POLYNOMIAL PARTITIONING ON VARIETIES AND POINT-HYPERSURFACE INCIDENCES IN FOUR DIMENSIONS

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Abstract. We present a polynomial partitioning theorem for finite sets of points in the real locus of a complex algebraic variety. This result generalizes the polynomial partitioning theorem on the Euclidean space of Guth and Katz, and its extension to hypersurfaces by Zahl and by Kaplan, Matoušek, Sharir and Safernová.

We also present a bound for the number of incidences between points and hypersurfaces in the four-dimensional Euclidean space. It is an application of our partitioning theorem together with the refined bounds for the number of connected components of a semi-algebraic set by Barone and Basu.

Contents

1. Introduction

The polynomial partitioning method was introduced by Guth and Katz in [GK10]. Using this method and the Elekes' framework as exposed in [ES11], they made a breakthrough in a long-standing problem of Erdős on the number of distinct distances between a n points in the plane, by nearly proving the distinct distances conjecture. This method gives a nonlinear decomposition of the Euclidean space, which plays a role analogous to that of cuttings or trapezoidal decompositions in the more classical Clarkson-Shor type divide-and-conquer arguments for such problems, see for instance [CEG+90].

The Guth-Katz polynomial partitioning method can be summarized in the following result. For a polynomial $g \in \mathbb{R}[x_1, \dots, x_d]$, we denote by $V(g)$ its set of zeros in $\mathbb{C}^d$ and, for a finite set $\mathcal{Q}$, we denote by $\operatorname{card}(\mathcal{Q})$ its cardinality.

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Theorem 1.1 (Guth and Katz [GK10]). *Let $\mathcal{P} \subset \mathbb{R}^d$ be a finite subset. Given $\ell \geq 1$, there is a nonzero polynomial $g \in \mathbb{R}[x_1, \ldots, x_d]$ of degree bounded by ℓ such that, for each connected component C of $\mathbb{R}^d \setminus V(g)$,*

$$\mathrm{card}(\mathcal{P} \cap C) = O_d\Big(\frac{\mathrm{card}(\mathcal{P})}{\ell^d}\Big),$$

where the implicit constant in the O-notation depends only on d.

Subsequently, this partitioning method has proved very successful producing other new results and simpler proofs of known results in discrete geometry, see for instance [KMS12, KMSS12, Zah13, ST12]. Let us point out that, to apply Theorem 1.1 in a concrete situation, one also needs to couple it with a suitable bound for the number of connected components of the semi-algebraic set $\mathbb{R}^d \setminus V(g)$. This is provided by classical works of Oleĭnik, Petrovskiĭ, Thom and Milnor on the Betti numbers of semi-algebraic varieties [PO49, Tho65, Mil64].

Notice that Theorem 1.1 allows for the possibility that many, or even all, of the points in $\mathcal{P}$ are contained in the hypersurface $V(g)$. This leads to a dichotomy: the open pieces of the decomposition, that is, the connected components of $\mathbb{R}^d \setminus V(g)$, are handled by induction, whereas a separate argument is needed for handling the points contained in $V(g)$.

To avoid this dichotomy, it is natural to try to extend this result to subvarieties of the affine space. By doing so, one can hope for a uniform technique for proving incidence type theorems. For hypersurfaces, such an extension was achieved independently by Zahl and by Kaplan, Matoušek, Sharir and Safernová, and applied to get new incidence results in $\mathbb{R}^3$ [Zah13, KMSS12].

Obtaining a polynomial partitioning theorem on varieties was identified as a major obstacle to extend the method to the higher dimensions, see for instance the discussion in [KMS12, §3]. Our main objective in this paper is, precisely, to present such a result.

Given an irreducible algebraic variety $X \subset \mathbb{C}^d$ we denote by $\dim(X)$ and $\deg(X)$ its dimension and degree, respectively. Given $\delta \geq 1$, we say that X is *locally set-theoretically defined at degree* δ if it is an irreducible component of the zero set of a family of polynomials of degree bounded by δ (Definition 2.1).

The following is a somewhat simplified version of our polynomial partitioning theorem, see Theorem 3.1 and Remark 3.2.

Theorem 1.2. *There is a constant $c = c(d)$ with the following property. Let $X \subset \mathbb{C}^d$ be an irreducible variety of dimension e and locally set-theoretically defined at degree $\delta \geq 1$. Let $\mathcal{P} \subset \mathbb{R}^d \cap X$ be a finite subset and $\ell \geq c\delta$. Then there is a polynomial $g \in \mathbb{R}[x_1, \ldots, x_d]$ of degree bounded by ℓ with* $\dim(X \cap V(g)) = e - 1$ *such that, for each connected component C of $\mathbb{R}^d \setminus V(g)$,*

$$\mathrm{card}(\mathcal{P} \cap C) = O_d\Big(\frac{\mathrm{card}(\mathcal{P})}{\deg(X)\ell^e}\Big).$$

As for the Guth-Katz theorem, the proof of this result is based on the ham sandwich theorem obtained by Stone and Tukey from the Borsuk-Ulam theorem, see [GK10, §4]. The new key ingredient is the systematic use of upper and lower bounds for Hilbert functions due to Chardin [Cha89], Chardin and Philippon [CP99] and Sombra [Som97].

If we denote by δ the minimal integer such that X is set-theoretically locally defined at degree δ, then

$$\delta \leq \deg(X) \leq \delta^{d-e},$$

see (2.1). Hence, for $X = \mathbb{C}^d$, Theorem 3.1 reduces to Theorem 1.1 and, when X is a hypersurface, it reduces to the polynomial partitioning theorems in [Zah13, KMSS12], up to constant factors.

Remark 1.3. The polynomial partition method also applies to problems in computational geometry, in particular to range searching with semi-algebraic sets. Concurrently with this paper, Matoušek and Safernová have also obtained a polynomial partitioning theorem for varieties [MS14, Theorem 1.1], focused on obtaining more efficient range searching algorithms. Their result is weaker than ours. Nevertheless, it is strong enough for their application to range searching.

As a test case for Theorem 3.1, we consider the problem of bounding the number of point-hypersurface incidences. Given a set $\mathcal{P}$ of points of $\mathbb{R}^d$ and a set $\mathcal{V}$ of subvarieties of $\mathbb{R}^d$ or of $\mathbb{C}^d$, we denote by $I(\mathcal{P}, \mathcal{V})$ their number of incidences, that is, the number of pairs (p, V) with $p \in \mathcal{P}$ and $V \in \mathcal{V}$ such that $p \in V$.

The following fundamental result was proved by Szeméredi and Trotter in 1983, in response to a problem of Erdős.

Theorem 1.4 (Szeméredi and Trotter [ST83])**.** *Let $\mathcal{P}$ be a set of m points of $\mathbb{R}^2$ and $\mathcal{L}$ a set of n lines in $\mathbb{R}^2$. Then*

$$I(\mathcal{P}, \mathcal{L}) = O(m^{\frac{2}{3}} n^{\frac{2}{3}} + m + n).$$

The Szemerédi-Trotter theorem has led to an extensive study of incidences of points and curves in the plane, and of points and varieties in higher dimensions. In particular, it was extended by Pach and Sharir to incidences between points in the plane and curves having a bounded degree of freedom [PS98].

Later on, Zahl obtained an analogous result for the incidences between points in $\mathbb{R}^3$ and algebraic surfaces having a bounded degree of freedom [Zah13]. A similar result was independently obtained by Kaplan, Matoušek, Sharir and Safernová for the incidences between points in $\mathbb{R}^3$ and unit spheres [KMSS12]. Both results are a consequence of their polynomial partitioning theorem on hypersurfaces.

We present the following bound for the incidences between points in $\mathbb{R}^4$ and threefolds.

Theorem 1.5. *Let $k, c \geq 1$, and let $\mathcal{P}$ be a finite set of points of $\mathbb{R}^4$ and $\mathcal{H}$ a finite set of hypersurfaces of $\mathbb{C}^4$ satisfying the following conditions:*

(a) *the degree of the hypersurfaces in $\mathcal{H}$ is bounded by c;*
(b) *the intersection of any family of four distinct hypersurfaces in $\mathcal{H}$ is finite;*
(c) *for any subset of k distinct points in $\mathcal{P}$, the number of hypersurfaces in $\mathcal{H}$ containing them is bounded by c.*

Set $m = \mathrm{card}(\mathcal{P})$ and $n = \mathrm{card}(\mathcal{H})$. Then

$$I(\mathcal{P}, \mathcal{H}) = O_{k,c}(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + m + n).$$

In particular, we obtain the following bound for point-hyperplane incidences in four dimensions.

Corollary 1.6. *Let $\mathcal{P}$ be a set of m points of $\mathbb{R}^4$ and $\mathcal{L}$ a finite set of n hyperplanes of $\mathbb{R}^4$. Then*

$$I(\mathcal{P},\mathcal{L}) = O(m^{\frac{4}{5}}n^{\frac{4}{5}} + m + n).$$

Theorem 1.5 is an application of Theorem 1.2 together with the refined bounds for the number of connected components of a semi-algebraic set due to Barone and Basu [BB12, BB13]. Our whole approach is strongly inspired in the treatment of the unit distance problem in three dimensions in [Zah13, KMSS12].

