through the work of several researchers (Koiran, Bürgisser, Cucker, Meer to name a few) a well-established complexity theory over the reals as well as complex numbers have been built up, which mirrors closely the discrete case.

Indeed, one of the main attractions of the Blum-Shub-Smale computational model is that it provides a framework to prove complexity results over more general structures than just finite fields, with the hope that such results will help to unravel the algebro-geometric underpinnings of the basic separation questions amongst complexity classes. It is also often interesting to investigate complex (as well as real) analogues of results in discrete complexity theory, because doing so reveals underlying geometric and topological phenomena not visible in the discrete case. From this viewpoint it is quite natural to seek complex (as well as real) analogues of Toda's theorem; and as we will see in this paper (see also [2]), Toda's theorem properly interpreted over the real and complex numbers gives an unexpected connection between two important but distinct strands of algorithmic algebraic geometry – namely, *decision problems* involving quantifier elimination on one hand, and the problems of *computing topological invariants* of constructible sets on the other. Indeed, the original result of Toda, together with its real and complex counter-parts seem to suggest a deeper connection of a model-theoretic nature, between the problems of efficient quantifier-elimination and efficient computation of certain discrete invariants of definable sets in a structure, which might be an interesting problem on its own to explore further in the future.

1.2. **Recent Work.** There has been a large body of recent research on obtaining appropriate real (as well as complex) analogues of results in discrete complexity theory, especially those related to counting complexity classes (see [20, 6, 8, 7]). In [2] a real analogue of Toda's theorem was proved (with a compactness hypothesis). In this paper we prove a similar result in the complex case. Even though the basic approach is similar in both cases, the topological tools in the complex case are different enough to merit a separate treatment. This is elaborated further in the next section (the main difficulty in extending the real arguments in [2] to the complex case is that we can no longer use inequalities in our constructions).

1.3. **Definitions of complexity classes.** In order to formulate our result it is first necessary to define precisely complex counter-parts of the discrete polynomial time hierarchy **PH** and the discrete complexity class **#P**, and this is what we do next.

1.3.1. *Complex counter-parts of* **PH** *and* **#P**. For the rest of the paper C will denote an algebraically closed field of characteristic zero (there is no essential loss in assuming that $\mathrm{C} = \mathbb{C}$) (indeed by a transfer argument it suffices to prove all our results in this case). By a ***complex machine*** we will mean a machine in the sense of Blum-Shub-Smale [5]) over the ground field C.

Notational convention. Since in what follows we will be forced to deal with multiple blocks of variables in our formulas, we follow a notational convention by which we denote blocks of variables by bold letters with superscripts (e.g. $\mathbf{X}^i$ denotes the i-th block), and we use non-bold letters with subscripts to denote single variables (e.g. X^i_j denotes the j-th variable in the i-th block). We use $\mathbf{x}^i$ to denote a specific value of the block of variables $\mathbf{X}^i$.

Definition 1.3. We will call a quantifier-free first-order formula (in the language of fields), $\phi(\mathbf{X}^1; \cdots ; \mathbf{X}^\omega)$, having several blocks of variables $(\mathbf{X}^1, \ldots, \mathbf{X}^\omega)$ to be ***multi-homogeneous*** if each polynomial appearing in it is multi-homogeneous in the blocks of variables $(\mathbf{X}^1, \ldots, \mathbf{X}^\omega)$ and such that ϕ is satisfied whenever any one of the blocks $\mathbf{X}^i = 0$. Recall that a polynomial $P \in \mathrm{C}[\mathbf{X}^1; \cdots ; \mathbf{X}^\omega]$ is multi-homogeneous of multi-degree $(d_1, \ldots, d_\omega)$ if and only if it satisfies the identity

$$P(\lambda_1 \mathbf{X}^1; \cdots ; \lambda_\omega \mathbf{X}^\omega) = \lambda^{d_1} \cdots \lambda_\omega^{d_\omega} P(\mathbf{X}^1; \cdots ; \mathbf{X}^\omega).$$

Clearly such a formula defines a ***constructible*** subset of $\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\omega}_{\mathrm{C}}$ where the block $\mathbf{X}^i$ is assumed to have $k_i + 1$ variables. If $\omega = 1$, that is there is only one block of variables, then we call ϕ a ***homogeneous*** formula.

Notation 1.4 (Realization)**.** More generally, let

$$\Phi(\mathbf{X}^1; \ldots ; \mathbf{X}^\sigma) \stackrel{\text{def}}{=} (\mathrm{Q}_1 \mathbf{Y}^1) \cdots (\mathrm{Q}_\omega \mathbf{Y}^\omega) \phi(\mathbf{X}^1; \cdots ; \mathbf{X}^\sigma; \mathbf{Y}^1; \cdots ; \mathbf{Y}^\omega)$$

be a (quantified) multi-homogeneous formula, with $\mathrm{Q}_i \in \{\exists, \forall\}, 1 \leq i \leq \omega$, ϕ a quantifier-free multi-homogeneous formula, and $\mathbf{X}^i$ (resp. $\mathbf{Y}^j$) is a block of $k_i + 1$

(resp. ℓ_j+1) variables. We denote by $\mathcal{R}(\Phi) \subset \mathbb{P}_{\mathrm{C}}^{k_1} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k_\sigma}$ the constructible set which is the ***realization*** of the formula Φ; i.e.,

$$\mathcal{R}(\Phi(\mathbf{X})) = \{(\mathbf{x}^1, \ldots, \mathbf{x}^\sigma) \in \mathbb{P}_{\mathrm{C}}^{k_1} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k_\sigma} \mid$$
$$(\mathrm{Q}_1\mathbf{y}^1 \in \mathbb{P}_{\mathrm{C}}^{\ell_1}) \cdots (\mathrm{Q}_\omega\mathbf{y}^\omega \in \mathbb{P}_{\mathrm{C}}^{\ell_\omega})\phi(\mathbf{x}^1; \cdots ; \mathbf{x}^\sigma; \mathbf{y}^1; \cdots ; \mathbf{y}^\omega)\}.$$

Sometimes, in order to emphasize the block structure in a multi-homogeneous formula, we will write the quantifications as $(\exists \mathbf{Y} \in \mathbb{P}_{\mathrm{C}}^{\ell})$ (resp. $(\forall \mathbf{Y} \in \mathbb{P}_{\mathrm{C}}^{\ell})$) instead of just $(\exists \mathbf{Y})$ (resp. $(\forall \mathbf{Y})$). This is purely notational and does not affect the syntax of the formula.

Notation 1.5 (Negation of a multi-homogeneous formula)**.** It is clear that the property of multi-homogeneity is preserved by the Boolean operations of conjunction and disjunction. In order for it to be preserved also under negation, we will adopt the convention that the negation, $\neg\Phi(\mathbf{X}^1; \cdots ; \mathbf{X}^\omega)$, of a multi-homogeneous formula $\Phi(\mathbf{X}^1; \cdots ; \mathbf{X}^\omega)$ is by definition equal to

$$\tilde{\Phi} \vee \bigvee_{1 \leq i \leq \omega} (X^i = 0)$$

where $\tilde{\Phi}$ is the usual negation of ϕ as a Boolean formula. It is clear that defined this way, $\neg\Phi$ is multi-homogeneous, and

$$\mathcal{R}(\neg\Phi) = \mathbb{P}_{\mathrm{C}}^{k_1} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k_\omega} \setminus \mathcal{R}(\Phi).$$

We say that two multi-homogeneous formulas, Φ and Ψ, are ***equivalent*** if $\mathcal{R}(\Phi) = \mathcal{R}(\Psi)$. Clearly, equivalent multi-homogeneous formulas must have identical number of blocks of free variables, and the corresponding block sizes must also be equal.

Since the notion of multi-homogeneous formulas might look a bit unusual at first glance from the point of view of logic, we illustrate below how to homogenize non-homogeneous formulas by considering the following simple example (which is a building block for the "repeated squaring" technique used to prove doubly exponential lower bounds for (real) quantifier elimination [10]).

Example 1.6. Let $\Phi(X)$ be the following (existentially) quantified non-homogeneous formula expressing the fact, that $X^4 = 1$.

$$\Phi(X) \stackrel{\text{def}}{=} \exists Y (Y^2 - 1 = 0) \wedge (Y - X^2 = 0).$$

A multi-homogeneous version of the same formula is given by:

$$\Phi^h(X_0 : X_1) \stackrel{\text{def}}{=} \exists((Y_0 : Y_1) \in \mathbb{P}_{\mathrm{C}}^1)(Y_1^2 - Y_0^2 = 0) \wedge (X_0^2 Y_1 - X_1^2 Y_0 = 0).$$

Notice that the quantifier-free bi-homogeneous formula

$$\Psi^h(X_0 : X_1; Y_0 : Y_1) \stackrel{\text{def}}{=} (Y_1^2 - Y_0^2 = 0) \wedge (X_0^2 Y_1 - X_1^2 Y_0 = 0)$$

defines a constructible subset of $\mathbb{P}_{\mathrm{C}}^1 \times \mathbb{P}_{\mathrm{C}}^1$, and that the affine part of the constructible subset of $\mathbb{P}_{\mathrm{C}}^1$ defined by Φ^h coincides with the constructible subset of C^1 defined by $\Phi(X)$.

1.4. **Complex analogue of PH.** The definition of the polynomial hierarchy over C mirrors that of the discrete case (see [27]) very closely.

Definition 1.7 (The class $\mathbf{P}_{\mathrm{C}}$)**.** A sequence

$$(T_n \subset \mathrm{C}^n)_{n>0}$$

of constructible subsets is said to belong to the class $\mathbf{P}_{\mathrm{C}}$ if there exists a B-S-S machine M over C (see [5, 4]), such that for all $\mathbf{x} \in \mathrm{C}^n$, the machine M decides membership of $\mathbf{x}$ in T_n in time bounded by a polynomial in n.

More generally, suppose that $k(n)$ is some fixed polynomial which is non-negative and increasing. Let $(T_n \subset \mathrm{C}^{k(n)})_{n>0}$ be a sequence of constructible sets. We will say that $(T_n \subset \mathrm{C}^{k(n)})_{n>0}$ belongs to $\mathbf{P}_{\mathrm{C}}$ if the sequence $(S_n \subset \mathrm{C}^n)_{n>0}$ belongs to $\mathbf{P}_{\mathrm{C}}$, where S_n is defined by

$$\begin{aligned} S_{k(n)} &= T_{k(n)}, \text{ for all } n > 0, \\ S_m &= \emptyset, \text{ otherwise.} \end{aligned}$$

Definition 1.8 (The classes $\mathbf{\Sigma}_{\mathrm{C},\omega}$ and $\mathbf{\Pi}_{\mathrm{C},\omega}$)**.** Let $\omega \geq 0$ be a fixed integer. A sequence

$$(S_n \subset \mathrm{C}^n)_{n>0}$$

of constructible subsets is said to be in the complexity class $\mathbf{\Sigma}_{\mathrm{C},\omega}$, if for each $n > 0$, the constructible set S_n is described by a first order formula

$$(\mathrm{Q}_1 \mathbf{Y}^1) \cdots (\mathrm{Q}_\omega \mathbf{Y}^\omega) \phi_n(X_1, \ldots, X_n, \mathbf{Y}^1, \ldots, \mathbf{Y}^\omega), \tag{1.1}$$

with ϕ_n a quantifier free formula in the first order theory of C, and for each $i, 1 \leq i \leq \omega$, $\mathbf{Y}^i = (Y^i_1, \ldots, Y^i_n)$ is a block of n variables, $\mathrm{Q}_i \in \{\exists, \forall\}$, with $\mathrm{Q}_j \neq \mathrm{Q}_{j+1}, 1 \leq j < \omega$, $\mathrm{Q}_1 = \exists$, and the sequence

$$\left(T_n \subset \mathrm{C}^n \times \underbrace{\mathrm{C}^n \times \cdots \times \mathrm{C}^n}_{\omega \text{ times}}\right)_{n>0}$$

of constructible subsets defined by the quantifier-free formulas $(\phi_n)_{n>0}$ belongs to the class $\mathbf{P}_{\mathrm{C}}$.

Similarly, the complexity class $\mathbf{\Pi}_{\mathrm{C},\omega}$ is defined as in Definition 1.8, with the difference that the alternating quantifiers in (1.1) start with $\mathrm{Q}_1 = \forall$.

Remark 1.9. Notice that in Definition 1.8 there is no loss of generality in assuming that the sizes of the blocks of variables $\mathbf{X}, \mathbf{Y}^1, \ldots, \mathbf{Y}^\omega$ are all equal. To be more precise, suppose that the size of block $\mathbf{X}$ is $k(n)$, and that of $\mathbf{Y}^i$ is $k_i(n)$ for $1 \leq i \leq \omega$, where $k(n), k_1(n), \ldots, k_\omega(n)$ are fixed non-negative polynomials. Let

$$\left(S_n \subset \mathrm{C}^{k(n)}\right)_{n>0}$$

be a sequence of constructible subsets described by a first order formula

$$(\mathrm{Q}_1 \mathbf{Y}^1) \cdots (\mathrm{Q}_\omega \mathbf{Y}^\omega) \phi_n(\mathbf{X}, \mathbf{Y}^1, \ldots, \mathbf{Y}^\omega), \tag{1.2}$$

with ϕ_n a quantifier free formula in the first order theory of C, and $\mathrm{Q}_i \in \{\exists, \forall\}$, with $\mathrm{Q}_j \neq \mathrm{Q}_{j+1}, 1 \leq j < \omega$, such that the sequence

$$\left(T_n \subset \mathrm{C}^{k(n)} \times \mathrm{C}^{k_1(n)} \times \cdots \times \mathrm{C}^{k_\omega(n)}\right)_{n>0}$$

of constructible subsets defined by the quantifier-free formulas $(\phi_n)_{n>0}$ belongs to $\mathbf{P}_{\mathrm{C}}$.