Our result is a particular case of a conjectural bound for the number of point-hypersurface incidences in $\mathbb{R}^d$ (Conjecture 4.1). Currently, our main obstacle for proving this conjecture in full generality is the absence of a suitable bound for the number of connected components of a semi-algebraic set: to apply efficiently our polynomial partitioning theorem on a variety, one also needs to couple it with a bound for the number of connected components depending on the degree of that variety, instead of the Bézout number of a set of defining equations. We propose a conjecture in this direction (Conjecture 2.7) which, if true, would be an important step in proving Conjecture 4.1.

Remark 1.7. After the results of this paper were announced in the talk [Som14] at the IPAM workshop "Tools from algebraic geometry", a proof by Fox, Pach, Suk, Sheffer and Zahl of a weaker version of Conjecture 4.1 with an extra factor n^{ε} was announced in Sheffer's blog [She14].

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2. Preliminaries on Hilbert functions and semi-algebraic geometry

Throughout this paper, we denote by $\mathbb{N}$ the set of nonnegative integers. Bold letters denote finite sets or sequences of objects, where the type and number should be clear from the context: for instance, $\boldsymbol{x}$ might denote the group of variables $\{x_1,\dots,x_d\}$ so that, for instance, $\mathbb{R}[\boldsymbol{x}]$ denotes the polynomial ring $\mathbb{R}[x_1,\dots,x_d]$.

Given functions $f,g\colon \mathbb{N}\to\mathbb{N}$, the Landau symbol $f=O(g)$ means that there exists $c\geq 0$ such that $f(l)\leq c\,g(l)$ for all $l\in\mathbb{N}$. If we want to emphasize the dependence of the constant c on parameters, say d and k, we will write $f=O_{d,k}(g)$.

2.1. **Hilbert functions.** Given a homogeneous ideal $I\subset\mathbb{C}[z_0,\dots,z_d]$, the quotient $\mathbb{C}[z_0,\dots,z_d]/I$ is a graded $\mathbb{C}$-algebra. The *Hilbert function* of I is the function $\mathrm{H}_I\colon\mathbb{N}\to\mathbb{N}$ given, for $\ell\in\mathbb{N}$, by the dimension of the ℓ-th graded piece of this quotient, that is

$$\mathrm{H}_I(\ell) = \dim_{\mathbb{C}}\left(\mathbb{C}[z_0,\dots,z_d]/I\right)_{\ell}.$$

By Hilbert's theorem, there is a polynomial $\mathrm{P}_I\in\mathbb{Q}[t]$ and an integer $\ell_0\in\mathbb{N}$ such that

$$\mathrm{H}_I(\ell) = \mathrm{P}_I(\ell) \quad \text{for } \ell\geq\ell_0.$$

Let $\mathbb{P}^d(\mathbb{C})$ denote the d-dimensional projective space over complex numbers. Let X be an irreducible subvariety of $\mathbb{P}^d(\mathbb{C})$ of dimension e and degree $\deg(X)$, and denote by $I(X)\subset\mathbb{C}[z_0,\dots,z_d]$ its defining ideal. Then $\mathrm{P}_{I(X)}$ is a polynomial of degree e and leading coefficient equal to $\deg(X)/e!$.

In Theorem 2.2 below, we collect the upper and lower bounds for Hilbert functions that we will use later on. Because of our applications, we will restrict to ideals coming from irreducible projective varieties, although these bounds are valid in greater generality. These results find also applications in Diophantine approximation and transcendental number theory, and in computer algebra.

Definition 2.1. Let $X \subset \mathbb{P}^d(\mathbb{C})$ be an irreducible variety and $\delta \geq 1$. We say that X is *set-theoretically locally defined at degree* δ if there exists a family of homogeneous polynomials $g_1, \ldots, g_t \in \mathbb{C}[z_0, \ldots, z_d]$ of degree bounded by δ such that X is an irreducible component of the zero set $V(g_1, \ldots, g_t)$.

With notation as in Definition 2.1, let δ be minimal such that X is set-theoretically locally defined at degree δ. Then

$$\delta \leq \deg(X) \leq \delta^{d-e}. \tag{2.1}$$

The first inequality follows by considering the image of X under generic linear maps from $\mathbb{P}^d(\mathbb{C})$ onto $\mathbb{P}^{e+1}(\mathbb{C})$, whereas the second one follows from Bézout theorem.

Theorem 2.2. *Let $X \subset \mathbb{P}^d(\mathbb{C})$ be an irreducible variety of dimension $e \geq 0$.*

(a) *For $\ell \geq 0$,*

$$\mathrm{H}_{I(X)}(\ell) \leq \deg(X) \binom{\ell + e}{e}.$$

(b) *For $\ell \geq 0$,*

$$\mathrm{H}_{I(X)}(\ell) \geq \binom{\ell + e + 1}{e + 1} - \max\left\{0, \binom{\ell - \deg(X) + e + 1}{e + 1}\right\}.$$

(c) *Suppose that X is locally set-theoretically defined at degree $\delta \geq 1$. Then, for $\ell \geq (d-e)(\delta-1)+1$,*

$$\mathrm{H}_{I(X)}(\ell) \geq \deg(X) \binom{\ell - (d-e)(\delta - 1) + e}{e}.$$

Proof. These statements are particular cases of [Cha89, Théorème], [Som97, Theorem 2.4] and [CP99, Corollaire 3], respectively. □

Corollary 2.3. *There is a constant $c_1 = c_1(d) > 0$ with the following property. Let $X \subset \mathbb{P}^d(\mathbb{C})$ be an irreducible variety of dimension $e \geq 0$, and $\delta \geq 1$ minimal such that X is locally set-theoretically defined at degree δ. Then, for $\ell \geq 0$,*

$$\mathrm{H}_{I(X)}(\ell) \geq \begin{cases} c_1 \ell^{e+1} & \text{if } \ell \leq 2(d-e)\delta - 1, \\ c_1 \deg(X) \ell^e & \text{if } \ell \geq 2(d-e)\delta. \end{cases}$$

Proof. By Theorem 2.2(b), for $\ell \geq 0$,

$$\begin{aligned} \mathrm{H}_{I(X)}(\ell) &\geq \binom{\ell + e + 1}{e + 1} - \max\left\{0, \binom{\ell - \deg(X) + e + 1}{e + 1}\right\} \\ &= \sum_{j=0}^{\deg(X)-1} \max\left\{0, \binom{\ell - j + e}{e}\right\}. \end{aligned}$$

Then there is a constant $c_1 = c_1(d) > 0$ such that

$$\mathrm{H}_{I(X)}(\ell) \geq \sum_{j=0}^{\deg(X)-1} \frac{1}{e!} \max\{0, \ell - j\}^e \geq c_1(\ell^{e+1} - \max\{0, \ell - \deg(X)\}^{e+1}).$$

This implies the first bound for $\ell \leq 2(d-e)\deg(X)-1$. By (2.1), this bound also holds for $\ell \leq 2(d-e)\delta-1$. The second bound for $\ell \geq (d-e)\delta$ follows directly from Theorem 2.2(c). □

The following result is an easy consequence of Theorem 2.2(a), see [Cha89, Corollaire 3] for details.

Proposition 2.4. *Let $d \geq 2$ and $X \subset \mathbb{P}^d(\mathbb{C})$ an irreducible variety of codimension 2. Then there are coprime polynomials $f_1, f_2 \in I(X)$ such that*

$$\deg(f_1)\deg(f_2) \leq d(d-1)\deg(X).$$

2.2. **Connected components of semi-algebraic sets.** As explained in the introduction, the polynomial partitioning method has to be coupled with bounds for the number of connected components of semi-algebraic sets. When partitioning the Euclidean space $\mathbb{R}^d$, the appropriate bound follows from the Oleĭnik-Petrovskiĭ-Thom-Milnor's bounds for the Betti numbers of a semi-algebraic varieties [PO49, Tho65, Mil64]: with notation as in Theorem 1.1, the number of connected components of $\mathbb{R}^d \setminus V(g)$ is bounded by $\ell(2\ell-1)^{d-1} = (O(\ell))^d$.

In our situation, we will need the Barone-Basu bound below for the number of connected components, with a refined dependence on the sequence of degrees of the defining equations [BB12, BB13].

Given $f_1, \dots, f_e \in \mathbb{R}[x_1, \dots, x_d]$, we denote by $V(f_1, \dots, f_e)$ its set of common zeros in $\mathbb{C}^d$. For a subvariety $X \subset \mathbb{C}^d$, we denote by $X(\mathbb{R}) = X \cap \mathbb{R}^d$ its set of real points. For a semi-algebraic subset $S \subset \mathbb{R}^d$, we denote by $\mathrm{cc}(S)$ the collection of connected components of S. The 0th Betti number $\mathrm{b}_0(S)$ coincides with the cardinality of the collection $\mathrm{cc}(S)$.