Let $\tilde{k}(n)$ be any non-negative polynomial which majorizes $k(n), k_1(n), \ldots, k_\omega(n)$, and let $\tilde{\mathbf{X}} = (\mathbf{X}, \mathbf{X}'), \tilde{\mathbf{Y}}^i = (\mathbf{Y}^i, \mathbf{Y}^{i\prime}), 1 \leq i \leq \omega$, be blocks of variables obtained from the blocks $\mathbf{X}, \mathbf{Y}^i$, of size $\tilde{k}(n)$ by padding by an appropriate number of extra variables, $\mathbf{X}', \mathbf{Y}^{i\prime}$, respectively. By identifying the subspace of $\mathrm{C}^{\tilde{k}(n)}$ defined by setting the variables in the block $\mathbf{X}'$ to 0, with $\mathrm{C}^{k(n)}$ (and thus identifying S_n with its image under the corresponding inclusion in $\mathrm{C}^{\tilde{k}(n)}$), we have a sequence

$$\left(S_n \subset \mathrm{C}^{\tilde{k}(n)}\right)_{n>0}$$

of constructible subsets described by the formula

$$(\mathrm{Q}_1 \tilde{\mathbf{Y}}^1) \cdots (\mathrm{Q}_\omega \tilde{\mathbf{Y}}^\omega) \phi_n(\mathbf{X}, \mathbf{Y}^1, \ldots, \mathbf{Y}^\omega) \wedge (\mathbf{X}' = 0). \tag{1.3}$$

It is clear that the sequence

$$\left(T_n \subset \mathrm{C}^{k(n)} \times \mathrm{C}^{k_1(n)} \times \cdots \times \mathrm{C}^{k_\omega(n)}\right)_{n>0}$$

belongs to the class $\mathbf{P}_{\mathrm{C}}$ if and only if the sequence

$$\left(\tilde{T}_n \subset \mathrm{C}^{\tilde{k}(n)} \times \mathrm{C}^{\tilde{k}(n)} \times \cdots \times \mathrm{C}^{\tilde{k}(n)}\right)_{n>0}$$

defined by

$$\tilde{\phi}_n(\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}^1, \ldots, \tilde{\mathbf{Y}}^\omega) := \phi_n(\mathbf{X}, \mathbf{Y}^1, \ldots, \mathbf{Y}^\omega) \wedge (\mathbf{X}' = 0).$$

belongs to the class $\mathbf{P}_{\mathrm{C}}$. In other words (up to padding by some additional variables $\mathbf{X}'$ as above) there is no loss of generality in assuming that all the block sizes are equal.

Since, adding an additional block of quantifiers on the outside (with new variables that do not appear in the quantifier-free formula ϕ_n) does not change the set defined by a quantified formula we have the following inclusions:

$$\mathbf{\Sigma}_{\mathrm{C},\omega} \subset \mathbf{\Pi}_{\mathrm{C},\omega+1}, \text{ and } \mathbf{\Pi}_{\mathrm{C},\omega} \subset \mathbf{\Sigma}_{\mathrm{C},\omega+1}.$$

Note that by the above definition the class $\mathbf{\Sigma}_{\mathrm{C},0} = \mathbf{\Pi}_{\mathrm{C},0}$ is the class $\mathbf{P}_{\mathrm{C}}$, the class $\mathbf{\Sigma}_{\mathrm{C},1} = \mathbf{NP}_{\mathrm{C}}$ and the class $\mathbf{\Pi}_{\mathrm{C},1} = \text{co-}\mathbf{NP}_{\mathrm{C}}$.

Definition 1.10 (Complex polynomial hierarchy)**.** The complex polynomial time hierarchy is defined to be the union

$$\mathbf{PH}_{\mathrm{C}} \stackrel{\text{def}}{=} \bigcup_{\omega \geq 0} (\mathbf{\Sigma}_{\mathrm{C},\omega} \cup \mathbf{\Pi}_{\mathrm{C},\omega}) = \bigcup_{\omega \geq 0} \mathbf{\Sigma}_{\mathrm{C},\omega} = \bigcup_{\omega \geq 0} \mathbf{\Pi}_{\mathrm{C},\omega}.$$

As in the real case studied in [2] for technical reasons we need to restrict to compact constructible sets. However, unlike in [2] where the compact languages consisted of closed semi-algebraic subsets of spheres, in this paper we consider closed subsets of projective spaces instead. This is a much more natural choice for defining compact complex complexity classes.

We now define the compact analogue of $\mathbf{PH}_{\mathrm{C}}$ that we will denote $\mathbf{PH}_{\mathrm{C}}^c$. Unlike in the non-compact case, we will assume all variables vary over certain compact sets (namely complex projective spaces of varying dimensions).

We first need to be precise about what we mean by a complexity class of sequences of constructible subsets of complex projective spaces.

Notation 1.11 (Affine cones)**.** For any constructible subset $S \subset \mathbb{P}^k_{\mathrm{C}}$ we denote by $C(S) \subset \mathrm{C}^{k+1}$ the affine cone over S. More generally, if $S \subset \mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\omega}_{\mathrm{C}}$ is a constructible subset, then $C(S) \subset \mathrm{C}^{k_1+1} \times \cdots \times \mathrm{C}^{k_\omega+1}$ will denote the union of $L^1 \times \cdots \times L^\omega$ such that each $L^i \subset \mathrm{C}^{k_i+1}$ is a line through the origin, such that the point in $\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\omega}_{\mathrm{C}}$ represented by $(L^1, \ldots, L^\omega)$ is in S.

Definition 1.12. We say that a sequence

$$\left(S_n \subset \underbrace{\mathbb{P}^n_{\mathrm{C}} \times \cdots \times \mathbb{P}^n_{\mathrm{C}}}_{n \text{ times}}\right)_{n>0}$$

of constructible subsets is in the complexity class $\mathbf{P}_{\mathrm{C}}$, if the sequence of affine cones

$$\left(C(S_n) \subset \underbrace{\mathrm{C}^{n+1} \times \cdots \times \mathrm{C}^{n+1}}_{n \text{ times}}\right)_{n>0} \quad \text{belongs to the complexity class } \mathbf{P}_{\mathrm{C}}.$$

Remark 1.13. The subspaces spanned by the increasing sequence of standard basis elements of

$$\mathrm{C} = \langle e_0 \rangle \subset \mathrm{C}^2 = \langle e_0, e_1 \rangle \subset \cdots \subset \mathrm{C}^{n+1} = \langle e_0, \ldots, e_n \rangle \subset \cdots$$

after projectivization gives a flag

$$\mathbb{P}^0_{\mathrm{C}} \subset \mathbb{P}^1_{\mathrm{C}} \subset \cdots \subset \mathbb{P}^n_{\mathrm{C}} \subset \cdots$$

For $0 \leq m \leq n$, let $\iota_{m,n} : \mathbb{P}^m_{\mathrm{C}} \hookrightarrow \mathbb{P}^n_{\mathrm{C}}$ denote the corresponding inclusion.

Now, if $(S_n \subset \mathbb{P}^n_{\mathrm{C}})_{n>0}$ is a sequence of constructible sets, we can after identifying $\mathbb{P}^n_{\mathrm{C}}$ with the subspace

$$\mathbb{P}^n_{\mathrm{C}} \times \underbrace{\iota_{0,n}(\mathbb{P}^0_{\mathrm{C}}) \times \cdots \times \iota_{0,n}(\mathbb{P}^0_{\mathrm{C}})}_{n-1 \text{ times}}$$

of $\underbrace{\mathbb{P}^n_{\mathrm{C}} \times \cdots \times \mathbb{P}^n_{\mathrm{C}}}_{n \text{ times}}$ identify the sequence $(S_n \subset \mathbb{P}^n_{\mathrm{C}})_{n>0}$ with the sequence

$$(\tilde{S_n} \subset \underbrace{\mathbb{P}^n_{\mathrm{C}} \times \cdots \times \mathbb{P}^n_{\mathrm{C}}}_{n \text{ times}})_{n>0}.$$

where

$$\tilde{S_n} = S_n \times \underbrace{\iota_{0,n}(\mathbb{P}^0_{\mathrm{C}}) \times \cdots \times \iota_{0,n}(\mathbb{P}^0_{\mathrm{C}})}_{n-1 \text{ times}}.$$

We will (by abuse of language) say that the sequence $(S_n \subset \mathbb{P}^n_{\mathrm{C}})_{n>0}$ belongs to the class $\mathbf{P}_{\mathrm{C}}$ if the sequence

$$(\tilde{S_n} \subset \underbrace{\mathbb{P}^n_{\mathrm{C}} \times \cdots \times \mathbb{P}^n_{\mathrm{C}}}_{n \text{ times}})_{n>0}.$$

belongs to class $\mathbf{P}_{\mathrm{C}}$.

More generally, suppose that $m(n)$ is a non-negative polynomial in n and, $(k_i(n))_{i>0}$ a sequence of non-negative polynomials such that there exists a polynomial $k(n)$ which majorizes $m(n), k_1(n), k_2(n), \ldots, k_{m(n)}(n)$ for all $n > 0$. For example, we could have $m(n) = n$, and $k_i(n) = in$. Clearly, in this case the polynomial $k(n) = n^2 + 1$ majorizes $m(n), k_1(n), k_2(n), \ldots, k_{m(n)}(n)$ are for all $n > 0$.

We say that a sequence

$$\left(S_n \subset \mathbb{P}_{\mathrm{C}}^{k_1(n)} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k_{m(n)}(n)}\right)_{n>0}$$

is in $\mathbf{P}_{\mathrm{C}}$, if the sequence

$$\left(T_n \subset \underbrace{\mathbb{P}_{\mathrm{C}}^n \times \cdots \times \mathbb{P}_{\mathrm{C}}^n}_{n \text{ times}}\right)_{n>0}$$

is in $\mathbf{P}_{\mathrm{C}}$, where

$$T_{k(n)} = \tilde{S_n} \text{ for all } n > 0,$$
$$T_n = \emptyset, \text{ otherwise}$$

and

$$\tilde{S}_n \subset \underbrace{\mathbb{P}_{\mathrm{C}}^{k(n)} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k(n)}}_{k(n) \text{ times}}$$

is defined by

$$\tilde{S}_n = \iota_{k_1(n),k(n)} \times \cdots \times \iota_{k_{m(n)},k(n)}(S_n) \times \underbrace{\iota_{0,k(n)}(\mathbb{P}_{\mathrm{C}}^0) \times \cdots \times \iota_{0,k(n)}(\mathbb{P}_{\mathrm{C}}^0)}_{k(n)-m(n) \text{ times}}.$$

Definition 1.14 (Compact projective version of $\mathbf{\Sigma}_{\mathrm{C},\omega}$)**.** We say that a sequence

$$\left(S_n \subset \underbrace{\mathbb{P}_{\mathrm{C}}^n \times \cdots \times \mathbb{P}_{\mathrm{C}}^n}_{n \text{ times}}\right)_{n>0}$$

of constructible subsets is in the complexity class $\mathbf{\Sigma}_{\mathrm{C},\omega}^c$, if for each $n > 0$, S_n is described by a first order formula

$$(\mathrm{Q}_1 \mathbf{Y}^1 \in \mathbb{P}_{\mathrm{C}}^n) \cdots (\mathrm{Q}_\omega \mathbf{Y}^\omega \in \mathbb{P}_{\mathrm{C}}^n) \phi_n(\mathbf{X}^1; \cdots; \mathbf{X}^n; \mathbf{Y}^1; \cdots; \mathbf{Y}^\omega),$$

with ϕ_n a quantifier-free first order multi-homogeneous formula defining a *closed* (in the Zariski topology) subset of

$$\underbrace{\mathbb{P}_{\mathrm{C}}^n \times \cdots \times \mathbb{P}_{\mathrm{C}}^n}_{n \text{ times}} \times \underbrace{\mathbb{P}_{\mathrm{C}}^n \times \cdots \times \mathbb{P}_{\mathrm{C}}^n}_{\omega \text{ times}},$$

$\mathrm{Q}_i \in \{\exists, \forall\}$, $\mathrm{Q}_1 = \exists$, and the sequence of constructible sets $(T_n)_{n>0}$ defined by the formulas $(\phi_n)_{n>0}$ belongs to the class $\mathbf{P}_{\mathrm{C}}$.

Remark 1.15. As remarked before (cf. Remark 1.9), it is not essential to have all the block sizes to be equal in the above definition as long as all the number and the sizes of the blocks are polynomially bounded, and we will by a slight abuse of language allow polynomially bounded number of blocks with polynomially bounded, but not necessarily equal, block sizes in what follows without further remark.

Example 1.16. We give a very natural example of a language in $\mathbf{\Sigma}_{\mathrm{C},1}^c$ (i.e. the compact version of $\mathbf{NP}_{\mathrm{C}}$). Let $k(n,d) = \binom{n+d}{d}$ and identify

$$\underbrace{\mathbb{P}_{\mathrm{C}}^{k(n,d)-1} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k(n,d)-1}}_{n+1 \text{ times}}$$

with systems of $n+1$ homogeneous polynomials in $n+1$ variables of degree d. Let

$$S_{n,d} \subset \underbrace{\mathbb{P}_{\mathrm{C}}^{k(n,d)-1} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k(n,d)-1}}_{n+1 \text{ times}}$$

be defined by

$$S_{n,d} = \{(P_1; \cdots ; P_{n+1}) \mid P_i \in \mathbb{P}_{\mathrm{C}}^{k(n,d)-1} \text{ and } \exists\, \mathbf{x} = (x_0 : \cdots : x_n) \in \mathbb{P}_{\mathrm{C}}^n \text{ with } P_1(\mathbf{x}) = \cdots = P_{n+1}(\mathbf{x}) = 0\}.$$

In other words, $S_{n,d}$ is the set of systems of $(n+1)$ homogeneous polynomial equations of degree d, which have a zero in $\mathbb{P}_{\mathrm{C}}^n$. Then it is clear from the definition of the class $\mathbf{\Sigma}_{\mathrm{C},1}^c$ that for any fixed $d > 0$,

$$\left(S_{n,d} \subset \underbrace{\mathbb{P}_{\mathrm{C}}^{k(n,d)-1} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k(n,d)-1}}_{n+1 \text{ times}} \right)_{n>0} \in \mathbf{\Sigma}_{\mathrm{C},1}^c.$$

Note that it is *not known* if for any fixed d

$$\left(S_{n,d} \subset \underbrace{\mathbb{P}_{\mathrm{C}}^{k(n,d)-1} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k(n,d)-1}}_{n+1 \text{ times}} \right)_{n>0}$$

is $\mathbf{NP}_{\mathrm{C}}$-complete, while the non-compact version of this language i.e. the language consisting of systems of polynomials having a zero in C^n (instead of $\mathbb{P}_{\mathrm{C}}^n$), has been shown to be $\mathbf{NP}_{\mathrm{C}}$-complete for $d \geq 2$ [4].

We define analogously the class $\mathbf{\Pi}_{\mathrm{C},\omega}^c$, and finally define:

Definition 1.17. The ***compact projective polynomial hierarchy*** over C is defined to be the union

$$\mathbf{PH}_{\mathrm{C}}^c \stackrel{\text{def}}{=} \bigcup_{\omega \geq 0} (\mathbf{\Sigma}_{\mathrm{C},\omega}^c \cup \mathbf{\Pi}_{\mathrm{C},\omega}^c) = \bigcup_{\omega \geq 0} \mathbf{\Sigma}_{\mathrm{C},\omega}^c = \bigcup_{\omega \geq 0} \mathbf{\Pi}_{\mathrm{C},\omega}^c.$$

Notice that the constructible subsets belonging to any language in $\mathbf{PH}_{\mathrm{C}}^c$ are all compact (in fact Zariski closed subsets of complex projective spaces).

Remark 1.18. The compact classes introduced above might be of interest in their own right. As remarked earlier it is not known whether the compact language

$$\left(S_{n,d} \subset \underbrace{\mathbb{P}_{\mathrm{C}}^{k(n,d)-1} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k(n,d)-1}}_{n+1 \text{ times}} \right)_{n>0}$$

in Example 1.16 is $\mathbf{NP}_{\mathrm{C}}$-complete. It is important to resolve this question in order to understand whether the hardness of solving polynomial systems over C is due to the non-compactness of the (affine) solution space, or due to some intrinsic algebraic reasons.

1.4.1. *Complex projective analogue of* $\#\mathbf{P}$. We now define the complex analogue of $\#\mathbf{P}$ (cf. the class $\#\mathbf{P}_{\mathrm{R}}^{\dagger}$ defined in [2] in the real case).

We first need a notation.

Notation 1.19 (Poincaré polynomial)**.** In case $\mathrm{C} = \mathbb{C}$, for any constructible subset $S \subset \mathbb{P}_{\mathrm{C}}^{k}$ we denote by $b_i(S)$ the i-th Betti number (that is the rank of the singular homology group $\mathrm{H}_i(S) \stackrel{\text{def}}{=} \mathrm{H}_i(S, \mathbb{Q})$) of S.

We also let $P_S \in \mathbb{Z}[T]$ denote the ***Poincaré polynomial*** of S, namely

$$P_S(T) \stackrel{\text{def}}{=} \sum_{i \geq 0} b_i(S)\, T^i. \tag{1.4}$$

Remark 1.20. Since we are only going to be concerned with the Betti numbers of constructible sets, we do not lose any information by considering homology groups with coefficients in $\mathbb{Q}$ rather than in $\mathbb{Z}$, noting that

$$\mathrm{H}_i(S, \mathbb{Q}) = \mathrm{H}_i(S, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Note also that in this case by the universal coefficient theorem for cohomology [26], we have that the cohomology groups

$$\mathrm{H}^i(S) \stackrel{\text{def}}{=} \mathrm{H}^i(S, \mathbb{Q}) \cong \mathrm{Hom}(\mathrm{H}_i(S, \mathbb{Q}), \mathbb{Q}).$$

Remark 1.21. Over an arbitrary algebraically closed field C of characteristic 0, ordinary singular homology is not well defined. We use a modified homology theory (which agrees with singular homology in case $\mathrm{C} = \mathbb{C}$ and which is homotopy invariant) as done in [1] in case of semi-algebraic sets over arbitrary real closed fields (see [1], page 279). Note that by taking real and imaginary parts, every constructible set over C is a semi-algebraic set over an appropriate real closed subfield – namely, a maximal real subfield of C.

For the rest of the paper we will assume $\mathrm{C} = \mathbb{C}$, noting that all the results generalize to arbitrary algebraically closed fields of characteristic 0 using the transfer principle.

Definition 1.22 (The class $\#\mathbf{P}_{\mathrm{C}}^{\dagger}$)**.** We say a sequence of constructible functions

$$(f_n : \mathbb{P}_{\mathrm{C}}^{n} \to \mathbb{Z}[T])_{n>0}$$

is in the class $\#\mathbf{P}_{\mathrm{C}}^{\dagger}$, if there exists a sequence

$$\left(S_n \subset \mathbb{P}_{\mathrm{C}}^{n} \times \underbrace{\mathbb{P}_{\mathrm{C}}^{n} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{n}}_{n \text{ times}} \right)_{n>0} \in \mathbf{P}_{\mathrm{C}},$$

such that

$$f_n(\mathbf{x}) = P_{S_{n,\mathbf{x}}}$$

for each $\mathbf{x} \in \mathbb{P}_{\mathrm{C}}^{n}$, where $S_{n,\mathbf{x}} = S_n \cap \pi_n^{-1}(\mathbf{x})$ and

$$\pi_n : \mathbb{P}_{\mathrm{C}}^{n} \times \underbrace{\mathbb{P}_{\mathrm{C}}^{n} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{n}}_{n \text{ times}} \to \mathbb{P}_{\mathrm{C}}^{n}$$

is the projection along the last co-ordinates.

Remark 1.23. We make a few remarks about the class $\#\mathbf{P}_{\mathrm{C}}^{\dagger}$ defined above. First of all notice that the class $\#\mathbf{P}_{\mathrm{C}}^{\dagger}$ is quite robust. For instance, given two sequences $(f_n)_{n>0}, (g_n)_{n>0} \in \#\mathbf{P}_{\mathrm{C}}^{\dagger}$ it follows (by taking disjoint union of the corresponding constructible sets) that $(f_n + g_n)_{n>0} \in \#\mathbf{P}_{\mathrm{C}}^{\dagger}$, and also $(f_n g_n)_{n>0} \in \#\mathbf{P}_{\mathrm{C}}^{\dagger}$ (by taking Cartesian product of the corresponding constructible sets and using the multiplicative property of the Poincaré polynomials, which itself is a consequence of the Kunneth formula in homology theory [26].)

Remark 1.24. The connection between counting points of varieties and their Betti numbers is more direct over fields of positive characteristic via the zeta function. The zeta function of a variety defined over $\mathbb{F}_p$ is the exponential generating function of the sequence whose n-th term is the number of points in the variety over $\mathbb{F}_{p^n}$. The zeta function of such a variety turns out to be a rational function in one variable (a deep theorem of algebraic geometry first conjectured by Andre Weil [30] and proved by Dwork [13] and Deligne [11, 12]), and its numerator and denominator are products of polynomials whose degrees are the Betti numbers of the variety with respect to a certain (ℓ-adic) co-homology theory. The point of this remark is that the problems of "counting" varieties and computing their Betti numbers, are connected at a deeper level, and thus our choice of definition for a complex analogue of $\#\mathbf{P}$ is not altogether ad hoc.

Remark 1.25. A different definition of the class $\#\mathbf{P}_{\mathrm{C}}^{\dagger}$ (more in line with previous work of Bürgisser et al. [8]) would be obtained by replacing in Definition 1.22 the Poincaré polynomial, $P_S(T)$, by the Euler-Poincaré characteristic i.e. the value of P_S at $T = -1$. The Euler-Poincaré characteristic is additive (at least when restricted to complex varieties), and thus has some attributes of being a discrete analogue of volume. But at the same time it should be noted that the Euler-Poincaré characteristic is a rather weak invariant – for instance, it does not determine the number of connected components of a given variety. Also notice that in the case of finite fields referred to in Remark 1.24, all the Betti numbers, not just their alternating sum, enter (as degrees of factors) in the rational expression for the zeta function of a variety. While it would certainly be a much stronger reduction result if one could obtain a Toda-type theorem using only the Euler-Poincaré characteristic instead of the whole Poincaré polynomial, it is at present unclear if such a theorem can be proven (see also Section 5(C)).

2. Statements of the main theorems

We can now state the main result of this paper.

Theorem 2.1 (Complex analogue of Toda's theorem)**.**

$$\mathbf{PH}_{\mathrm{C}}^{c} \subset \mathbf{P}_{\mathrm{C}}^{\#\mathbf{P}_{\mathrm{C}}^{\dagger}}.$$

Remark 2.2. Note that following the usual convention $\mathbf{P}_{\mathrm{C}}^{\#\mathbf{P}_{\mathrm{C}}^{\dagger}}$ denotes the class of languages accepted by a B-S-S machine over C in polynomial time with access to an oracle which can compute functions in $\#\mathbf{P}_{\mathrm{C}}^{\dagger}$.

Remark 2.3. We leave it as an open problem to prove Theorem 2.1 with $\mathbf{PH}_{\mathrm{C}}$ instead of $\mathbf{PH}_{\mathrm{C}}^{c}$ on the left hand side. However, we also note that many theorems of complex algebraic geometry take their most satisfactory form in the case of complete varieties, which is the setting considered in this paper.

As a consequence of our method, we obtain a reduction (Theorem 2.6) that might be of independent interest. We first define the following two problems:

Definition 2.4 (Compact general decision problem with at most ω quantifier alternations ($\mathbf{GDP}^{\mathbf{c}}_{\mathrm{C},\omega}$))**.** The input and output for this problem are as follows.

- **Input.** A sentence Φ
$$(\mathrm{Q}_1\mathbf{X}^1 \in \mathbb{P}^{k_1}_{\mathrm{C}}) \cdots (\mathrm{Q}_\omega \mathbf{X}^\omega \in \mathbb{P}^{k_\omega}_{\mathrm{C}})\phi(\mathbf{X}^1;\ldots;\mathbf{X}^\omega),$$
where for each $i, 1 \leq i \leq \omega$, $\mathrm{Q}_i \in \{\exists,\forall\}$, with $\mathrm{Q}_j \neq \mathrm{Q}_{j+1}, 1 \leq j < \omega$, and ϕ is a quantifier-free multi-homogeneous formula defining a *closed* subset S of $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_\omega}$.
- **Output.** True or False depending on whether Φ is true or false.

Definition 2.5 (Computing the Poincaré polynomial of constructible sets (***Poincaré***))**.** The input and output for this problem are as follows.

- **Input.** A quantifier-free homogeneous formula defining a constructible subset $S \subset \mathbb{P}^k_{\mathrm{C}}$.
- **Output.** The Poincaré polynomial $P_S(T)$.

Theorem 2.6. *For every $\omega > 0$, there is a deterministic polynomial time reduction in the Blum-Shub-Smale model of $\mathbf{GDP}^{\mathbf{c}}_{\mathrm{C},\omega}$ to **Poincaré**.*

Remark 2.7. We remark that (in contrast to the real case) in the complex case, we are able to prove a slightly stronger result than stated above in Theorem 2.6. Our proof of Theorem 2.6 gives a polynomial time reduction of $\mathbf{GDP}^{\mathbf{c}}_{\mathrm{C},\omega}$ to the problem of computing the *pseudo-Poincaré* polynomial (defined below, see Eqn. 3.2) of constructible sets. The pseudo-Poincaré polynomial is easily computable from the Poincaré polynomial.

2.1. **Outline of the main ideas and contributions.** The basic idea behind the proof of a real analogue of Toda's theorem in [2] is a topological construction, which given a semi-algebraic set $X \subset \mathrm{R}^m \times \mathrm{R}^n$, $p \geq 0$, and $\mathrm{pr}_1 : \mathrm{R}^m \times \mathrm{R}^n \subset \mathrm{R}^n$ the projection on R^m constructs *efficiently* a semi-algebraic set, $D^p(X)$, such that

$$b_i(\mathrm{pr}_1(X)) = b_i(D^p(X)), \quad 0 \leq i < p. \tag{2.1}$$

Moreover, membership in $D^p(X)$ can be tested efficiently if the same is true for X. Note that this last property will not hold in general for the set $\mathrm{pr}_1(X)$ itself (unless of course $\mathbf{P}_{\mathrm{R}} = \mathbf{NP}_{\mathrm{R}}$).

The topological construction used in the definition of $D^p(X)$ in [2] is the iterated fibered join, $J^p_{\mathrm{pr}_1}(X)$, of a semi-algebraic set X with itself over a projection map pr_1. There is also an induced surjective map $J^p_{\mathrm{pr}_1}(X) \to \mathrm{pr}_1(X)$ which we denote by $\mathrm{pr}_1^{(p)}$. The fibers of this induced map $\mathrm{pr}_1^{(p)} : J^p_{\mathrm{pr}_1}(X) \to \mathrm{pr}_1(X)$, over a point $\mathbf{x} \in \mathrm{pr}_1(X)$, are then ordinary $(p+1)$-fold joins of the fiber $(\mathrm{pr}_1^{(p)})^{-1}(\mathbf{x})$, and by connectivity properties of the join are p-connected. It is now possible to prove using a version of the Vietoris-Beagle theorem that the map $\mathrm{pr}_1^{(p)}$ is a p-equivalence (see [2] for the precise definition of p-equivalence). The main construction in [2] was to realize efficiently the fibered join $J^p_{\mathrm{pr}_1}(X)$ up to homotopy by a semi-algebraic set. This construction however is semi-algebraic in nature, i.e. it uses real inequalities in an essential way and thus does not generalize in a straightforward way to the complex case. Thus, a different construction is needed in the complex case.

In the complex case, the role of the fibered join is played by the *complex join fibered over a projection* $\mathrm{pr}_1 : \mathrm{C}^m \times \mathrm{C}^n \to \mathrm{C}^m$ defined below (see Definition 3.21). The fibers of the $(p+1)$-fold complex join fibered over a projection pr_1, $J^p_{\mathrm{C}}(X)$, of a compact constructible set X are not quite p-connected as in the real case, but are reasonably nice – namely they are homologically equivalent to a projective space of dimension p (see Proposition 3.16). This allows us to relate the Poincaré polynomial of X with that of its image $\mathrm{pr}_1(X)$, even though the relation is not as straightforward as in the real case (see Theorem 3.23 below).