Theorem 2.5. *There is a constant $c = c(d)$ such that, given a family of polynomials $f_1, \dots, f_e, g \in \mathbb{R}[x_1, \dots, x_d]$ with $\deg(f_1) \leq \dots \leq \deg(f_e)$, and for all $i, 1 \leq i \leq e$, $\dim(V(f_1, \dots, f_i)) = d - i$, $\deg(f_i) \leq \deg(g)$. Then both*

$$\mathrm{b}_0(V(f_1, \dots, f_e)(\mathbb{R}) \setminus V(g)) \quad \textit{and} \quad \mathrm{b}_0(V(f_1, \dots, f_e, g)(\mathbb{R}))$$

are bounded by $c\deg(f_1)\dots\deg(f_e)\deg(g)^{d-e}$.

Proof. The semi-algebraic set $V(f_1, \dots, f_e)(\mathbb{R}) \setminus V(g)$ is the union of the realization of the sign conditions ± 1 of g on $V(f_1, \dots, f_e)(\mathbb{R})$. Similarly, $V(f_1, \dots, f_e, g)(\mathbb{R})$ is the realization of the sign condition 0 of g on the same real algebraic variety.

The result then follows from [BB13, Theorem 16] together with the following observation. For each i, $1 \leq i \leq e$, $\dim(V(f_1, \dots, f_i))$ bounds from above the (real) dimension of $V(f_1, \dots, f_i)(\mathbb{R})$. □

We will also use the technical result below. Given $p \in \mathbb{R}^d$ and $r > 0$, we denote by $\mathrm{B}(p, r)$ the open ball in $\mathbb{R}^d$ with center p and radius r. Given a subvariety $W \subset \mathbb{C}^d$ and a hypersurface $H \subset \mathbb{C}^d$, we denote by $B(W, H)$ the subset of $W(\mathbb{R})$ of points $p \in W(\mathbb{R})$ having an open neighborhood, in the Euclidean topology of $W(\mathbb{R})$, contained in H. We also set $G(W, H) = W(\mathbb{R}) \setminus B(W, H)$.

Proposition 2.6. *Let $W \subset \mathbb{C}^4$ be a variety and $H, K \subset \mathbb{C}^4$ two hypersurfaces. Let $b \in \mathbb{R}[x_1, x_2, x_3, x_4]$ be a polynomial defining H. Then there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_0$ and any variety $\widetilde{W} \subset \mathbb{C}^4$ containing W, the number of connected components C of $\mathbb{R}^4 \setminus K$ such that $C \cap G(W, H) \cap H \neq \emptyset$ is bounded by*

$$\mathrm{b}_0((\widetilde{W}(\mathbb{R}) \cap V(b^2 - \varepsilon)) \setminus K).$$

Proof. Consider the collection of connected components

$$\mathcal{C} = \{C \in \mathrm{cc}(\mathbb{R}^4 \setminus K) \mid C \cap G(W,H) \cap H \neq \emptyset\}.$$

For each $C \in \mathcal{C}$ choose a point $x_C \in C \cap G(W,H) \cap H$. Since $\mathcal{C}$ is a finite set, the set of points $\{x_C\}_{C\in\mathcal{C}}$ is also finite. For each $C \in \mathcal{C}$ and $r > 0$, consider also the semi-algebraic set given by

$$U_r(x_C) = \mathrm{B}(x_C, r) \cap (W(\mathbb{R}) \setminus K).$$

By the definition of $G(W,H)$, $U_r(x_c)$ is not contained in H. Semi-algebraic sets are locally contractible because of their local conical structure, see for instance [BPR06, Theorem 5.48]. Hence, there exists $r_C > 0$ such that, for all $0 < r \leq r_C$, the semi-algebraic set $U_r(x_C)$ is contractible and, in particular, connected. Set $r_0 = \min_C r_C$.

Choose also $y_C \in U_{r_0}(x_C) \setminus H$ and a semi-algebraic path $\gamma_C : [0,1] \to U_r(\mathbf{x}_C)$ with $\gamma(0) = x_C$ and $\gamma(1) = y_C$. We have that $b^2(y_C) > 0$ because $y_C \notin H$. We set $\varepsilon_0 = \min_C b^2(y_C)$.

By the intermediate value theorem, for all $C \in \mathcal{C}$ and $0 < \varepsilon \leq \varepsilon_0$, there exists $0 < t_C \leq 1$ such that $b^2(z_C) = \varepsilon$ with $z_C = \gamma_C(t_C)$. By construction,

$$z_C \in (W(\mathbb{R}) \cap V(b^2 - \varepsilon)) \setminus K \subset (\widetilde{W}(\mathbb{R}) \cap V(b^2 - \varepsilon)) \setminus K.$$

Moreover, $z_C \in C$ because this point is connected by a path to x_C. For $C, C' \in \mathcal{C}$ with $C \neq C'$, the points z_C and $z_{C'}$ belong to distinct connected components of $\widetilde{W}(\mathbb{R}) \cap V(b^2 - \varepsilon) \setminus K$. Hence, the map $C \mapsto z_C$ induces an injection between the collections of connected components $\mathcal{C}$ and $\mathrm{cc}(\widetilde{W}(\mathbb{R}) \cap V(b^2 - \varepsilon)) \setminus K$, which proves the proposition. □

To apply our polynomial partitioning theorem on a variety in higher dimensions, one also needs to couple it with a bound for the number of connected components depending on the degree of that variety, instead of the Bézout number of a set of defining equations. In this direction, we propose the following conjecture.

Conjecture 2.7. *Let $X \subset \mathbb{C}^d$ be an irreducible variety of dimension e and locally set-theoretically defined at degree $\delta \geq 1$. Let $g \in \mathbb{R}[x_1, \dots, x_d]$ be a polynomial of degree $\ell \geq \delta$. Then there exists a subvariety $Y \subset \mathbb{C}^d$ containing X as an irreducible component such that*

$$\mathrm{b}_0(Y(\mathbb{R}) \setminus V(g)) \leq c \deg(X) \ell^e$$

for a constant $c = c(d)$.

If this statement turns out to be true, it would be an important step in proving Conjecture 4.1 on the number of point-hypersurface incidences in higher dimensions.

3. Partitioning finite sets on varieties

In this section, we state (and prove) the partitioning theorem on a variety in terms of sign conditions.

Given a subset $\mathcal{P} \subset \mathbb{R}^d$, a finite set of polynomials $\mathcal{G} \subset \mathbb{R}[x_1, \dots, x_d]$ and a choice of signs $\varepsilon \in \{\pm 1\}^{\mathcal{G}}$, we consider the subset of $\mathcal{P}$ defined by

$$\mathcal{P}(\varepsilon) = \{p \in \mathcal{P} \mid \varepsilon_q q(p) > 0 \text{ for all } q \in \mathcal{G}\}. \tag{3.1}$$

Given a nonzero polynomial $g \in \mathbb{R}[x_1, \dots, x_d]$, we denote by $\mathrm{irr}(g) \subset \mathbb{R}[x_1, \dots, x_d]$ a complete and irredundant set of irreducible factors of g. These irreducible factors are unique up to non-zero scalars factors in $\mathbb{R}^\times$. To fix their indeterminacy, we choose them to be monic with respect to some fixed monomial order on $\mathbb{R}[x_1, \dots, x_d]$. With this convention, the set $\mathrm{irr}(g)$ is uniquely defined and

$$g = \lambda \prod_{q \in \mathrm{irr}(g)} q^{e_q}$$

with $\lambda \in \mathbb{R}^\times$ and $e_q \in \mathbb{N}$.

We denote by $\mathbb{R}[x_1, \dots, x_d]_{\leq \ell}$ the subset of $\mathbb{R}[x_1, \dots, x_d]$ of polynomials of degree bounded by ℓ. Recall that, for a subvariety $X \subset \mathbb{C}^d$, we denote by $X(\mathbb{R}) = X \cap \mathbb{R}^d$ its set of real points.

Theorem 3.1. *There is a constant $c = c(d)$ with the following property. Let $X \subset \mathbb{C}^d$ be an irreducible variety of dimension e and locally set-theoretically defined at degree $\delta \geq 1$. Let $\mathcal{P} \subset X(\mathbb{R})$ be a finite subset and $\ell \geq c\delta$. Then there exists $g \in \mathbb{R}[x_1, \dots, x_d]_{\leq \ell}$ such that $\dim(X \cap V(g)) = e - 1$ and, for each $\varepsilon \in \{\pm 1\}^{\mathrm{irr}(g)}$,*

$$\mathrm{card}(\mathcal{P}(\varepsilon)) = O_d\Big(\frac{\mathrm{card}(\mathcal{P})}{\deg(X)\ell^e}\Big).$$

Remark 3.2. For an arbitrary subset $S \subset \mathbb{R}^d$ containing $\mathcal{P}$, each set of the form $\mathcal{P}(\varepsilon)$ is necessarily contained in a connected component of S. Hence, the number of nonempty sets defined by sign conditions is bounded by the number of connected components of $S \backslash V(g)$. Hence, Theorem 1.2 in the introduction follows immediately from Theorem 3.1 above.