We remark that Theorem 3.23 can be used to express directly the Betti numbers of the image under projection of a projective variety in terms of those another projective variety obtained directly without having to perform effective quantifier elimination (which has exponential complexity). The description of this second variety is *much simpler and algebraic* in nature compared to the one used in [2] in the real semi-algebraic case, and thus might be of independent interest. Theorem 3.23 can also be viewed as an improvement over the descent spectral sequence argument used in [14] to bound the Betti numbers of projections (of semi-algebraic sets) in the complex projective case. A similar construction using the projective join is also available in the real case (using $\mathbb{Z}/2\mathbb{Z}$ coefficients) but we omit its description in the current paper.

Finally, we believe that the compact projective versions of the complex complexity classes introduced in this paper deserve further investigations on their own (see Remark 1.18 below), since many numerical algorithms for computing solutions of complex polynomial systems assume some form of compactness (see, for instance, [25, 3, 9]).

The rest of the paper is organized as follows. In Section 3 we state and prove the necessary ingredients from algebraic topology needed to prove the main theorems. In Section 4 we prove the main results of the paper. Finally, in Section 5, we pose some open problems and discuss possible extensions to the current work.

3. Topological Ingredients

In this section we state and prove the main topological ingredients necessary for the proof of the main theorems.

3.1. Alexander-Lefschetz duality.

We will need the classical Alexander-Lefschetz duality theorem in order to relate the Betti numbers of a closed constructible subset $S \subset \mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}}$ with those of its complement $\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}} \setminus S$.

Theorem 3.1 (Alexander-Lefschetz duality). *Let $S \subset \mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}}$ be a closed constructible subset. Then for each odd i, $1 \leq i \leq 2k+1$ with $k = k_1 + \cdots + k_\ell$, we have that*

(3.1)
$$b_{i-1}(S) - b_{i-2}(S) = b_{2k-i}(\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}} - S) - b_{2k-i+1}(\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}} - S) + b_{i-1}(\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}}).$$

Proof. Lefshetz duality theorem [26] gives for each $i, 0 \leq i \leq 2k$,

$$b_i(\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}} - S) = b_{2k-i}(\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}}, S).$$

The theorem now follows from the long exact sequence of homology,

$$\cdots \to \mathrm{H}_i(S) \to \mathrm{H}_i(\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}}) \to \mathrm{H}_i(\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}}, S) \to \mathrm{H}_{i-1}(S) \to \cdots$$

after noting that $\mathrm{H}_i(\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}}) = 0$, for all $i \neq 0, 2, 4, \ldots, 2k$. $\square$

For technical reasons (see Corollary 3.4 below) we need to consider the even and odd parts of the Poincaré polynomial of constructible sets.

Given $P = \sum_{i \geq 0} a_i T^i \in \mathbb{Z}[T]$, we write

$$P \stackrel{\text{def}}{=} P^{\text{even}}(T^2) + T P^{\text{odd}}(T^2),$$

where

$$P^{\text{even}}(T) = \sum_{i \geq 0} a_{2i} T^i,$$

and

$$P^{\text{odd}}(T) = \sum_{i \geq 0} a_{2i+1} T^i.$$

We introduce for any $S \subset \mathbb{P}^n_{\mathrm{C}}$, a related polynomial, $Q_S(T)$, which we call the ***pseudo-Poincaré polynomial*** of S defined as follows.

$$Q_S(T) \stackrel{\text{def}}{=} \sum_{j \geq 0} (b_{2j}(S) - b_{2j-1}(S)) T^j. \tag{3.2}$$

In other words,

$$Q_S = P_S^{\text{even}} - T P_S^{\text{odd}}. \tag{3.3}$$

We introduce below notation for several operators on polynomials that we will use later.

Notation 3.2 (Operators on polynomials)**.** For any polynomial $Q = \sum_{i \geq 0} a_i T^i \in \mathbb{Z}[T]$ with $\deg(Q) \leq n$, we will denote by:

(A) $\mathrm{Rec}_n(Q)$ the polynomial $T^n Q(\frac{1}{T})$;
(B) for $0 \leq m \leq n$, $\mathrm{Trunc}_m(Q)$ the polynomial $\sum_{0 \leq i \leq m} a_i T^i \in \mathbb{Z}[T]$; and,
(C) $M_P(Q)$ the polynomial PQ, for any polynomial $P \in \mathbb{Z}[T]$.

Remark 3.3. Notice that all the operators introduced above are computable in polynomial time.

Using the notation introduced above we have the following easy corollary of Theorem 3.1.

Corollary 3.4. *Let $S \subset \mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}}$ be any closed (resp. open) constructible subset, and $k = k_1 + \cdots + k_\ell$. Then,*

$$Q_S = Q_{\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}}} - \mathrm{Rec}_k(Q_{\mathbb{P}^{k_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{k_\ell}_{\mathrm{C}} - S}).$$

3.2. **The complex join of subsets of complex projective spaces.** We first give a purely geometric definition of the complex join of two sets followed by one using co-ordinates. The geometric definition is useful in understanding the topological properties of the join proved later. The definition involving co-ordinates and formulas is necessary for the complexity theoretic arguments.

Let V, W be finite dimensional vector spaces over C and let $X \subset \mathbb{P}(V)$ and $Y \subset \mathbb{P}(W)$ be two arbitrary (not necessarily constructible) subsets. Note that $\mathbb{P}(V) \cong \mathbb{P}(V \oplus \mathbf{0}) \subset \mathbb{P}(V \oplus W)$ and $\mathbb{P}(W) \cong \mathbb{P}(\mathbf{0} \oplus W) \subset \mathbb{P}(V \oplus W)$ are two disjoint subspaces of $\mathbb{P}(V \oplus W)$ and thus X and Y are embedded as disjoint subsets of $\mathbb{P}(V \oplus W)$. With the above notation

Definition 3.5 (Geometric definition of complex join)**.** The ***complex join***, $J_{\mathrm{C}}(X,Y)$, is defined to be the union of projective lines in $\mathbb{P}(V \oplus W)$ which meets both X and Y if X and Y are both non-empty. We let $J_{\mathrm{C}}(X,Y) = Y$ if X is empty, and $J_{\mathrm{C}}(X,Y) = X$ if Y is empty.

We now give a definition of the complex join which involve co-ordinates which we are going to use in this paper.

Let $X \subset \mathbb{P}^k_{\mathrm{C}}$ and $Y \subset \mathbb{P}^\ell_{\mathrm{C}}$ be two constructible sets defined by homogeneous formulas $\Phi(X_0,\ldots,X_k)$ and $\Psi(Y_0,\ldots,Y_\ell)$ respectively, where $(X_0 : \cdots : X_k)$ (respectively $(Y_0 : \cdots : Y_\ell)$) are homogeneous co-ordinates in $\mathbb{P}^k_{\mathrm{C}}$ (respectively $\mathbb{P}^\ell_{\mathrm{C}}$).

Definition 3.6 (Complex join in terms of co-ordinates)**.** The complex join, $J_{\mathrm{C}}(X,Y)$, of X and Y is the constructible subset of $\mathbb{P}^{k+\ell+1}_{\mathrm{C}}$ defined by the formula

$$J_{\mathrm{C}}(\Phi,\Psi) \stackrel{def}{=} \phi(Z_0,\cdots,Z_k) \wedge \psi(Z_{k+1},\cdots,Z_{k+\ell+1}),$$

where $(Z_0 : \cdots : Z_{k+\ell+1})$ are homogeneous coordinates in $\mathbb{P}^{k+\ell+1}_{\mathrm{C}}$.

Remark 3.7. Firstly, notice that the realization, $\mathcal{R}(J_{\mathrm{C}}(\Phi,\Psi))$, does not depend on the formulas ϕ and ψ used to define X and Y respectively. Also, notice that if X and Y are *both* empty then so is $J_{\mathrm{C}}(X,Y)$. Indeed, if $X = \emptyset$ (respectively, $Y = \emptyset$) then $J_{\mathrm{C}}(X,Y)$ is isomorphic to Y (respectively, X). To see this notice that by definition (cf. Definition 1.3) the homogeneous formula $\Phi(\mathbf{X})$ (resp. $\Psi(\mathbf{Y})$) is true whenever $\mathbf{X}$ (resp. $\mathbf{Y}$) is the $\mathbf{0}$-vector. Now consider the following two constructible subsets of $\mathbb{P}^{k+\ell+1}_{\mathrm{C}}$.

$$\tilde{X} = \{(\mathbf{x} : 0 : \cdots : 0) \mid \mathbf{x} \in X\},$$
$$\tilde{Y} = \{(0 : \cdots : 0 : \mathbf{y}) \mid \mathbf{y} \in Y\}.$$

We have that $\tilde{X}$ (resp. $\tilde{Y}$) is isomorphic to X (resp. Y). Moreover $\tilde{X}$ and $\tilde{Y}$ are contained in $J_{\mathrm{C}}(X,Y)$, since as remarked earlier $\Psi(\mathbf{Y})$ (resp. $\Phi(\mathbf{X})$) is true whenever $\mathbf{Y}$ (resp. $\mathbf{X}$) is the 0-vector. Moreover, $\tilde{X}$ (resp. $\tilde{Y}$) is equal to $J_{\mathrm{C}}(X,Y)$ in case Y (resp. X) is empty.

Also, clearly $\tilde{X}$ and $\tilde{Y}$ are disjoint, and if X and Y are both non-empty then, $J_{\mathrm{C}}(X,Y)$ is obtained by taking the union of projective lines in $\mathbb{P}^{k+\ell+1}_{\mathrm{C}}$ meeting both $\tilde{X}$ and $\tilde{Y}$.

Example 3.8. It is easy to check from the above definition that the join, $J_{\mathrm{C}}(\mathbb{P}^k_{\mathrm{C}},\mathbb{P}^\ell_{\mathrm{C}})$, of two projective spaces is again a projective space, namely $\mathbb{P}^{k+\ell+1}_{\mathrm{C}}$.

Remark 3.9. The projective join as defined above is a classical object in algebraic geometry. Amongst many other applications, the complex suspension of a projective variety X (i.e. the complex join $J_{\mathrm{C}}(X,\mathbb{P}^0_{\mathrm{C}})$) plays an important role in defining Lawson homology of projective varieties [17]. Within the area of computational complexity theory, the projective join of a variety with a point was used in [22] for proving hardness of the problem of computing Betti numbers of complex varieties.

Definition 3.10. For $p \geq 0$, we denote by $J^p_{\mathrm{C}}(X)$ the $(p+1)$-fold iterated complex join of X with itself.

More precisely,

$$J^0_{\mathrm{C}}(X) := X,$$
$$J^{p+1}_{\mathrm{C}}(X) := J_{\mathrm{C}}(J^p_{\mathrm{C}}(X),X), \text{ for } p \geq 1.$$

If $X \subset \mathbb{P}^k_{\mathrm{C}}$ is defined by a first-order homogeneous formula $\Phi(X_0, \ldots, X_k)$, then $J^p_{\mathrm{C}}(X) \subset \mathbb{P}^{(p+1)(k+1)-1}_{\mathrm{C}}$ is defined by the homogeneous formula

$$J^p_{\mathrm{C}}(\Phi)(X^0_0, \ldots, X^0_k, \ldots, X^p_0, \ldots, X^p_k) \stackrel{\text{def}}{=} \bigwedge_{i=0}^{p} \phi(X^i_0, \ldots, X^i_k).$$

where $(X^0_0 : \cdots : X^p_k)$ are homogeneous co-ordinates in $\mathbb{P}^{(p+1)(k+1)-1}_{\mathrm{C}}$.

Note that by Remark 3.7, if X is empty then $J^p_{\mathrm{C}}(X)$ is empty for every $p \geq 0$.

3.3. **Properties of the topological join.**

We also need to introduce the ***topological join*** of two spaces. The following is classical.

Definition 3.11. The ***join***, $X * Y$, of two topological spaces X and Y is defined by

$$X * Y \stackrel{\text{def}}{=} X \times Y \times \Delta^1 / \sim, \tag{3.4}$$

where $\Delta^1 = \{(t_0, t_1) \mid t_0, t_1 \geq 0, t_0 + t_1 = 1\}$ denotes the standard geometric realization of the 1-dimensional simplex, and

$$(x, y, t_0, t_1) \sim (x', y', t_0, t_1)$$

if and only if $t_0 = 1, x = x'$ or $t_1 = 1, y = y'$.

Intuitively, $X * Y$ is obtained by joining each point of X with each point of Y by an interval.

We will need the well-known fact that the iterated join of a topological space is highly connected. In order to make this statement precise we first define

Definition 3.12 (p-equivalence)**.** A map $f : A \to B$ between two topological spaces is called a ***p-equivalence*** if the induced homomorphism

$$f_* : \mathrm{H}_i(A) \to \mathrm{H}_i(B)$$

is an isomorphism for all $0 \leq i < p$, and an epimorphism for $i = p$, and we say that A is ***p-equivalent*** to B.

The following is well known. (see, for instance, [18, Proposition 4.4.3]).

Theorem 3.13. *Let X be a non-empty compact semi-algebraic set. Then, the $(p+1)$-fold join $\underbrace{X * \cdots * X}_{(p+1)\ \text{times}}$ is p-equivalent to a point.*

We will need a particular property of projection maps that we are going to consider later in the paper.

Notation 3.14. For any constructible set A, we denote by $\mathrm{K}(A)$ the collection of all compact (in the Euclidean topology) subsets of A.

Definition 3.15. Let $f : A \to B$ be a map between two constructible sets A and B. We say that f ***compact covering*** if for any $L \in \mathrm{K}(f(A))$, there exists $K \in \mathrm{K}(A)$ such that $f(K) = L$.

3.4. **Topological properties of the complex join.**

Proposition 3.16. *Let $X \subset \mathbb{P}^k_{\mathrm{C}}$ be a non-empty semi-algebraic subset and $p > 0$. Let*

$$\iota : J^p_{\mathrm{C}}(X) \hookrightarrow \mathbb{P}^{(p+1)(k+1)-1}_{\mathrm{C}}$$

denote the inclusion map. Then the induced homomorphisms

$$\iota_* : \mathrm{H}_j(J^p_{\mathrm{C}}(X)) \to \mathrm{H}_j(\mathbb{P}^{(p+1)(k+1)-1}_{\mathrm{C}})$$
$$\iota^* : \mathrm{H}^j(\mathbb{P}^{(p+1)(k+1)-1}_{\mathrm{C}}) \to \mathrm{H}^j(J^p_{\mathrm{C}}(X))$$

are isomorphisms for $0 \leq j < p$.

Before proving Proposition 3.16 we first fix some notation.

Notation 3.17 (Hopf fibration)**.** For any $k \geq 0$, we will denote by $\pi : \mathrm{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k_{\mathrm{C}}$ the tautological line bundle over $\mathbb{P}^k_{\mathrm{C}}$, and by

$$\tilde{\pi} : \mathbf{S}^{2k+1} \to \mathbb{P}^k_{\mathrm{C}},$$

the ***Hopf fibration***, namely the restriction of π to the unit sphere in C^{k+1} defined by the equation $|z_1|^2 + \cdots + |z_{k+1}|^2 = 1$. Finally for any subset $S \subset \mathbb{P}^k_{\mathrm{C}}$, we will denote by $\widetilde{S}$ the subset $\tilde{\pi}^{-1}(S) \subset \mathbf{S}^{2k+1}$. Restricting the map $\tilde{\pi}$ to $\widetilde{S}$ we obtain the restriction of the Hopf fibration to the base S i.e. we have the following commutative diagram.

$$\begin{tikzcd}
\widetilde{S} \arrow[r, hook, "\iota"] \arrow[d, "\tilde{\pi}"] & \mathbf{S}^{2k+1} \arrow[d, "\tilde{\pi}"] \\
S \arrow[r, hook, "\iota"] & \mathbb{P}^k_{\mathrm{C}}
\end{tikzcd}$$

We need the following lemma.

Lemma 3.18. *Let $X \subset \mathbb{P}^k_{\mathrm{C}}, Y \subset \mathbb{P}^\ell_{\mathrm{C}}$ be semi-algebraic subsets. Then $\widetilde{J_{\mathrm{C}}(X,Y)} \subset \mathbf{S}^{2(k+\ell)+3}$ is homeomorphic to the (topological) join $\widetilde{X} * \widetilde{Y}$.*

Proof. Consider $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ and the projective line $L \subset J_{\mathrm{C}}(X,Y)$ joining $\mathbf{x}$ and $\mathbf{y}$. It is easy to see that the preimage $\tilde{L} = \tilde{\pi}^{-1}(L) \cong \mathbf{S}^3$ is a topological join of $\tilde{\pi}^{-1}(\mathbf{x})$ and $\tilde{\pi}^{-1}(\mathbf{y})$ (each homeomorphic to $\mathbf{S}^1$). Now since $\tilde{X}$ (resp. $\tilde{Y}$) is fibered by the various $\tilde{\pi}^{-1}(\mathbf{x})$ (resp. $\tilde{\pi}^{-1}(\mathbf{y})$), it follows that $\widetilde{J_{\mathrm{C}}(X,Y)}$ is homeomorphic to $\widetilde{X} * \widetilde{Y}$. $\square$

Proof of Proposition 3.16. We first treat the cases $p = 1, 2$.

$p = 1$: It is an easy exercise to show that the join, $J^1_{\mathrm{C}}(X) = J_{\mathrm{C}}(X,X)$ is non-empty and connected whenever X is non-empty. This proves the proposition in this case.

$p = 2$: It is easy to see that $J^2_{\mathrm{C}}(X) = J_{\mathrm{C}}(J^1_{\mathrm{C}}(X), X)$ is non-empty and connected, whenever X is non-empty. It is only a slightly more difficult exercise to prove that $\mathrm{H}_1(J^2_{\mathrm{C}}(X))$ (and in fact, $\mathrm{H}_1(J^p_{\mathrm{C}}(X))$ for all $p > 1$) vanishes. This follows from the statement that $\mathrm{H}_1(J_{\mathrm{C}}(Y,Z)) = 0$ whenever Y is connected, since $J^p_{\mathrm{C}}(X) = J_{\mathrm{C}}(J^{p-1}_{\mathrm{C}}(X), X)$ and we have that $J^{p-1}_{\mathrm{C}}(X)$ is connected for $p > 1$. Proving that $\mathrm{H}_1(J_{\mathrm{C}}(Y,Z)) = 0$ whenever Y is connected, after an application of Mayer-Vietoris exact sequence, reduces to proving that

$\mathrm{H}_1(J_{\mathrm{C}}(Y,Z)) = 0$ whenever both Y and Z are connected. This can be checked by a direct calculation using the fact that the topological join $Y * Z$ is simply connected whenever Y, Z are both connected. Note that, this also proves that $J^p_{\mathrm{C}}(X)$ is simply connected for all $p > 1$.

Now let $p \geq 2$. It follows from repeated applications of Lemma 3.18 that $\widetilde{J^p_{\mathrm{C}}(X)}$ is homeomorphic to

$$\underbrace{\widetilde{X} * \cdots * \widetilde{X}}_{(p+1) \text{ times}}.$$

We also have the commutative square

$$\begin{tikzcd}
\widetilde{J^p_{\mathrm{C}}(X)} \arrow[r, hook, "i"] \arrow[d, "\tilde{\pi}"] & \mathbf{S}^{2(p+1)(k+1)-1} \arrow[d, "\tilde{\pi}"] \\
J^p_{\mathrm{C}}(X) \arrow[r, hook, "i"] & \mathbb{P}_{\mathrm{C}}^{(p+1)(k+1)-1}
\end{tikzcd}$$

and a corresponding square

$$\begin{tikzcd}
\mathrm{H}_*(\widetilde{J^p_{\mathrm{C}}(X)}) \arrow[r, "\iota_*"] \arrow[d, "\tilde{\pi}_*"] & \mathrm{H}_*(\mathbf{S}^{2(p+1)(k+1)-1}) \arrow[d, "\tilde{\pi}_*"] \\
\mathrm{H}_*(J^p_{\mathrm{C}}(X)) \arrow[r, "\iota_*"] & \mathrm{H}_*(\mathbb{P}_{\mathrm{C}}^{(p+1)(k+1)-1})
\end{tikzcd}$$

of induced homomorphisms in the homology groups.

It follows from Theorem 3.13 that if $X \neq \emptyset$, then

$$\mathrm{H}_0(\widetilde{J^p_{\mathrm{C}}(X)}) \cong \mathbb{Q},$$
$$\mathrm{H}_i(\widetilde{J^p_{\mathrm{C}}(X)}) \cong 0, \quad 0 < i < p.$$

Since, for $p > 1$, $J^p_{\mathrm{C}}(X)$ is simply connected (see above) $\widetilde{J^p_{\mathrm{C}}(X)}$ is a simple $\mathbf{S}^1$-bundle (i.e. a $\mathbf{S}^1$-bundle with a simply connected base) over $J^p_{\mathrm{C}}(X)$.

It now follows by a standard argument (which we expand below) involving the spectral sequence of the bundle $\tilde{\pi} : \widetilde{J^p_{\mathrm{C}}(X)} \to J^p_{\mathrm{C}}(X)$, that for $0 \leq i < p$,

$$\begin{aligned}
\mathrm{H}_i(J^p_{\mathrm{C}}(X)) &\cong \mathbb{Q}, \text{ for } i \text{ even}, \\
\mathrm{H}_i(J^p_{\mathrm{C}}(X)) &\cong 0 \text{ for } i \text{ odd}.
\end{aligned} \tag{3.5}$$

(The above claim also follows from the Gysin sequence of the $\mathbf{S}^1$-bundle $\tilde{\pi}$: $\widetilde{J^p_{\mathrm{C}}(X)} \to J^p_{\mathrm{C}}(X)$ but we give an independent proof below).

Consider the E^2-term of the (homological) spectral sequence of the bundle

$$\tilde{\pi} : \widetilde{J^p_{\mathrm{C}}(X)} \to J^p_{\mathrm{C}}(X).$$

For $i, j \geq 0$, we have that

$$E^2_{i,j} = \mathrm{H}_i(J^p_{\mathrm{C}}(X)) \otimes \mathrm{H}_j(\mathbf{S}^1).$$

From this we deduce that

$$E^2_{i,0} = E^2_{i,1} = \mathrm{H}_i(J^p_{\mathrm{C}}(X)).$$

Also, from the fact that

$$\mathrm{H}_0(\widetilde{J^p_{\mathrm{C}}(X)}) = \mathbb{Q},$$

we get that

$$E^2_{0,0} = \mathbb{Q},$$

and hence,

$$E^2_{0,1} = \mathbb{Q}$$

as well. Moreover, we have that

$$E^3_{i,j} = E^4_{i,j} = \cdots = E^\infty_{i,j}$$

for all $i \geq 0$ and $j = 0, 1$. Now from the fact that the spectral sequence E^r converges to the homology of $\widetilde{J^p_{\mathrm{C}}(X)}$ we deduce that

$$\begin{aligned} E^3_{i,j} &= 0 \text{ for } 0 \leq i \leq p-1 \text{ and all } j, \\ E^3_{0,0} &= \mathbb{Q}. \end{aligned}$$

This implies that the differential

$$d_2 : E^2_{i,0} \to E^2_{i-2,1}$$

is an isomorphism for $1 \leq i \leq p-1$. Together with the fact that

$$E^2_{i,0} = E^2_{i,1} = \mathrm{H}_i(J^p_{\mathrm{C}}(X)),$$

this immediately implies (3.5). The claim that ι_* is an isomorphism follows directly from the above. The dual statement about ι^* follows immediately from the universal coefficient theorem for cohomology (see e.g. [26]). $\square$

In our application we will need the following (rather technical) generalization of Proposition 3.16. Let $p, \alpha_0, \ldots, \alpha_\omega \geq 0$, $N = \prod_{0 \leq j \leq \omega}(\alpha_j + 1)$. Let I denote the set of tuples $(i_0, \ldots, i_\omega)$ with $0 \leq i_j \leq \alpha_j, 0 \leq j \leq \omega$, and for each tuple $(i_0, \ldots, i_\omega) \in I$, let $\pi_{(i_0,\ldots,i_\omega)}$ denote the projection

$$\bigtimes_{(j_0,\ldots,j_\omega)\in I} \mathbb{P}^k_{\mathrm{C}} \longrightarrow \mathbb{P}^k_{\mathrm{C}}$$

defined by

$$(\mathbf{x}_{(j_0,\ldots,j_\omega)})_{(j_0,\ldots,j_\omega)\in I} \mapsto \mathbf{x}_{(i_0,\ldots,i_\omega)},$$

and for any subset $X \subset \mathbb{P}^k_{\mathrm{C}}$ we denote

$$X^{(i_0,\ldots,i_\omega)} = \pi^{-1}_{(i_0,\ldots,i_\omega)}(X).$$

Proposition 3.19. *Let $X \subset \mathbb{P}^k_{\mathrm{C}}$ be a semi-algebraic subset. Also, let for each $i, 0 \leq i \leq \omega$, $\Lambda^i \in \{\bigcap, \bigcup\}$, and let $S \subset \bigtimes_{(j_0,\ldots,j_\omega)\in I} \mathbb{P}^k_{\mathrm{C}}$ denote the semi-algebraic subset*

$$S \stackrel{def}{=} \Lambda^0_{0\leq i_0\leq\alpha_0} \cdots \Lambda^\omega_{0\leq i_\omega\leq\alpha_\omega} (J^p_{\mathrm{C}}(X))^{(i_0,\ldots,i_\omega)},$$

with

$$\iota : S \hookrightarrow \bigtimes_{(j_0,\ldots,j_\omega)\in I} \mathbb{P}^{(p+1)(k+1)-1}_{\mathrm{C}}$$

denoting the inclusion map. Then, the induced homomorphisms

$$\iota_* : \mathrm{H}_j(S) \to \mathrm{H}_j\Big(\bigtimes_{(j_0,\ldots,j_\omega)\in I} \mathbb{P}_{\mathrm{C}}^{(p+1)(k+1)-1}\Big)$$
$$\iota^* : \mathrm{H}^j\Big(\bigtimes_{(j_0,\ldots,j_\omega)\in I} \mathbb{P}_{\mathrm{C}}^{(p+1)(k+1)-1}\Big) \to \mathrm{H}^j(S)$$

are isomorphisms for $0 \le j < p$.

Proof. Notice that, if $\omega = 0$ and $\Lambda^0 = \bigcap$, then

$$\bigcap_{0 \le i_0 \le \alpha_0} J_{\mathrm{C}}^p(X)^{(i_0)} = \bigtimes_{(j_0,\ldots,j_\omega)\in I} J_{\mathrm{C}}^p(X),$$

and the claim follows in this case from Proposition 3.16 and the Kunneth formula.

If $\omega = 0$ and $\Lambda^0 = \bigcup$, the claim follows from the previous case and a standard argument using the Mayer-Vietoris double complex.

The general case is easily proved using induction on ω. □

3.5. **Complex join fibered over a projection and its properties.** In our application we need the complex join fibered over certain projections. We first give a geometric definition followed by one involving co-ordinates.

Let V, W be finite dimensional C-vector spaces and $A \subset \mathbb{P}(V) \times \mathbb{P}(W)$ a subset. Let $\mathrm{pr}_1 : \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V)$ denote the projection on the first component. Then, for $p \ge 0$, the ***p-fold complex join of A fibered over the projection*** pr_1 is defined by

Definition 3.20 (Geometric definition of complex join fibered over a projection)**.**

$$J_{\mathrm{C},\mathrm{pr}_1}^p(A) \quad = \quad \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \mathbb{P}_{\mathrm{C}}^k, \mathbf{y} \in J_{\mathrm{C}}^p(A_{\mathbf{x}})\},$$

were $A_{\mathbf{x}} = \mathrm{pr}_1^{-1}(\mathbf{x}) \cap A$.

We now give a definition in terms of co-ordinates.