Given $\ell \in \mathbb{N}$, we denote by v_ℓ the Veronese embedding $\mathbb{C}^d \hookrightarrow \mathbb{C}^{\binom{\ell+d}{d}-1}$ given, for a point $\boldsymbol{p} = (p_1, \dots, p_d) \in \mathbb{C}^d$, by

$$v_\ell(\boldsymbol{p}) = (\boldsymbol{p}^{\boldsymbol{a}})_{\boldsymbol{a}} \tag{3.2}$$

where $\boldsymbol{a} = (a_1, \dots, a_d) \in \mathbb{N}^d$ runs over all nonzero vectors of length $|\boldsymbol{a}| = \sum_i a_i$ bounded by ℓ, and where $\boldsymbol{p}^{\boldsymbol{a}}$ denotes the monomial $p_1^{a_1} \dots p_d^{a_d}$. We also denote by ι the standard inclusion $\mathbb{C}^d \to \mathbb{P}^d(\mathbb{C})$ given by

$$\iota(\boldsymbol{p}) = (1 : p_1 : \dots : p_d).$$

For a subset $E \subset \mathbb{R}^d$, we denote by $\mathrm{aff}(E)$ the smallest affine subspace of $\mathbb{R}^d$ containing E. We also denote by $I(\iota(E)) \subset \mathbb{C}[z_0, \dots, z_d]$ the homogeneous ideal of polynomials vanishing identically on the subset $\iota(E) \subset \mathbb{P}^d(\mathbb{C})$.

Lemma 3.3. *Let $E \subset \mathbb{R}^d$ be a subset. Then $\dim_{\mathbb{R}}(\mathrm{aff}(v_\ell(E))) = \mathrm{H}_{I(\iota(E))}(\ell) - 1$.*

Proof. Consider the vector space $\mathbb{R}^{\binom{\ell+d}{d}}$ with coordinates indexed by the vectors of $\mathbb{N}^{d+1}$ of length ℓ. Consider also the pairing

$$\mathbb{R}[z_0, \dots, z_d]_\ell \times \mathbb{R}^{\binom{\ell+d}{d}} \longrightarrow \mathbb{R}, \quad \Big(\sum_{|\boldsymbol{b}|=\ell} \alpha_{\boldsymbol{b}} \boldsymbol{z}^{\boldsymbol{b}}, \boldsymbol{w}\Big) \longmapsto \sum_{\boldsymbol{b}} \alpha_{\boldsymbol{b}} \boldsymbol{w}_{\boldsymbol{b}},$$

where $\boldsymbol{b}$ runs over all vectors of $\mathbb{N}^{d+1}$ of length ℓ.

Set $I = I(\iota(E)) \cap \mathbb{R}[z_0, \dots, z_d]$. The graded part I_ℓ coincides with the annihilator of $\{1\} \times v_\ell(E)$ or, equivalently, with the annihilator of the linear span in $\mathbb{R}^{\binom{\ell+d}{d}}$ of this subset, denoted $\mathrm{lin}(\{1\} \times v_\ell(E))$.

Since the pairing is nondegenerate, the dimension of these linear spaces is complementary. Moreover, $\{1\} \times \mathrm{aff}(v_\ell(E)) = \mathrm{lin}(\{1\} \times v_\ell(E)) \cap V(w_{\mathbf{0}} - 1)$. Hence,

$$\mathrm{H}_{I(\iota(E))}(\ell) = \dim_{\mathbb{R}}(\mathbb{R}[\boldsymbol{z}]_\ell) - \dim_{\mathbb{R}}(I_\ell) \\ = \dim_{\mathbb{R}}(\mathrm{lin}(\{1\} \times v_\ell(E))) = \dim_{\mathbb{R}}(\mathrm{aff}(v_\ell(E))) + 1,$$

which proves the result. □

Proof of Theorem 3.1. By (2.1), we can suppose without loss of generality that $\delta \leq \deg(X)$. For $i = 0, \dots, \log_2(\deg(X)\ell^e)$ set

$$l_i = \begin{cases} c_1^{-1} 2^{\frac{i}{e+1}} & \text{if } i < (e+1)\log(2c_1(d-e)\delta), \\ c_1^{-1}\Big(\dfrac{2^i}{\deg(X)}\Big)^{\frac{1}{e}} & \text{if } i \geq (e+1)\log(2c_1(d-e)\delta), \end{cases}$$

with $c_1 = c_1(d)$ as in Corollary 2.3.

Let v_{l_i} be the Veronese map of degree l_i as in (3.2) and set $A_i \subset \mathbb{R}^{\binom{l_i+e}{e}} - 1$ for the affine hull of the image of $X(\mathbb{R})$ under v_{l_i}. By Lemma 3.3,

$$\dim_{\mathbb{R}}(A_i) = \mathrm{H}_{I(\iota(X(\mathbb{R})))}(l_i) - 1, \tag{3.3}$$

where $I(\iota(X(\mathbb{R})))$ is the ideal of polynomials vanishing on the image under ι of the set of real points of X. Since $\iota(X(\mathbb{R})) \subset \iota(X)$, we have that $I(\iota(X(\mathbb{R}))) \supset I(\iota(X))$ and so

$$\mathrm{H}_{\overline{\iota(X(\mathbb{R}))}}(l_i) \leq \mathrm{H}_{\iota(X)}(l_i). \tag{3.4}$$

We consider first the case when (3.4) is an equality for all i. Since the affine variety X is irreducible and locally set-theoretically defined at degree δ, the same holds for the projective variety $\overline{\iota(X)}$. Hence, by Corollary 2.3, there is a constant $c_1 = c_1(d)$ such that

$$\mathrm{H}_{I(\overline{\iota(X(\mathbb{R}))})}(l_i) - 1 \geq \begin{cases} c_1 l_i^{e+1} & \text{if } l_i \leq 2(d-e)\delta - 1, \\ c_1 \deg(X) l_i^e & \text{if } l_i \geq 2(d-e)\delta. \end{cases} \tag{3.5}$$

Hence $\mathrm{H}_{I(\overline{\iota(X(\mathbb{R}))})}(l_i) - 1 \geq 2^i$.

As in the Guth-Katz polynomial partitioning theorem in [GK10], we will inductively subdivide the set of points $\mathcal{P}$. We start with $\mathcal{C}_0 = \{\mathcal{P}\}$. Having constructed $\mathcal{C}_i$ with at most 2^i sets, we apply the ham sandwich theorem to the image of these sets under the map v_{l_i}. By (3.3) and (3.5), these images lie in A_i, which is an affine space of dimension $\geq 2^i$. Hence, there is a nonzero linear form on A_i that bisects each of these images or, equivalently, there is a polynomial $g_i \in \mathbb{R}[x_1, \dots, x_d]_{\leq l_i}$ that bisects each of the sets in $\mathcal{C}_i$.

For each $\mathcal{Q} \in \mathcal{C}_i$, we put $\mathcal{Q}^+$ and $\mathcal{Q}^-$ for the sets of points of $\mathcal{Q}$ at which $g_i > 0$ and $g_i < 0$, respectively. We then put

$$\mathcal{C}_{i+1} = \bigcup_{\mathcal{Q} \in \mathcal{C}_i} \{\mathcal{Q}^+, \mathcal{Q}^-\}.$$

Each of the sets in $\mathcal{C}_t$ has size at most

$$\frac{\mathrm{card}(\mathcal{P})}{2^t} = \frac{\mathrm{card}(\mathcal{P})}{2^t} \leq c_2 \frac{\mathrm{card}(\mathcal{P})}{\deg(X)\ell^e}$$

for a constant $c_2 = c_2(d)$.

We set $g = \prod_{i=0}^{t} g_i$. Hence, the collection of sets $\mathcal{C}_t$ is defined by the sign conditions given by the g_i's and, *a fortiori*, by $\mathrm{irr}(g)$. Moreover, the sets in $\mathcal{C}_t$ verify the bound in the statement.

It remains to bound the degree of g. Set $s = (e+1)\log(2c_1(d-e)\delta)$ and $t = \log_2(\deg(X)\ell^e)$. Then

$$\begin{aligned}
\deg(g) &= \sum_{i=0}^{t} \deg(g_i) \\
&\leq \sum_{i=0}^{s-1} c_1^{-1} 2^{\frac{i}{e+1}} + \sum_{i=s}^{t} c_1^{-1} \Big(\frac{2^i}{\deg(X)}\Big)^{\frac{1}{e}} \\
&\leq c_3 \Big(2^{\frac{s}{e+1}} + \Big(\frac{2^t}{\deg(X)}\Big)^{\frac{1}{e}} \Big) \\
&\leq c_4(\delta + \ell)
\end{aligned}$$

for a constant $c_4 = c_4(d)$, which implies the statement in this case.

Finally, consider the case when the inequality (3.4) is strict for some i. Then there exists $g_i \in I(\overline{X(\mathbb{R})}) \setminus I(X)$ with $\deg(g_i) \leq l_i$. Hence, g_i cuts X properly and contains its set of real points $X(\mathbb{R})$. In particular, $\mathcal{P} \subset V(g_i)$. It follows that $g = g_i$ has the appropriate degree and $\mathcal{P}(\varepsilon) = \emptyset$ for all $\varepsilon \in \{\pm 1\}^{\mathrm{irr}(g)}$. □

4. Point-hypersurface incidences

In this section we prove Theorem 1.5. To this end, we use three levels of polynomial partitioning. This leads to a stratification of the Euclidean space $\mathbb{R}^4$ into semi-algebraic pieces of various dimensions. We bound separately the number of incidences contributed by the points of the set $\mathcal{P}$ in each piece. The contribution from each level of the partitioning is the essentially same, up to constant factors, as the claimed bound.