Definition 3.21 (Complex join fibered over a projection in terms of co-ordinates)**.** Let $A \subset \mathbb{P}_{\mathrm{C}}^k \times \mathbb{P}_{\mathrm{C}}^\ell$ be a constructible set defined by a first-order multi-homogeneous formula,

$$\Phi(X_0, \ldots, X_k; Y_0, \ldots, Y_\ell)$$

and let $\mathrm{pr}_1 : \mathbb{P}_{\mathrm{C}}^k \times \mathbb{P}_{\mathrm{C}}^\ell \to \mathbb{P}_{\mathrm{C}}^k$ be the projection map to the first component. For $p \ge 0$, the p-fold complex join of A fibered over the map pr_1, $J_{\mathrm{C},\mathrm{pr}_1}^p(A) \subset \mathbb{P}_{\mathrm{C}}^k \times \mathbb{P}_{\mathrm{C}}^{(\ell+1)(p+1)-1}$, is defined by the formula

(3.6)
$$J_{\mathrm{C},\mathrm{pr}_1}^p(\Phi)(X_0, \ldots, X_k; Y_0^0, \ldots, Y_\ell^0, \ldots, Y_0^p, \ldots, Y_\ell^p) \overset{def}{=} \bigwedge_{i=0}^p \phi(X_0, \ldots, X_k; Y_0^i, \ldots, Y_\ell^i).$$

Remark 3.22. The projection map

$$\mathrm{pr}_1 : \mathbb{P}_{\mathrm{C}}^k \times \mathbb{P}_{\mathrm{C}}^{(\ell+1)(p+1)-1} \to \mathbb{P}_{\mathrm{C}}^k$$

clearly restricts to a surjection

$$\mathrm{pr}_1^{(p)} : J_{\mathrm{C},\mathrm{pr}_1}^p(A) \to \mathrm{pr}_1(A)$$

sending $(x_0 : \cdots : x_k; y_0^0 : \cdots : y_\ell^p) \in J_{\mathrm{C},\mathrm{pr}_1}^p(A)$ to $(x_0 : \cdots : x_k) \in \mathrm{pr}_1(A)$.

Now, let $A \subset \mathbb{P}^k_{\mathbb{C}} \times \mathbb{P}^\ell_{\mathbb{C}}$ be a semi-algebraic subset $\mathrm{pr}_1 : \mathbb{P}^k_{\mathbb{C}} \times \mathbb{P}^\ell_{\mathbb{C}} \to \mathbb{P}^k_{\mathbb{C}}$ be the projection on the first component.

Suppose that pr_1 restricted to A is a compact covering. The following theorem relates the Poincaré polynomial of $J^p_{\mathbb{C},\mathrm{pr}_1}(A)$ to that of the image $\mathrm{pr}_1(A)$.

Theorem 3.23. *For every $p \geq 0$, we have that*

$$P_{\mathrm{pr}_1(A)} \quad = \quad (1-T^2) P_{J^p_{\mathbb{C},\mathrm{pr}_1}(A)} \quad \mathrm{mod}\ T^p. \tag{3.7}$$

Remark 3.24. Note that the compact covering property is crucial for Theorem 3.23. to hold. In our applications, pr_1 is going to be either an open or a closed map, and will thus automatically have the compact covering property.

Proof. We first assume that A is semi-algebraic and compact, and let B denote $\mathrm{pr}_1(A) \times \mathbb{P}^{(p+1)(\ell+1)-1}_{\mathbb{C}}$. We have the following commutative square.

$$\begin{tikzcd}
J^p_{\mathbb{C},\mathrm{pr}_1}(A) \arrow[r, hook, "i"] \arrow[d, "\mathrm{pr}_1^{(p)}"] & B \arrow[d, "\mathrm{pr}_1"] \\
\mathrm{pr}_1(A) \arrow[r, "\mathrm{Id}"] & \mathrm{pr}_1(A)
\end{tikzcd}$$

The diagram above induces a morphism, $\phi_r^{i,j} : E_r^{i,j} \to {}'E_r^{i,j}$ between the Leray-Serre spectral sequences of the two vertical maps in the above diagram. Here, E_r (resp. ${}'E_r$) denotes the Leray-Serre spectral sequence of the map $\mathrm{pr}_1 : B \to \mathrm{pr}_1(A)$ (resp. $\mathrm{pr}_1^{(p)} : J^p_{\mathbb{C},\mathrm{pr}_1}(A) \to \mathrm{pr}_1(A)$). The spectral sequence, E_r, degenerates at the E_2-term where

$$E_2^{i,j} = \mathrm{H}^i(\mathrm{pr}_1(A), R^j \mathrm{pr}_{1*} \mathbb{Q}_B),$$

and $\mathbb{Q}_B$ denotes the constant sheaf with stalk $\mathbb{Q}$ on B, and $R^* \mathrm{pr}_{1*}$ denotes the higher direct image functor. (The above formulation of Leray-Serre spectral sequence of a map is standard; we refer the reader to [15, Théorème 4.17.1] for a purely sheaf theoretic statement without reference to higher derived images.)

Similarly we have

$${}'E_2^{i,j} = \mathrm{H}^i(\mathrm{pr}_1(A), R^j \mathrm{pr}_{1*}^{(p)} \mathbb{Q}_{J^p_{\mathbb{C},\mathrm{pr}_1}(A)}).$$

We also have that for each $\mathbf{x} \in \mathrm{pr}_1(A)$

$$(R^j \mathrm{pr}_{1*} \mathbb{Q}_B)_{\mathbf{x}} \cong \mathrm{H}^j(\mathbb{P}^{(p+1)(\ell+1)-1}_{\mathbb{C}}) \cong \mathrm{H}^j((\mathrm{pr}_1^{(p)})^{-1}(\mathbf{x})) \cong (R^j \mathrm{pr}_{1*}^{(p)} \mathbb{Q}_{J^p_{\mathbb{C},\mathrm{pr}_1}(A)})_{\mathbf{x}},$$

where the first and the last isomorphisms are consequences of the proper base change theorem (see [15, Remarque 4.17.1]) noting that $\mathrm{pr}_1, \mathrm{pr}_1^{(p)}$ are both proper maps, and the middle one is a consequence of Proposition 3.16.

It follows that the sheaves $R^j \mathrm{pr}_1 \mathbb{Q}_B$ and $R^j \mathrm{pr}_1^{(p)} \mathbb{Q}_{J^p_{\mathbb{C},\mathrm{pr}_1}(A)}$ are isomorphic by the sheaf map induced by the inclusions

$$(\mathrm{pr}_1^{(p)})^{-1}(\mathbf{x}) \hookrightarrow \{\mathbf{x}\} \times \mathbb{P}^{(p+1)(\ell+1)-1}_{\mathbb{C}}, \mathbf{x} \in \mathrm{pr}_1(A)$$

and hence,

$$\phi_2^{i,j} : E_2^{i,j} \to {}'E_2^{i,j}$$

are isomorphisms for $i + j < p$.

It now follows from a general result about spectral sequences (see [19, page. 66]) that $E_\infty^{i,j} \cong {}'E_\infty^{i,j}$ for $0 \leq i+j < p$. This implies that $\mathrm{H}^q(J^p_{\mathrm{C},\mathrm{pr}_1}(A)) \cong \mathrm{H}^q(\mathrm{pr}_1(A) \times \mathbb{P}_{\mathrm{C}}^{(p+1)(\ell+1)-1})$ for $0 \leq q < p$, and thus

$$P_{J^p_{\mathrm{C},\mathrm{pr}_1}(A)} = P_{\mathrm{pr}_1(A)\times\mathbb{P}_{\mathrm{C}}^{(p+1)(\ell+1)-1}} \mod T^p. \tag{3.8}$$

We also have that

$$\begin{aligned} P_{\mathrm{pr}_1(A)\times\mathbb{P}_{\mathrm{C}}^{(p+1)(\ell+1)-1}} &= P_{\mathrm{pr}_1(A)} \times P_{\mathbb{P}_{\mathrm{C}}^{(p+1)(\ell+1)-1}} \\ &= P_{\mathrm{pr}_1(A)} \times (1 + T^2 + \cdots + T^{2((p+1)(\ell+1)-1)}) \\ &= P_{\mathrm{pr}_1(A)} \times (1-T^2)^{-1} \mod T^p. \end{aligned} \tag{3.9}$$

Equation (3.7) now follows from Equations (3.8) and (3.9).

The general case follows by taking direct limit over all compact subsets of A. More precisely, for $K_1 \subset K_2$ compact subsets of A, we have for $0 \leq q < p$ the following commutative square after switching to homology (cf. Remark 1.20). (Note that following Definition 3.20 the complex join fibered over a projection is defined for arbitrary not necessarily constructible subsets of A.)

$$\begin{tikzcd} \mathrm{H}_q(J^p_{\mathrm{C},\mathrm{pr}_1}(K_1)) \arrow[r, "\iota_*"] \arrow[d, "\cong"] & \mathrm{H}_q(J^p_{\mathrm{C},\mathrm{pr}_1}(K_2)) \arrow[d, "\cong"] \\ \mathrm{H}_q(\mathrm{pr}_1(K_1) \times \mathbb{P}_{\mathrm{C}}^{(p+1)(\ell+1)-1}) \arrow[r, "\iota_*"] & \mathrm{H}_q(\mathrm{pr}_1(K_2) \times \mathbb{P}_{\mathrm{C}}^{(p+1)(\ell+1)-1}) \end{tikzcd}$$

where the vertical maps are isomorphisms by the previous case. If we take the direct limit as K ranges in $\mathrm{K}(A)$, we obtain the following:

$$\begin{tikzcd} \varinjlim \mathrm{H}_q(J^p_{\mathrm{C},\mathrm{pr}_1}(K)) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{H}_q(J^p_{\mathrm{C},\mathrm{pr}_1}(A)) \arrow[d] \\ \varinjlim \mathrm{H}_q(\mathrm{pr}_1(K) \times \mathbb{P}_{\mathrm{C}}^{(p+1)(\ell+1)-1}) \arrow[r, "\cong"] & \mathrm{H}_q(\mathrm{pr}_1(A) \times \mathbb{P}_{\mathrm{C}}^{(p+1)(\ell+1)-1}) \end{tikzcd}$$

The isomorphism on the top level comes from the fact that homology and direct limit commute [26]. For the bottom isomorphism, we need the additional fact that since we assume that pr_1 is a compact covering we have

$$\varinjlim\{\mathrm{H}_q(\mathrm{pr}_1(K)\times\mathbb{P}_{\mathrm{C}}^{(p+1)(\ell+1)-1}) \mid K \in \mathrm{K}(A)\} = \varinjlim\{\mathrm{H}_q(L\times\mathbb{P}_{\mathrm{C}}^{(p+1)(\ell+1)-1}) \mid L \in \mathrm{K}(\mathrm{pr}_1(A))\}.$$

This proves that the right vertical arrow is also an isomorphism. □

Using the same notation as in Theorem 3.23 and Eqn. (3.2) we have the following easy corollary of Theorem 3.23.

Corollary 3.25. *Let $p = 2m+1$ with $m \geq 0$. Then*

$$Q_{\mathrm{pr}_1(A)} = (1-T)\, Q_{J^p_{\mathrm{C},\mathrm{pr}_1}(A)} \mod T^{m+1}. \tag{3.10}$$

Proof. The corollary follows directly from Theorem 3.23 and the fact that for any polynomial $P \in \mathbb{Z}[T]$ we have

$$\begin{aligned} ((1-T^2)P)^{\mathrm{even}} &= (1-T)(P)^{\mathrm{even}}, \\ ((1-T^2)P)^{\mathrm{odd}} &= (1-T)(P)^{\mathrm{odd}}. \end{aligned}$$

□

As before we need a slightly more general version of Theorem 3.23 as well as Corollary 3.25.

Let $\alpha_0, \ldots, \alpha_\sigma \geq 0$, and $N = \prod_{0 \leq j \leq \omega}(\alpha_j + 1)$. Let ϕ be a homogeneous formula defining a constructible subset of $\mathbb{P}^{k_0} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k_\sigma} \times \mathbb{P}_{\mathrm{C}}^{\ell}$. Also, let for each $i, 0 \leq i \leq \sigma$, $\Lambda^i \in \{\bigvee, \bigwedge\}$, and let Φ denote the multi-homogeneous formula defined by

$$\Phi \stackrel{\text{def}}{=} \underset{0 \leq i_0 \leq \alpha_0}{\Lambda^0} \cdots \underset{0 \leq i_\sigma \leq \alpha_\sigma}{\Lambda^\sigma} \phi(\mathbf{X}^0; \cdots; \mathbf{X}^\sigma; \mathbf{Y}_{i_0, \ldots, i_\sigma}).$$

Let

$$A = \mathcal{R}(\Phi) \subset \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k_\sigma} \times \underbrace{\mathbb{P}_{\mathrm{C}}^{\ell} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{\ell}}_{N \text{ times}},$$

and let $\mathrm{pr}_{[0,\sigma]} : \mathbb{P}_{\mathrm{C}}^{k_0} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k_\sigma} \times \underbrace{\mathbb{P}_{\mathrm{C}}^{\ell} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{\ell}}_{N \text{ times}} \to \mathbb{P}_{\mathrm{C}}^{k_0} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k_\sigma}$ be the projection onto the first $\sigma+1$ components, and suppose that $\mathrm{pr}_{[0,\sigma]}$ restricted to A is a compact covering.

For $p \geq 0$, let

$$J^p_{\mathrm{C}, \mathrm{pr}_{[0,\sigma]}}(A) \subset \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k_\sigma} \times \underbrace{\mathbb{P}_{\mathrm{C}}^{(\ell+1)(p+1)-1} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{(\ell+1)(p+1)-1}}_{N \text{ times}}$$

be defined by the formula

$$J^p_{\mathrm{C}, \mathrm{pr}_{[0,\sigma]}}(\Phi) \stackrel{\text{def}}{=} \underset{0 \leq i_0 \leq \alpha_0}{\Lambda^0} \cdots \underset{0 \leq i_\sigma \leq \alpha_\sigma}{\Lambda^\omega} J^p_{\mathrm{C}, \mathrm{pr}_{[0,\sigma]}} \phi(\mathbf{X}^0; \cdots; \mathbf{X}^\sigma; \mathbf{Y}_{i_0, \ldots, i_\sigma}). \tag{3.11}$$

Theorem 3.26. *For every $p \geq 0$, we have that*

$$P_{\mathrm{pr}_{[0,\sigma]}(A)} \quad = \quad (1 - T^2)^N P_{J^p_{\mathrm{C}, \mathrm{pr}_{[0,\sigma]}}(A)} \mod T^p. \tag{3.12}$$

Proof. The proof is identical to that of Theorem 3.23 above using Proposition 3.19 instead of Proposition 3.16 and noticing that by the Kunneth formula for homology, the Poincaré polynomial of

$$\underbrace{\mathbb{P}_{\mathrm{C}}^{(\ell+1)(p+1)-1} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{(\ell+1)(p+1)-1}}_{N \text{ times}}$$

equals $(1 - T^2)^{-N} \mod T^p$. □

As before we have the following corollary.

Corollary 3.27. *Let $p = 2m + 1$ with $m \geq 0$. Then*

$$Q_{\mathrm{pr}_{[0,\sigma]}(A)} \quad = \quad (1 - T)^N \, Q_{J^p_{\mathrm{C}, \mathrm{pr}_{[0,\sigma]}}(A)} \mod T^{m+1}. \tag{3.13}$$

It is clear from the definition that the complex joins of languages in $\mathbf{P}_{\mathrm{C}}$ also belong to the complexity class $\mathbf{P}_{\mathrm{C}}$. We record this observation formally in the following proposition.

Proposition 3.28 (Polynomial time membership testing)**.** *Suppose that the sequence of constructible sets $(S_n \subset \mathbb{P}_{\mathrm{C}}^{k(n)} \times \mathbb{P}_{\mathrm{C}}^{\ell(n)})_{n>0} \in \mathbf{P}_{\mathrm{C}}$, and $\mathbf{X}_n = (X_0 : \cdots :$*

$X_{k(n)})$ $\mathbf{Y}_n = (Y_0 : \cdots : Y_{\ell(n)})$ are homogeneous co-ordinates of $\mathbb{P}_{\mathrm{C}}^{k(n)}$ and $\mathbb{P}_{\mathrm{C}}^{\ell(n)}$ respectively. Let $p(n)$ be a non-negative polynomial, and for each $n > 0$ let

$$\mathrm{pr}_1 : \mathbb{P}_{\mathrm{C}}^{k(n)} \times \mathbb{P}_{\mathrm{C}}^{\ell(n)} \to \mathbb{P}_{\mathrm{C}}^{k(n)}$$

denote the projection on the first component.

Then,

$$\left(J^{p(n)}_{\mathrm{C},\mathrm{pr}_1}(S_n) \subset \mathbb{P}_{\mathrm{C}}^{k(n)} \times \mathbb{P}_{\mathrm{C}}^{(p(n)+1)(\ell(n)+1)-1}\right)_{n>0} \in \mathbf{P}_{\mathrm{C}}.$$

Proof. Obvious from the definition of $(J^{p(n)}_{\mathrm{C},\mathrm{pr}_1}(S_n))_{n>0}$. □

4. Proof of the main theorem

We are now in a position to prove Theorem 2.1. The proof relies on the following key proposition.

Proposition 4.1. *Let $m(n), k_1(n), \ldots, k_\omega(n)$ be polynomials, and let*

$$(\Phi_n(\mathbf{X}, \mathbf{Y}))_{n>0}$$

be a sequence of multi-homogeneous formulas

$$\Phi_n(\mathbf{X}, \mathbf{Y}) \overset{def}{=} (\mathrm{Q}_1 \mathbf{Z}^1 \in \mathbb{P}_{\mathrm{C}}^{k_1}) \cdots (\mathrm{Q}_\omega \mathbf{Z}^\omega \in \mathbb{P}_{\mathrm{C}}^{k_\omega}) \phi_n(\mathbf{X}; \mathbf{Y}; \mathbf{Z}^1; \cdots; \mathbf{Z}^\omega),$$

having free variables $(\mathbf{X}; \mathbf{Y}) = (X_0, \ldots, X_{k(n)}; Y_0, \ldots, Y_{m(n)})$, with

$$\mathrm{Q}_1, \ldots, \mathrm{Q}_\omega \in \{\exists, \forall\},$$

and ϕ_n a multi-homogeneous quantifier-free formula defining a closed (resp. open) constructible subset

$$S_n \subset \mathbb{P}_{\mathrm{C}}^{k} \times \mathbb{P}_{\mathrm{C}}^{m} \times \mathbb{P}_{\mathrm{C}}^{k_1} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{k_\omega}.$$

Suppose also that

$$\left(\mathcal{R}(\phi_n(\mathbf{X}; \mathbf{Y}; \mathbf{Z}^1; \cdots; \mathbf{Z}^\omega))\right)_{n>0} \in \mathbf{P}_{\mathrm{C}}.$$

Then there exist:

(A) *a sequence of quantifier-free multi-homogeneous formulas*

$$\left(\Theta_n(\mathbf{X}; \mathbf{V}^1; \cdots; \mathbf{V}^N)\right)_{n>0},$$

with $\mathbf{V}^i = (V_0, \ldots, V_{p_i})$, and $N, p_1, \ldots, p_N$ polynomials in n, such that for all $\mathbf{x} \in \mathbb{P}_{\mathrm{C}}^{k}$ $\Theta_n(\mathbf{x}; \mathbf{V}^1; \cdots; \mathbf{V}^N)$ defines a constructible subset $T_n \subset \mathbb{P}_{\mathrm{C}}^{p_1} \times \cdots \times \mathbb{P}_{\mathrm{C}}^{p_N}$, with

$$(T_n)_{n>0} \in \mathbf{P}_{\mathrm{C}};$$

(B) *polynomial time computable maps*

$$F_n : \mathbb{Z}[T] \to \mathbb{Z}[T],$$

such that for all $\mathbf{x} \in \mathbb{P}_{\mathrm{C}}^{k}$

$$Q_{\mathcal{R}(\Phi_n(\mathbf{x}; \mathbf{Y}))} = F_n(Q_{\mathcal{R}(\Theta_n(\mathbf{x}; \mathbf{V}^1; \cdots; \mathbf{V}^N))}).$$

The idea behind the proof of Proposition 4.1 is to use induction on the number, ω, of quantifier blocks. When $\omega = 0$, the proposition is obvious. When $\omega > 0$, then using Corollary 3.25, we can construct a new formula (say Φ'_n) such that Φ' has one less block of quantifiers, but such that $Q_{\mathcal{R}(\Phi_n)}$ is easily computable from $Q_{\mathcal{R}(\Phi'_n)}$. One can then use induction to finish the proof. However, a technical complication arises due to the fact that in the projective situation (unlike in the affine case) we cannot immediately replace two adjacent blocks of the same quantifier by a single block. This is the logical manifestation of the elementary fact that the product of two projective spaces is not itself a projective space. In order to overcome this difficulty and carry through the inductive step properly, we need to prove a slightly stronger, but technically more involved proposition, which we state next. Proposition 4.1 will be an immediate corollary of this more general proposition.

Proposition 4.2. *Let $\sigma, \omega \geq 0$ be constants, and*

$$a_0(n), \alpha_0(n), a_1(n), \alpha_1(n), \ldots, a_\sigma(n), \alpha_\sigma(n),$$
$$k(n), k_1(n), \ldots, k_\omega(n)$$

fixed polynomials in n taking non-negative values for $n \in \mathbb{N}$.

Let $\mathbf{X} = (X_0 : \cdots : X_{k(n)})$ denote a block of $k(n)+1$ variables. For $0 \leq j \leq \sigma$, let $\mathbf{W}^j$ denote the tuple of variables

$$(\ldots, \mathbf{W}^j_{i_0,\ldots,i_j}, \ldots), 0 \leq i_0 \leq \alpha_0, \ldots, 0 \leq i_j \leq \alpha_j,$$

where each $\mathbf{W}^j_{i_0,\ldots,i_j}$ is a block of $a_j(n)+1$ variables.

Let

$$\left(\Phi_n(\mathbf{X}; \mathbf{W}^0; \mathbf{W}^1; \ldots; \mathbf{W}^\sigma)\right)_{n>0},$$

be a sequence of multi-homogeneous formulas defined by

$$\Phi_n(\mathbf{X}; \mathbf{W}^0; \mathbf{W}^1; \cdots; \mathbf{W}^\sigma) \stackrel{def}{=}$$
$$\underset{0 \leq i_0 \leq \alpha_0}{\Lambda^0} \cdots \underset{0 \leq i_n \leq \alpha_\sigma}{\Lambda^\sigma} (\mathrm{Q}_1 \mathbf{Z}^1 \in \mathbb{P}^{k_1}_{\mathrm{C}}) \cdots (\mathrm{Q}_\omega \mathbf{Z}^\omega \in \mathbb{P}^{k_\omega}_{\mathrm{C}})$$
$$\phi_n(\mathbf{X}; \mathbf{W}^0_{i_0}; \mathbf{W}^1_{i_0,i_1}; \cdots; \mathbf{W}^\sigma_{i_0,\ldots,i_\sigma}; \mathbf{Z}^1; \cdots; \mathbf{Z}^\omega),$$

with

$$\Lambda^0, \ldots, \Lambda^\sigma \in \{\bigvee, \bigwedge\},$$

$$\mathrm{Q}_1, \ldots, \mathrm{Q}_\omega \in \{\exists, \forall\},$$

and each ϕ_n a multi-homogeneous quantifier-free formula, multi-homogeneous in the blocks of variables, $\mathbf{X}, \mathbf{Z}^1, \ldots, \mathbf{Z}^\omega$ and $(W^j_{i_0,\ldots,i_j,0}, \ldots, W^j_{i_0,\ldots,i_j,\alpha_j})$ for $0 \leq j \leq \sigma$. Suppose also that each ϕ_n defines a closed (resp. open) constructible set, and that

$$(\mathcal{R}(\phi_n))_{n>0} \in \mathbf{P}_{\mathrm{C}}.$$

Then there exists:

(A) *a sequence of quantifier-free multi-homogeneous formulas*

$$\left(\Theta_n(\mathbf{X}; \mathbf{V}^1; \cdots; \mathbf{V}^N)\right)_{n>0},$$

with $\mathbf{V}^i = (V_0, \ldots, V_{p_i})$, and $N, p_1, \ldots, p_N$ polynomials in n, such that $\Theta_n(\mathbf{x}; \mathbf{V}^1; \cdots; \mathbf{V}^N)$ defines a constructible subset $T_n \subset \mathbb{P}^{p_1}_{\mathrm{C}} \times \cdots \times \mathbb{P}^{p_N}_{\mathrm{C}}$, with

$$(T_n)_{n>0} \in \mathbf{P}_{\mathrm{C}};$$

(B) *polynomial time computable maps*

$$F_n : \mathbb{Z}[T] \to \mathbb{Z}[T],$$

such that

$$Q_{\mathcal{R}(\Phi_n(\mathbf{x};\cdot))} = F_n(Q_{\mathcal{R}(\Theta_n(\mathbf{x};\mathbf{V}^1;\cdots;\mathbf{V}^N))}).$$

Proof of Proposition 4.2. The proof is by induction on ω.

If $\omega = 0$, we let $\Theta_n = \Phi_n$, and F_n to be the identity map. Since there are no quantifiers, for each $n \geq 0$ the constructible set defined by Θ_n and Φ_n are the same, and thus the Betti numbers of the sets defined by Θ_n and Φ_n are equal.

If $\omega > 0$, we have the following two cases.

(A) Case 1, $\mathrm{Q}_1 = \exists$: First note that Φ_n defines a constructible subset of $\mathbb{P}_{\mathrm{C}}^{k(n)} \times U_n$, where

$$U_n = (\mathbb{P}_{\mathrm{C}}^{(a_0+1)(\alpha_0+1)-1})^{m_0} \times \cdots \times (\mathbb{P}_{\mathrm{C}}^{(a_j+1)(\alpha_j+1)-1})^{m_j} \times \cdots \times (\mathbb{P}_{\mathrm{C}}^{(a_\sigma+1)(\alpha_\sigma+1)-1})^{m_\sigma},$$

where for $0 \leq j \leq \sigma$,

$$m_j(n) = \prod_{0 \leq i \leq j-1} (\alpha_i(n) + 1).$$

The formula Φ_n is equivalent to the following formula:

$$\Big(\cdots (\exists \mathbf{Z}^{1,i_0,\ldots,i_\sigma} \in \mathbb{P}_{\mathrm{C}}^{k_1}) \cdots \Big) \bar{\Phi}_n$$

where where the blocks of existential quantifiers in the beginning are indexed by the tuples

$$(i_0, \ldots, i_\sigma), 0 \leq i_0 \leq \alpha_0(n), \ldots, 0 \leq i_\sigma \leq \alpha_\sigma(n).$$

and the number of such blocks is

$$\alpha(n) = \prod_{i=0}^{\sigma} (\alpha_i(n) + 1),$$

and

$$\bar{\Phi}_n \stackrel{\mathrm{def}}{=} \bigwedge_{0 \leq i_0 \leq \alpha_0}^{0} \cdots \bigwedge_{0 \leq i_\sigma \leq \alpha_\sigma}^{\sigma}$$

$$(\mathrm{Q}_2 \mathbf{Z}^2 \in \mathbb{P}_{\mathrm{C}}^{k_2}) \cdots (\mathrm{Q}_\omega \mathbf{Z}^\omega \in \mathbb{P}_{\mathrm{C}}^{k_\omega}) \phi_n(\mathbf{X}; \mathbf{W}_{i_0}^0; \cdots, \mathbf{W}_{i_0,\ldots,i_\sigma}^\sigma; \mathbf{Z}_{i_0,\ldots,i_\sigma}^1; \mathbf{Z}^2 \cdots; \mathbf{Z}^\omega),$$

Let

$$m(n) = \sum_{j=0}^{\sigma} m_j(n)((a_j(n) + 1)(\alpha_j(n) + 1) - 1).$$

(Note that $m(n)$ is the (complex) dimension of U_n defined previously.)

Let

$$\mathrm{pr}_{1,2} : \mathbb{P}_{\mathrm{C}}^{k(n)} \times U_n \times (\mathbb{P}_{\mathrm{C}}^{k_1})^{\alpha(n)} \to \mathbb{P}_{\mathrm{C}}^{k(n)} \times U_n$$

denote the projection on the first two components.