Proof of Theorem 1.5. The procedure performed at each level is similar. For clarity and ease of exposition, we prefer to describe each of these level separately, even at the expense of repeating some of the arguments.

The *set of incidences* between $\mathcal{P}$ and $\mathcal{H}$ is the subset of $\mathcal{P} \times \mathcal{H}$ defined by

$$\mathcal{I}(\mathcal{P}, \mathcal{H}) = \{(p, H) \in \mathcal{P} \times \mathcal{H} \mid p \in H\}.$$

Hence $I(\mathcal{P}, \mathcal{H}) = \mathrm{card}(\mathcal{I}(\mathcal{P}, \mathcal{H}))$. For a subset $\mathcal{Q} \subset \mathcal{P}$, we denote by

$$\begin{aligned}
\mathcal{I}_{<k}(\mathcal{Q}, \mathcal{H}) &= \{(p, H) \in \mathcal{I}(\mathcal{Q}, \mathcal{H}) \mid \mathrm{card}(H \cap \mathcal{Q}) < k\}, \\
\mathcal{I}_{\geq k}(\mathcal{Q}, \mathcal{H}) &= \{(p, H) \in \mathcal{I}(\mathcal{Q}, \mathcal{H}) \mid \mathrm{card}(H \cap \mathcal{Q}) \geq k\}
\end{aligned}$$

the set of incidences between $\mathcal{Q}$ and hypersurfaces of $\mathcal{H}$ containing at most $k-1$ points of $\mathcal{Q}$ and at least k points of $\mathcal{Q}$, respectively. We also set $I_{<k}(\mathcal{Q}, \mathcal{H}) = \mathrm{card}(\mathcal{I}_{<k}(\mathcal{Q}, \mathcal{H}))$ and $I_{\geq k}(\mathcal{Q}, \mathcal{H}) = \mathrm{card}(\mathcal{I}_{\geq k}(\mathcal{Q}, \mathcal{H}))$. Clearly,

$$I(\mathcal{Q}, \mathcal{H}) = I_{<k}(\mathcal{Q}, \mathcal{H}) + I_{\geq k}(\mathcal{Q}, \mathcal{H}). \tag{4.1}$$

Henceforth, the dimension d of the ambient space is fixed to 4. Hence, all implicit constants in the O-notation depend only on the parameters k and c in the statement of the theorem. We denote all these implicit constants also by c, although this letter might refer to different constants along the proof.

First level partitioning. Let $D \geq 1$ to be fixed later on. By Theorem 3.1, there exists $f \in \mathbb{R}[x_1, x_2, x_3, x_4]_{\leq D} \setminus \{0\}$ such that, for each $\gamma \in \{\pm 1\}^{\mathrm{irr}(f)}$,

$$\mathrm{card}(\mathcal{P}(\gamma)) = O\Big(\frac{m}{D^4}\Big), \tag{4.2}$$

where $\mathcal{P}(\gamma)$ denotes the subset of $\mathcal{P}$ defined as in (3.1) by the choice of signs γ and the set of irreducible factors of f. Choose a minimal subset $\Sigma_1 \subset \{\pm 1\}^{\mathrm{irr}(f)}$ such that all nonempty subsets of the form $\mathcal{P}(\gamma)$ are realized by some element of Σ_1.

We partition $\mathcal{P}$ into the disjoint subsets $\mathcal{P}_0 = \mathcal{P} \cap V(f)(\mathbb{R})$ and $\mathcal{P}(\gamma)$, $\gamma \in \Sigma_1$. Set $m_0 = \mathrm{card}(\mathcal{P}_0)$ and $m_\gamma = \mathrm{card}(\mathcal{P}(\gamma))$ for each γ. Clearly,

$$m_0 + \sum_{\gamma \in \Sigma_1} m_\gamma = m. \tag{4.3}$$

We first bound the number of incidences with hypersurfaces that contain at least k points in one of the subsets $\mathcal{P}(\gamma)$. By the hypothesis (c), for each $\gamma \in \Sigma_1$ and a subset of k points of $\mathcal{P}(\gamma)$, there are at most c hypersurfaces in $\mathcal{H}$ containing these points. Hence,

$$I_{\geq k}(\mathcal{P}(\gamma), \mathcal{H}) \leq ck\binom{m_\gamma}{k} = O(m_\gamma^k). \tag{4.4}$$

The cardinality of Σ_1, that is, the number of nonempty subsets of the form $\mathcal{P}(\gamma)$, is bounded by $\mathrm{b}_0(\mathbb{R}^4 \setminus V(f))$, the number of connected components of $\mathbb{R}^4 \setminus V(f)$. By Theorem 2.5, this is bounded by $O(D^4)$. Together with (4.2) and (4.4), this implies that

$$\sum_\gamma I_{\geq k}(\mathcal{P}(\gamma), \mathcal{H}) = O\Big(\sum_\gamma \Big(\frac{m}{D^4}\Big)^k\Big) = O(m^k D^{4-4k}). \tag{4.5}$$

We now bound the number of incidences with hypersurfaces that contain at most $k-1$ points in every subset $\mathcal{P}(\gamma)$. For each $H \in \mathcal{H}$, the number of subsets $\mathcal{P}(\gamma)$ with nonempty intersection with H is bounded by $\mathrm{b}_0(H(\mathbb{R}) \setminus V(f))$. By Theorem 2.5, this number of connected components is bounded by $O(D^3)$, because the degree of H is bounded by a constant. Hence $\sum_\gamma I_{<k}(\mathcal{P}(\gamma), \{H\}) \leq (k-1)\,\mathrm{b}_0(H \setminus V(f)) = O(D^3)$. It follows that

$$\sum_\gamma I_{<k}(\mathcal{P}(\gamma), \mathcal{H}) = O(nD^3). \tag{4.6}$$

From (4.1), (4.5) and (4.6) we deduce that

$$I(\mathcal{P} \setminus \mathcal{P}_0, \mathcal{H}) = \sum_\gamma I(\mathcal{P}(\gamma), \mathcal{H}) = O(nD^3 + m^k D^{4-4k}). \tag{4.7}$$

We then set

$$D = \max\Big(1, \frac{m^{\alpha_1}}{n^{\beta_1}}\Big) \quad \text{with } \alpha_1 = \frac{k}{4k-1} \text{ and } \beta_1 = \frac{1}{4k-1}. \tag{4.8}$$

If $D = 1$, then $m^{\alpha_1} n^{-\beta_1} \leq 1$ or, equivalently, $m^k \leq n$. In this case, we deduce from (4.7) that $I(\mathcal{P} \setminus \mathcal{P}_0, \mathcal{H}) = O(n + m^k) = O(n)$. Otherwise,

$$I(\mathcal{P} \setminus \mathcal{P}_0, \mathcal{H}) = O(m^{3\alpha_1} n^{1-3\beta_1}) = O(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}}).$$

In either case,

$$I(\mathcal{P} \setminus \mathcal{P}_0, \mathcal{H}) = O(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + n). \tag{4.9}$$

Second level partitioning. Let $V(f) = \bigcup_{i \in I} V_i$ be the decomposition of the hypersurface $V(f)$ into irreducible components. Set $D_i = \deg(V_i)$ for each $i \in I$. Then

$$\sum_{i \in I} D_i = \deg(V(f)) \leq D. \tag{4.10}$$

We choose a partition of the finite set $\mathcal{P}_0 = \mathcal{P} \cap V(f)(\mathbb{R})$ into disjoint subsets $\mathcal{Q}_i$, $i \in I$, by assigning each point in $\mathcal{P}_0$ to one of the subsets $\mathcal{Q}_i$ corresponding to an irreducible component V_i it belongs to. Set $l_i = \mathrm{card}(\mathcal{Q}_i)$ for each $i \in I$. Then

$$\sum_i l_i = m_0. \tag{4.11}$$

Let $i \in I$ and choose $E_i \geq cD_i$, where c is the constant from Theorem 3.1 for $d = 4$. By Theorem 3.1, there exists $g_i \in \mathbb{R}[x_1, x_2, x_3, x_4]_{\leq E_i}$ such that $\dim(V_i \cap V(g_i)) = 2$ and, for each $\delta \in \{\pm 1\}^{\mathrm{irr}(g_i)}$,

$$\mathrm{card}(\mathcal{Q}_i(\delta)) = O\Big(\frac{l_i}{D_i E_i^3}\Big). \tag{4.12}$$

Choose a minimal subset $\Sigma_{2,i} \subset \{\pm 1\}^{\mathrm{irr}(g_i)}$ such that all nonempty subsets of the form $\mathcal{Q}_i(\delta)$ are realized by some element of $\Sigma_{2,i}$.