Consider the sequence

$$\left(J_{\mathrm{C},\mathrm{pr}_{1,2}}^{2m(n)+1}(\bar{\Phi}_n) \right)_{n>0}.$$

Note that by 3.11 we have

$$J_{\mathrm{C},\mathrm{pr}_{1,2}}^{2m(n)+1}(\bar{\Phi}_n) =$$

$$\bigwedge_{0 \leq i_0 \leq \alpha_0}^{0} \cdots \bigwedge_{0 \leq i_\sigma \leq \alpha_\sigma}^{\sigma} \bigwedge_{0 \leq i_{\sigma+1} \leq \alpha_{\sigma+1}}^{\sigma+1}$$

$$(\mathrm{Q}_2 \mathbf{Z}^2 \in \mathbb{P}_{\mathrm{C}}^{k_2}) \cdots (\mathrm{Q}_\omega \mathbf{Z}^\omega \in \mathbb{P}_{\mathrm{C}}^{k_\omega})$$

$$\phi_n(\mathbf{X}; \mathbf{W}_{i_0}^0; \cdots; \mathbf{W}_{i_0,\ldots,i_\sigma}^\sigma; \mathbf{W}_{i_0,\ldots,i_\sigma,i_{\sigma+1}}^{\sigma+1}; \mathbf{Z}^2; \cdots; \mathbf{Z}^\omega).$$

with $\Lambda^{\sigma+1} = \bigwedge$, $\alpha_{\sigma+1} = 2m+1$, and $\mathbf{W}^{\sigma+1}_{i_0,\ldots,i_\sigma,i_{\sigma+1}} = \mathbf{Z}^1_{i_0,\ldots,i_\sigma,i_{\sigma+1}}$. We will denote by $\mathbf{W}^{\sigma+1}$ the tuple

$$(\ldots, W^{\sigma+1}_{i_0,\ldots,i_\sigma,i_{\sigma+1}},\ldots), 0 \le i_0 \le \alpha_0, \ldots, 0 \le i_{\sigma+1} \le \alpha_{\sigma+1}.$$

Observe that the number of quantifiers in the formulas $J^{2m(n)+1}_{\mathrm{C},\mathrm{pr}_{1,2}}(\bar{\Phi}_n)$, is $\omega - 1$.

Moreover, $J^{2m(n)+1}_{\mathrm{C},\mathrm{pr}_{1,2}}(\bar{\Phi}_n)$ satisfy by Proposition 3.28 the required polynomial time hypothesis, and have the same shape as the formulas Φ_n. We can thus apply the induction hypothesis to this sequence to obtain a sequence $(\Theta_n)_{n>0}$, as well as a sequence of polynomial time computable maps $(G_n)_{n>0}$. By inductive hypothesis we can suppose that for each $\mathbf{x} \in \mathbb{P}^{k(n)}_{\mathrm{C}}$

$$Q_{\mathcal{R}(J^{2m(n)+1}_{\mathrm{C},\mathrm{pr}_{1,2}}(\bar{\Phi}_n)(\mathbf{x};\cdot))} = G_n(Q_{\mathcal{R}(\Theta_n(\mathbf{x};\cdot))}).$$

Using Corollary 3.27 and noticing that the map $\mathrm{pr}_{1,2}$ is either open or closed and hence a compact covering,

$$\begin{aligned} Q_{\mathcal{R}(\Phi_n(\mathbf{x};\cdot))} &= (1-T)^{\alpha(n)} Q_{\mathcal{R}(J^{2m(n)+1}_{\mathrm{C},\mathrm{pr}_{1,2}}(\bar{\Phi}_n)(\mathbf{x};\cdot))} \mod T^{m(n)+1} \\ &= (1-T)^{\alpha(n)} G_n(Q_{\mathcal{R}(\Theta_n(\mathbf{x};\cdot))}) \mod T^{m(n)+1}. \end{aligned}$$

We set

$$F_n = \mathrm{Trunc}_{m(n)} \circ M_{(1-T)^{\alpha(n)}} \circ G_n$$

(see Notation 3.2). This completes the induction in this case.

(B) Case 2, $\mathrm{Q}_1 = \forall$:

The formula Φ_n is equivalent to the following formula:

$$\Big(\cdots (\forall \mathbf{Z}^{1,i_0,\ldots,i_\sigma} \in \mathbb{P}^{k_1}_{\mathrm{C}}) \cdots\Big)\, \bar{\Phi}_n$$

where the blocks of universal quantifiers in the beginning are indexed by the tuples

$$(i_0,\ldots,i_\sigma), 0 \le i_0 \le \alpha_0(n), \ldots, 0 \le i_\sigma \le \alpha_\sigma(n),$$

the number of such blocks is

$$\alpha(n) = \prod_{i=0}^{\sigma} (\alpha_i(n)+1),$$

and

$$\bar{\Phi}_n \stackrel{\mathrm{def}}{=} \underset{0\le i_0\le\alpha_0}{\Lambda^0} \cdots \underset{0\le i_\sigma\le\alpha_\sigma}{\Lambda^\sigma}$$
$$(\mathrm{Q}_2\mathbf{Z}^2 \in \mathbb{P}^{k_2}_{\mathrm{C}}) \cdots (\mathrm{Q}_\omega \mathbf{Z}^\omega \in \mathbb{P}^{k_\omega}_{\mathrm{C}}) \phi_n(\mathbf{X}; \mathbf{W}^0_{i_0}; \cdots; \mathbf{W}^\sigma_{i_0,\ldots,i_\sigma}; \mathbf{Z}^1_{i_0,\ldots,i_\sigma}; \mathbf{Z}^2 \cdots; \mathbf{Z}^\omega).$$

Consider the sequence

$$\left(J^{2m+1}_{\mathrm{C},\mathrm{pr}_{1,2}}(\neg\bar{\Phi}_n)\right)_{n>0}.$$

Note that the formula

$$\begin{array}{c} J^{2m+1}_{\mathrm{C},\mathrm{pr}_{1,2}}(\neg\bar{\Phi}_n) = \\ \bar{\Lambda}^0_{0\le i_0\le\alpha_0} \cdots \bar{\Lambda}^{\sigma}_{0\le i_\sigma\le\alpha_\sigma} \Lambda^{\sigma+1}_{0\le i_{\sigma+1}\le\alpha_{\sigma+1}} \\ (\bar{Q}_2 \mathbf{Z}^2 \in \mathbb{P}^{k_2}_{\mathrm{C}}) \cdots (\bar{Q}_\omega \mathbf{Z}^\omega \in \mathbb{P}^{k_\omega}_{\mathrm{C}}) \\ \neg\phi_n(\mathbf{X}; \mathbf{W}^0_{i_0}; \cdots; \mathbf{W}^\sigma_{i_0,\ldots,i_\sigma}; \mathbf{W}^{\sigma+1}_{i_0,\ldots,i_{\sigma+1}}; \mathbf{Z}^2; \cdots; \mathbf{Z}^\omega) \end{array}$$

with $\Lambda^{\sigma+1} = \bigwedge$, $\alpha_{\sigma+1} = 2m+1$, $\mathbf{W}^{\sigma+1}_{i_0,\ldots,i_{\sigma+1}} = \mathbf{Z}^1_{i_0,\ldots,i_\sigma,i_{\sigma+1}}$, and

$$\begin{array}{c} \bar{\Lambda}^i = \bigvee \text{ if } \Lambda^i = \bigwedge, \bar{\Lambda}^i = \bigwedge \text{ if } \Lambda^i = \bigvee, \\ \bar{Q}_i = \exists \text{ if } \mathrm{Q}_i = \forall, \bar{Q}_i = \forall \text{ if } \mathrm{Q}_i = \exists. \end{array}$$

Observe that the number of quantifiers in the formulas $J^{2m+1}_{\mathrm{C},\mathrm{pr}_{1,2}}(\neg\bar{\Phi}_n)$, is $\omega - 1$.

Moreover, $J^{2m+1}_{\mathrm{C},\mathrm{pr}_{1,2}}(\neg\bar{\Phi}_n)$ satisfy by Proposition 3.28 the required polynomial time hypothesis, and have the same shape as the formulas Φ_n. We can thus apply the induction hypothesis to this sequence to obtain a sequence $(\Theta_n)_{n>0}$, as well as a sequence of polynomial time computable maps $(G_n)_{n>0}$. By inductive hypothesis we can suppose that for each $\mathbf{x} \in \mathbb{P}^{k(n)}_{\mathrm{C}}$

$$Q_{\mathcal{R}(J^{2m+1}_{\mathrm{C},\mathrm{pr}_{1,2}}(\neg\bar{\Phi}_n)(\mathbf{x};\cdot))} = G_n(Q_{\mathcal{R}(\Theta_n(\mathbf{x};\cdot))}).$$

Using Corollary 3.27 and noticing that the map $\mathrm{pr}_{1,2}$ is either open or closed and hence a compact covering, we have

$$\begin{aligned} Q_{\mathcal{R}(\neg\Phi_n(\mathbf{x};\cdot))} &= (1-T)^{\alpha(n)} Q_{\mathcal{R}(J^{2m+1}_{\mathrm{C},\mathrm{pr}_{1,2}}(\bar{\Phi}_n)(\mathbf{x};\cdot))} \mod T^{m(n)+1} \\ &= (1-T)^{\alpha(n)} G_n(Q_{\mathcal{R}(\Theta_n(\mathbf{x};\cdot))}) \mod T^{m(n)+1}. \end{aligned}$$

The sets $K_n = \mathcal{R}(\Phi_n(\mathbf{x};\cdot))$ are constructible and open (resp. closed); so by Corollary 3.4 (corollary to Theorem 3.1), we have

$$Q_{K_n}(T) = Q_{U_n} - \mathrm{Rec}_m(\mathrm{Trunc}_m(Q_{U_n - K_n})).$$

We set F_n to be the operator defined by

$$F_n(Q) = Q_{U_n} - \mathrm{Rec}_m(\mathrm{Trunc}_m(M_{(1-T)^{\alpha(n)}}(G_n(Q)))).$$

This completes the induction in this case as well.

□

Proof of Proposition 4.1. Proposition 4.1 is a special case of Proposition 4.2 with $\sigma = 0$, $\alpha_0 = 0$, and $\mathbf{Y} = \mathbf{W}^0$. □

Proof of Theorem 2.1. Follows immediately from Proposition 4.1 in the special case $m = 0$. In this case the sequence of formulas $(\Phi_n)_{n>0}$ corresponds to a language in the polynomial hierarchy and for each n, $\mathbf{x} = (x_0 : \cdots : x_{k(n)}) \in S_n \subset \mathbb{P}^{k(n)}_{\mathrm{C}}$ if and only if

$$F_n(Q_{\mathcal{R}(\Theta_n(\mathbf{x};\cdot))})(0) > 0$$

and this last condition can be checked in polynomial time using an oracle from the class $\#\mathbf{P}^{\dagger}_{\mathrm{C}}$. □

Remark 4.3. It is interesting to observe that in complete analogy with the proof of the classical Toda's theorem the proof of Theorem 2.1 also requires just one call to the oracle at the end.

Proof of Theorem 2.6. Follows from the proof of Proposition 4.1 since the formula Θ_n is clearly computable in polynomial time from the given formula Φ_n as long as the number of quantifier alternations ω is bounded by a constant. □

5. Future Directions

In this section, we sketch a few directions in which the work presented in this paper could be developed further.

(A) Remove the compactness hypothesis from the main theorem.

(B) The compact fragment of the polynomial hierarchy introduced in this paper, and especially the class $\mathbf{\Sigma}^c_{\mathrm{C},1}$ (which is the compact fragment of $\mathbf{NP}_{\mathrm{C}}$), is possibly interesting on their own, and it would be nice to develop a theory of compact reductions and compact hardness, and have $\mathbf{NP}^c_{\mathrm{C}}$-complete problems. The compact feasibility problem discussed in Example 1.16 is a good candidate for being a $\mathbf{NP}^c_{\mathrm{C}}$-complete problem.

(C) As remarked earlier, one would obtain a stronger reduction result if one could prove a Toda-type theorem using only the Euler-Poincaré characteristic instead of the whole Poincaré polynomial. This seems to be rather difficult. An intermediate goal could be to use the *virtual Poincaré polynomial*. The virtual Poincaré polynomial of a complex variety X is defined by

$$\mathcal{P}_X(T) = H_X(-T, -T),$$

where $H_X(u, v) \in \mathbb{Z}[u, v]$ is the Hodge-Deligne polynomial uniquely determined by the following properties.

(1) The map $X \mapsto H_X$ gives an additive and multiplicative invariant from the Grothendieck ring of equivalence classes of complex varieties to $\mathbb{Z}[u, v]$.

(2) $H_X(u, v)$ coincides with $\sum(-1)^{p+q} h^{p,q}(X) u^p v^q$ when X is smooth and projective, where $h^{p,q}(X)$ are the Hodge numbers.

Clearly, the virtual Poincaré polynomial is additive, and coincides with the ordinary Poincaré polynomial, P_X, in the case when X is smooth and projective. Thus, one could try to prove the results in this paper using the virtual Poincaré polynomial instead of the Poincaré polynomial. Unfortunately, the virtual Poincaré polynomial is an algebro-geometric, rather than topological invariant, and the topological methods used in this paper are not sufficient to obtain such a result. In particular, Theorem 3.23 does not hold for the virtual Poincaré polynomial except in the trivial case when $A = \mathbb{P}^k_{\mathrm{C}} \times \mathbb{P}^\ell_{\mathrm{C}}$.

(D) Theorem 3.23 can be used to bound the Betti numbers of the images of complex varieties under regular maps (in conjunction with tight bounds on the Betti numbers of complex projective varieties due to Katz [16]), instead of first using elimination methods, and then applying the bounds due to Katz. A similar method was used in [14] to obtain bounds on the Betti numbers of projections of semi-algebraic sets in the real case. One can also treat a complex variety as a real semi-algebraic set by separating the real and imaginary parts, but the direct complex method using Theorem 3.23 suggested above should yield better upper bounds on the Betti numbers of projections.

6. Acknowledgments

The author benefited from discussions with Donu Arapura, Andrei Gabrielov, and James McClure at various stages of this work. The author also thanks an anonymous referee for making several suggestions which helped to improve this paper.

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Department of Mathematics, Purdue University, West Lafayette, IN 47906, U.S.A.
E-mail address: sbasu@math.purdue.edu