We partition $\mathcal{Q}_i$ into the disjoint subsets $\mathcal{Q}_{i,0} = \mathcal{Q}_i \cap (V_i \cap V(g_i))(\mathbb{R})$ and $\mathcal{Q}_i(\delta)$, $\delta \in \Sigma_{2,i}$. We set $l_{i,0} = \mathrm{card}(\mathcal{Q}_{i,0})$ and $l_{i,\delta} = \mathrm{card}(\mathcal{Q}_i(\delta))$ for each δ. Clearly,

$$l_{i,0} + \sum_{\delta \in \Sigma_{2,i}} l_{i,\delta} = l_i \quad \text{and} \quad \sum_i l_{i,0} = \mathrm{card}\Big(\mathcal{P} \cap \bigcup_i W_i\Big). \tag{4.13}$$

We follow the same approach as in the previous case, and we first bound the number of incidences with hypersurfaces that contain at least k points in some subset of the form $\mathcal{Q}_i(\delta)$. Similarly as in (4.4), the hypothesis (c) implies that, for each δ,

$$I_{\geq k}(\mathcal{Q}_i(\delta), \mathcal{H}) \leq ck \binom{l_{i,\delta}}{k} = O(l_{i,\delta}^k). \tag{4.14}$$

The cardinality of $\Sigma_{2,i}$ is bounded by $\mathrm{b}_0(V_i(\mathbb{R}) \setminus V(g_i))$ which, by Theorem 2.5, is bounded by $O(D_i E_i^3)$. Together with (4.12) and (4.14), this implies that

$$\sum_\delta I_{\geq k}(\mathcal{Q}_i(\delta), \mathcal{H}) = O\Big(\sum_\delta \Big(\frac{l_i}{D_i E_i^3}\Big)^k\Big) = O(l_i^k D_i^{1-k} E_i^{3-3k}). \tag{4.15}$$

We now bound the number of incidences with hypersurfaces that contain at most $k-1$ points in every $\mathcal{Q}_i(\delta)$. Let $H \in \mathcal{H}$ and, for the moment, suppose that $V_i \not\subset H$. We have that number of subsets of the form $\mathcal{Q}_i(\delta)$ with nonempty intersection with H is bounded by $\mathrm{b}_0((H \cap V_i)(\mathbb{R}) \setminus V(g_i))$. By Theorem 2.5, this number is bounded by $O(D_i E_i^2)$, since $\dim(V_i) = 3$ and either $H \cap V_i$ is empty or of dimension 2, and the degree of H is bounded by a constant.

If we set $\mathcal{H}_i$ for the set of hypersurfaces of $\mathcal{H}$ not containing V_i, then

$$\sum_\delta I_{<k}(\mathcal{Q}_i(\delta), \mathcal{H}_i) = O(nD_i E_i^2).$$

On the other hand, by the hypothesis (b), there can be at most 3 hypersurfaces $H \in \mathcal{H}$ containing V_i, and each of these hypersurfaces contains the l_i points of $\mathcal{Q}_i$. Hence

$$I_{<k}(\mathcal{Q}_i \setminus \mathcal{Q}_{i,0}, \mathcal{H} \setminus \mathcal{H}_i) \leq I(\mathcal{Q}_i, \mathcal{H} \setminus \mathcal{H}_i) \leq 3l_i. \tag{4.16}$$

By (4.15) and (4.16),

$$I(\mathcal{Q}_i \setminus \mathcal{Q}_{i,0}, \mathcal{H}) = \sum_{\delta} I(\mathcal{Q}_i(\delta), \mathcal{H}) = O(nD_iE_i^2 + l_i^k D_i^{1-k} E_i^{3-3k} + l_i). \tag{4.17}$$

We set

$$E_i = \max\Big(cD_i, \Big(\frac{l_i}{D_i}\Big)^{\alpha_2} \frac{1}{n^{\beta_2}}\Big) \quad \text{with } \alpha_2 = \frac{k}{3k-1} \text{ and } \beta_2 = \frac{1}{3k-1}. \tag{4.18}$$

If $E_i = cD_i$, then $(\frac{l_i}{D_i})^{\alpha_2} n^{-\beta_2} \le cD_i$. In this case, the first term in the right-hand side of (4.17) controls the second one. Otherwise, both terms are equal up to a constant factor. We deduce from (4.17) that

$$I(\mathcal{Q}_i \setminus \mathcal{Q}_{i,0}, \mathcal{H}) = \begin{cases} O(nD_i^3 + l_i) & \text{if } E_i = cD_i, \\ O(l_i^{2\alpha_2} D_i^{1-2\alpha_2} n^{1-2\beta_2} + l_i) & \text{otherwise.} \end{cases} \tag{4.19}$$

By (4.10), we have that

$$\sum_i nD_i^3 \le nD^3 = O(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + n), \tag{4.20}$$

as the term nD^3 appears in (4.7) and is accounted for in (4.9). Using the Hölder inequality as well as (4.10) and (4.11),

$$\begin{aligned} \sum_i l_i^{2\alpha_2} D_i^{1-2\alpha_2} n^{1-2\beta_2} &\le n^{1-2\beta_2} \Big(\sum_i l_i\Big)^{2\alpha_2} \Big(\sum_i D_i\Big)^{1-2\alpha_2} \\ &\le n^{1-2\beta_2} m_0^{2\alpha_2} D^{1-2\alpha_2} \end{aligned} \tag{4.21}$$

We now substitute the value of D from (4.8) and those of $\alpha_1, \alpha_2, \beta_1$ and β_2 in the above expression. If $D = 1$, then $m^k \le n$ and so $n^{1-2\beta_2} m_0^{2\alpha_2} D^{1-2\alpha_2} = n^{1-2\beta_2} m_0^{2\alpha_2} \le n$. Otherwise,

$$n^{1-2\beta_2} m_0^{2\alpha_2} D^{1-2\alpha_2} \le n^{1-2\beta_2} m^{2\alpha_2} (m^{\alpha_1} n^{-\beta_1})^{1-2\alpha_2} = m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}}. \tag{4.22}$$

It follows from (4.19), (4.20), (4.21), (4.22) and (4.11) that

$$\begin{aligned} I\Big(\mathcal{P}_0 \setminus \bigcup_i \mathcal{Q}_{i,0}, \mathcal{H}\Big) &= \sum_i I(\mathcal{Q}_i \setminus \mathcal{Q}_{i,0}, \mathcal{H}) \\ &= O\Big(\sum_i nD_i^3 + \sum_i n^{1-2\beta_2} l_i^{2\alpha_2} D_i^{1-2\alpha_2} + \sum_i l_i\Big) \\ &= O(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + n + m_0). \end{aligned} \tag{4.23}$$

Third partitioning polynomials. For each $i \in I$, consider the surface $W_i = V_i \cap V(g_i) = V(f_i, g_i)$ and let $W_i = \bigcup_{j \in J_i} W_{i,j}$ be its decomposition into irreducible components. Set $\Delta_{i,j} = \deg(W_{i,j})$ for each j. By Bézout theorem,

$$\sum_{j \in J_i} \Delta_{i,j} = \deg(W_i) \le D_i E_i. \tag{4.24}$$

We denote by $W_i(\mathbb{R})_0$ and $W_{i,j}(\mathbb{R})_0$ the set of isolated points of $W_i(\mathbb{R})$ and of $W_{i,j}(\mathbb{R})$, respectively. We then choose an arbitrary partition of the set $\mathcal{Q}_{i,0} = \mathcal{Q}_i \cap W_i$ into disjoints subsets $\mathcal{R}_{i,j}$, $j \in J_i$, such that

$$\mathcal{R}_{i,j} \subset W_{i,j}(\mathbb{R}) \quad \text{and} \quad \mathcal{R}_{i,j} \cap W_{i,j}(\mathbb{R})_0 \subset W_i(\mathbb{R})_0. \tag{4.25}$$

Set $e_{i,j} = \mathrm{card}(\mathcal{R}_{i,j})$ for each j. Then

$$\sum_j e_{i,j} = l_{i,0}. \tag{4.26}$$

For each $j \in J_i$, the variety $W_{i,j}$ is locally set-theoretically defined at degree E_i. Let $F_{i,j} \geq c \max(D_i, E_i)$, where c is the constant from Theorem 3.1 for $d = 4$. By Theorem 3.1, there exists $h_{i,j} \in \mathbb{R}[x_1, x_2, x_3, x_4]_{\leq F_{i,j}}$ such that $\dim(W_{i,j} \cap V(h_{i,j})) = 1$ and, for each $\eta \in \{\pm 1\}^{\mathrm{irr}(h_{i,j})}$,

$$\mathrm{card}(\mathcal{R}_{i,j}(\eta)) = O\Big(\frac{e_{i,j}}{\Delta_{i,j} F_{i,j}^2}\Big). \tag{4.27}$$

Similarly as before, choose a minimal subset $\Sigma_{3,i,j} \subset \{\pm 1\}^{\mathrm{irr}(h_{i,j})}$ such that all nonempty subsets of the form $\mathcal{R}_{i,j}(\eta)$ are realized by some element of $\Sigma_{3,i,j}$.

We further consider a partition of $\mathcal{R}_{i,j}$ into the disjoint subsets $\mathcal{R}_{i,j,0} = \mathcal{R}_{i,j} \cap (W_{i,j} \cap V(h_{i,j}))(\mathbb{R})$ and $\mathcal{R}_{i,j,\eta} = \mathcal{R}_{i,j}(\eta)$, $\eta \in \Sigma_{3,i,j}$. Set also $e_{i,j,0} = \mathrm{card}(\mathcal{R}_{i,j,0})$ and $e_{i,j,\eta} = \mathrm{card}(\mathcal{R}_{i,j}(\eta))$ for each η. Hence,

$$e_{i,j,0} + \sum_{\eta \in \Sigma_{3,i,j}} e_{i,j,\eta} = e_{i,j}. \tag{4.28}$$

We first bound the number of incidences of $\mathcal{R}_{i,j} \setminus \mathcal{R}_{i,j,0}$ with hypersurfaces that contain at least k points in some subset of the form $\mathcal{R}_{i,j}(\eta)$. Similarly as for (4.4), the hypothesis (c) implies that, for each η,

$$I_{\geq k}(\mathcal{R}_{i,j}(\eta), \mathcal{H}) \leq ck \binom{e_{i,j,\eta}}{k} = O(e_{i,j,\eta}^k). \tag{4.29}$$

By Proposition 2.4, there exist coprime polynomials $\widetilde{f}_{i,j}, \widetilde{g}_{i,j} \in \mathbb{R}[x_1, x_2, x_3, x_4]$ such that $W_{i,j}$ is an irreducible component of the variety $\widetilde{W}_{i,j} = V(\widetilde{f}_{i,j}, \widetilde{g}_{i,j})$ and

$$\deg(\widetilde{f}_{i,j}) \deg(\widetilde{g}_{i,j}) = O(\Delta_{i,j}). \tag{4.30}$$

Furthermore, we can assume that $\deg(\widetilde{f}_i), \deg(\widetilde{g}_i) \leq E_i$ because $W_{i,j}$ is locally set-theoretically defined at degree E_i.

The number of nonempty subsets of the form $\mathcal{R}_{i,j}(\eta)$ is bounded by the number of connected components of $\widetilde{W}_{i,j}(\mathbb{R}) \setminus V(h_{i,j})$, as explained in Remark 3.2. By Theorem 2.5 and (4.30),

$$\mathrm{b}_0(\widetilde{W}_{i,j}(\mathbb{R}) \setminus V(h_{i,j})) = O(\deg(\widetilde{f}_{i,j}) \deg(\widetilde{g}_{i,j}) F_{i,j}^2) = O(\Delta_{i,j} F_{i,j}^2). \tag{4.31}$$

By (4.29), (4.27), and (4.31),

$$\sum_\eta I_{\geq k}(\mathcal{R}_{i,j}(\eta), \mathcal{H}) = O\Big(\sum_\eta \Big(\frac{e_{i,j}}{\Delta_{i,j} F_{i,j}^2}\Big)^k\Big) = O(e_{i,j}^k \Delta_{i,j}^{1-k} F_{i,j}^{2-2k}). \tag{4.32}$$

We now bound the number of incidences of $\mathcal{R}_{i,j} \setminus \mathcal{R}_{i,j,0}$ with hypersurfaces that contain at most $k-1$ points in a subset of the form $\mathcal{R}_{i,j}(\eta)$.

Given $H \in \mathcal{H}$, we denote by $B_{i,j}(H) \subset W_{i,j}(\mathbb{R})$ the semi-algebraic subset of points $p \in W_{i,j}(\mathbb{R})$ having an open neighborhood, in the Euclidean topology of $W_{i,j}(\mathbb{R})$, which is contained in H. We also set $G_{i,j}(H) = W_{i,j}(\mathbb{R}) \setminus B_{i,j}(H)$. Notice that $W_{i,j}(\mathbb{R})_0 \cap H \subset B_{i,j}(H)$.

For any finite subset $\mathcal{R} \subset W_{i,j}(\mathbb{R})$ we set

$$\mathcal{I}^{\mathrm{B}}(\mathcal{R}, \mathcal{H}) = \{(p, H) \in \mathcal{I}(\mathcal{R}, \mathcal{H}) \mid p \in B_{i,j}(H)\} \quad , \quad \mathcal{I}^{\mathrm{G}}(\mathcal{R}, \mathcal{H}) = \mathcal{I}(\mathcal{R}, \mathcal{H}) \setminus \mathcal{I}^{\mathrm{B}}(\mathcal{R}, \mathcal{H}).$$

We also set $I^{\mathrm{B}}(\mathcal{R},\mathcal{H}) = \mathrm{card}(\mathcal{I}^{\mathrm{B}}(\mathcal{R},\mathcal{H}))$ and $I^{\mathrm{G}}(\mathcal{R},\mathcal{H}) = \mathrm{card}(\mathcal{I}^{\mathrm{G}}(\mathcal{R},\mathcal{H}))$. Clearly,

$$I(\mathcal{R},\mathcal{H}) = I^{\mathrm{B}}(\mathcal{R},\mathcal{H}) + I^{\mathrm{G}}(\mathcal{R},\mathcal{H}).$$

We first treat the incidences in $G_{i,j}(H)$. Write $H = V(b)$ with $b \in \mathbb{R}[x_1, x_2, x_3, x_4]$. The number of nonempty subsets of the form $\mathcal{R}_{i,j}(\eta) \cap G_{i,j}(H)$ is bounded by the number of connected components of $\mathbb{R}^4 \setminus V(h_{i,j})$ having nonempty intersection with $H \cap G_{i,j}(H)$. By Proposition 2.6, this number of connected components is bounded by the number of connected components of the semi-algebraic set

$$(\widetilde{W}_{i,j} \cap V(b^2-\varepsilon))(\mathbb{R}) \setminus V(h_{i,j}) = V(b^2-\varepsilon, \widetilde{f}_{i,j}, \widetilde{g}_{i,j})(\mathbb{R}) \setminus V(h_{i,j}) \tag{4.33}$$

for any $\varepsilon > 0$ sufficiently small. For a generic choice of ε, we also have that $\dim(\widetilde{W}_{i,j} \cap V(b^2-\varepsilon)) = 1$. For any such a choice of ε, by Theorem 2.5 and (4.30), the number of connected components in the semi-algebraic set in (4.33) is bounded by

$$O(\deg(b^2-\varepsilon)\deg(\widetilde{f}_{i,j})\deg(\widetilde{g}_{i,j})\deg(h_{i,j})) = O(\Delta_{i,j}F_{i,j}).$$

Thus

$$\sum_{\eta} I^{\mathrm{G}}_{<k}(\mathcal{R}_{i,j}(\eta),\mathcal{H})_{<k} = O(n\Delta_{i,j}F_{i,j}). \tag{4.34}$$

Gathering together (4.32) and (4.34) we obtain that

$$\begin{aligned} I^{\mathrm{G}}(\mathcal{R}_{i,j} \setminus \mathcal{R}_{i,j,0},\mathcal{H}) &= \sum_{\eta} I^{\mathrm{G}}(\mathcal{R}_{i,j}(\eta),\mathcal{H}) \\ &= O\big(n\Delta_{i,j}F_{i,j} + e_{i,j}^k \Delta_{i,j}^{1-k} F_{i,j}^{2-2k}\big). \end{aligned} \tag{4.35}$$

We set

$$F_{i,j} = \max\Big(cE_i, \Big(\frac{e_{i,j}}{\Delta_{i,j}}\Big)^{\alpha_3}\frac{1}{n^{\beta_3}}\Big) \quad \text{with } \alpha_3 = \frac{k}{2k-1} \text{ and } \beta_3 = \frac{1}{2k-1}. \tag{4.36}$$

If $F_{i,j} = cE_i$, then $(\frac{e_{i,j}}{\Delta_{i,j}})^{\alpha_3} n^{-\beta_3} \le cE_i$. In this case, the first term in the right-hand side of (4.35) controls the second one and otherwise, both terms are equal up to a constant factor. Hence, we deduce that

$$I^{\mathrm{G}}(\mathcal{R}_{i,j} \setminus \mathcal{R}_{i,j,0},\mathcal{H}) = \begin{cases} O(n\Delta_{i,j}E_i) & \text{if } F_{i,j} = cE_i, \\ O(n^{1-\beta_3}e_{i,j}^{\alpha_3}\Delta_{i,j}^{1-\alpha_3}) & \text{otherwise.} \end{cases} \tag{4.37}$$

By (4.10) and Bézout theorem,

$$\sum_{i,j} n\Delta_{i,j}E_i \le \sum_i nD_iE_i^2 = O(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + n), \tag{4.38}$$

as shown when passing from (4.17) to (4.23). Else, applying the Hölder inequality together with (4.26) and (4.10),

$$\begin{aligned} \sum_{i,j} n^{1-\beta_3} e_{i,j}^{\alpha_3}\Delta_{i,j}^{1-\alpha_3} &\le n^{1-\beta_3}\Big(\sum_{i,j} e_{i,j}\Big)^{\alpha_3}\Big(\sum_{i,j}\Delta_{i,j}\Big)^{1-\alpha_3} \\ &\le n^{1-\beta_3} m^{\alpha_3}\Big(\sum_i D_iE_i\Big)^{1-\alpha_3}. \end{aligned} \tag{4.39}$$

Recall that $E_i = \max\big(cD_i, \big(\frac{l_i}{D_i}\big)^{\alpha_2} n^{-\beta_2}\big)$ as in (4.18). Hence

$$\begin{aligned}\sum_i D_i E_i &\le c\sum_i D_i^2 + n^{-\beta_2}\sum_i {l_i}^{\alpha_2} {D_i}^{1-\alpha_2}\\ &\le c\sum_i D_i^2 + n^{-\beta_2}\Big(\sum_i l_i\Big)^{\alpha_2}\Big(\sum_i D_i\Big)^{1-\alpha_2}\\ &\le cD^2 + n^{-\beta_2} m^{\alpha_2} D^{1-\alpha_2}. \end{aligned} \tag{4.40}$$

Recall also that $D = \max\big(1, m^{\alpha_1} n^{-\beta_1}\big)$ as in (4.8). If $D = 1$, then $m^{\alpha_1} n^{-\beta_1} \le 1$ or, equivalently, $m^k \le n$. In this case $\sum_i D_i E_i = O(1)$. Otherwise, substituting $D = m^{\alpha_1} n^{-\beta_1}$ in (4.40) and the sum $\sum_i D_i E_i$ into (4.39),

$$\begin{aligned} n^{1-\beta_3} m^{\alpha_3}\Big(\sum_i D_i E_i\Big)^{1-\alpha_3} &= O\big(n^{1-\beta_3} m^{\alpha_3}(n^{-\beta_2} m^{\alpha_2}(m^{\alpha_1} n^{-\beta_1})^{1-\alpha_2})^{1-\alpha_3}\big)\\ &= O\big(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}}\big). \end{aligned} \tag{4.41}$$

It follows from (4.37), (4.38), (4.39), (4.40) and(4.41), that

$$\begin{aligned} I^{\mathrm{G}}\Big(\bigcup_i \mathcal{Q}_{i,0} \setminus \bigcup_{i,j} \mathcal{R}_{i,j,0}, \mathcal{H}\Big) &= \sum_{i,j} I^{\mathrm{G}}(\mathcal{R}_{i,j} \setminus \mathcal{R}_{i,j,0}, \mathcal{H})\\ &= O\Big(\sum_{i,j}\Big(n\Delta_{i,j} F_{i,j} + e_{i,j}^k \Delta_{i,j}^{1-k} F_{i,j}^{2-2k}\Big)\Big) \qquad (4.42)\\ &= O\big(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + n\big). \qquad (4.43) \end{aligned}$$

Finally, we treat the incidences in $B_{i,j}(H)$. We first claim that for each $p \in \mathcal{R}_{i,j} \setminus W_i(\mathbb{R})_0$ there are at most 3 hypersurfaces in $\mathcal{H}$ such that $p \in B_{i,j}(H)$. To see this, observe that $p \in B_{i,j}(H)$ implies that H contains an open neighborhood $U \subset W_{i,j}(\mathbb{R})$ of p. Since p is not an isolated point of $W_{i,j}(\mathbb{R})$, if follows that U is of real dimension at least 1. The hypothesis (b) then implies that there are at most 3 such hypersurfaces. Hence,

$$I^{\mathrm{B}}(\mathcal{R}_{i,j} \setminus W_i(\mathbb{R})_0, \mathcal{H}) \le 3e_{i,j}. \tag{4.44}$$

The incidences of $\mathcal{Q}_i$ with hypersurfaces $H \in \mathcal{H}$ containing V_i are already accounted for in (4.16). Hence, we can suppose that V_i is not contained in H. In this case, by Theorem 2.5, $\mathrm{card}(H \cap W_i(\mathbb{R})_0) \le \mathrm{b}_0(V(b, f_i) \cap V(g_i)) = O(D_i E_i^2)$. Together with (4.16), this implies that

$$\sum_j I^{\mathrm{B}}(\mathcal{R}_{i,j} \cap W_i(\mathbb{R})_0, \mathcal{H}) = O(nD_i E_i^2 + l_i) \tag{4.45}$$

It follows from (4.44), (4.26) and (4.45) that

$$\sum_j I^{\mathrm{B}}(\mathcal{R}_{i,j}, \mathcal{H}) = O(nD_i E_i^2 + l_i).$$

This bound already appears in (4.17). The contribution of the sum of these terms over $i \in I$ is accounted for in (4.43), and can be absorbed into the bound (4.43), after adding the term m.

Curves and conclusion of the proof. Finally, we bound the number of incidences that occur on the curves $Y_{i,j} = W_{i,j} \cap V(h_{i,j})$.

For each i, j, set $\mathcal{R}_{i,j,0} = \mathcal{R}_{i,j} \cap Y_{i,j}(\mathbb{R})$. Let $Y_{i,j} = \bigcup_{l \in L_{i,j}} Y_{i,j,l}$ be the decomposition of $Y_{i,j}$ into irreducible components, and consider an arbitrary partition of $\mathcal{R}_{i,j,0}$ into disjoint subsets $\mathcal{S}_{i,j,l}$, $l \in L_{i,j}$, with $\mathcal{S}_{i,j,l} \subset Y_{i,j,l}(\mathbb{R})$ for all l.

Let $l \in L_{i,j}$ and $H \in \mathcal{H}$. If $Y_{i,j,l}$ is not contained in H, then the number of incidences between $\mathcal{S}_{i,j,l}$ and this hypersurface is bounded by $\mathrm{card}(Y_{i,j,l} \cap H)$. From Bézout theorem, we deduce that

$$I(\mathcal{S}_{i,j,l}, \{H\}) \leq \begin{cases} \deg(H)\deg(Y_{i,j,l}) & \text{if } Y_{i,j,l} \not\subset H, \\ \mathrm{card}(\mathcal{S}_{i,j,l}) & \text{if } Y_{i,j,l} \subset H. \end{cases} \tag{4.46}$$

The hypothesis (b) implies that, for each l, there are at most 3 hypersurfaces in $\mathcal{H}$ containing $Y_{i,j,l}$. It follows from (4.46) that

$$\begin{aligned} I(\mathcal{R}_{i,j,0}, \mathcal{H}) &= \sum_{l \in L_{i,j}} \sum_{H \in \mathcal{H}} I(\mathcal{S}_{i,j,l}, \{H\}) \\ &= O\Big(\sum_{l,H} \deg(Y_{i,j,l})\Big) + 3\sum_{l} \mathrm{card}(\mathcal{S}_{i,j,l}) \\ &= O(n \deg(Y_{i,j}) + \mathrm{card}(\mathcal{R}_{i,j,0})). \end{aligned} \tag{4.47}$$

By Bézout theorem, $\deg(Y_{i,j}) \leq \Delta_{i,j} F_{i,j}$. Using (4.47),

$$I\Big(\mathcal{P} \cap \bigcup_{i,j} Y_{i,j}, \mathcal{H}\Big) = \sum_{i,j} I(\mathcal{R}_{i,j,0}, \mathcal{H}) = O\Big(n \sum_{i,j} \Delta_{i,j} F_{i,j} + \sum_{i,j} \mathrm{card}(\mathcal{R}_{i,j,0})\Big). \tag{4.48}$$

The first sum in the right-hand side of (4.48) appears in (4.42) and is already accounted for in (4.43). By construction, the family of sets $\{\mathcal{R}_{i,j,0}\}_{i,j}$ is a partition of $\mathcal{P} \cap \bigcup_{i,j} Y_{i,j}$. Therefore, the sum of their cardinalities is bounded by m. Hence

$$I\Big(\mathcal{P} \cap \bigcup_{i,j} Y_{i,j}, \mathcal{H}\Big) = O\big(m^{1-\frac{k-1}{4k-1}} n^{1-\frac{3}{4k-1}} + m + n\big). \tag{4.49}$$

The statement now follows by summing up the contributions from (4.9), (4.23), (4.43) and (4.49). ☐

We close this paper by proposing the next conjecture for the number of point-hypersurface incidences in $\mathbb{R}^d$.

Conjecture 4.1. *Let $d, k, c \geq 1$, and let $\mathcal{P}$ be a finite set of points of $\mathbb{R}^d$ and $\mathcal{H}$ a finite set of hypersurfaces of $\mathbb{C}^d$ satisfying the following conditions:*

(a) *the degree of the hypersurfaces in $\mathcal{H}$ is bounded by c;*
(b) *the intersection of any family of d distinct hypersurfaces in $\mathcal{H}$ is finite;*
(c) *for any subset of k distinct points in $\mathcal{P}$, the number of hypersurfaces in $\mathcal{H}$ containing them is bounded by c.*

Set $m = \mathrm{card}(\mathcal{P})$ and $n = \mathrm{card}(\mathcal{H})$. Then

$$I(\mathcal{P}, \mathcal{H}) = O_{d,k,c}(m^{1-\frac{k-1}{dk-1}} n^{1-\frac{d-1}{dk-1}} + m + n).$$

This conjecture is suggested by the bound that follows from the first level of the polynomial partitioning method applied to this problem. This statement contains the Szeméredi-Trotter theorem 1.4, the results of Zahl and Kaplan, Matoušek, Sharir and Safernová in three dimensions [Zah13, KMSS12], and Theorem 1.5.

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