Spectral Sequences, Exact Couples and Persistent Homology of filtrations

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ABSTRACT. In this paper we study the relationship between a very classical algebraic object associated to a filtration of topological spaces, namely a spectral sequence introduced by Leray in the 1940's, and a more recently invented object that has found many applications – namely, its persistent homology groups. We show the existence of a long exact sequence of groups linking these two objects and using it derive formulas expressing the dimensions of each individual groups of one object in terms of the dimensions of the groups in the other object. The main tool used to mediate between these objects is the notion of exact couples first introduced by Massey in 1952.

1. Introduction

Given a topological space X (which for the purposes of the current paper will be taken to be a finite CW-complex) a finite filtration, $\mathcal{F}$ of X, is a sequence of subspaces

$$\emptyset = \cdots = X_{-1} = X_0 \subset X_1 \subset \cdots \subset X_N = X_{N+1} = \cdots = X$$

(we will denote the subspace X_s in the sequence by $\mathcal{F}_s X$). A very classical technique in algebraic topology for computing topological invariants of a space X is to consider a filtration $\mathcal{F}$ of X where the successive spaces $\mathcal{F}_s X$ capture progressively more and more of the topology of X. For example, in case X is a CW-complex one can take for $\mathcal{F}_s X$ the s-th skeleton $\mathrm{sk}_s(X)$ consisting of all cells of dimension at most s. More generally, given a cellular map $f \colon X \to Y$, one can take for $\mathcal{F}_s X$ the inverse image under f of $\mathrm{sk}_s(Y)$. One then associates to this sequence a sequence of algebraic objects which in nice situations is expected to "converge" (in an appropriate sense) to the topological invariant (such as the homology or cohomology groups) associated to X itself, directly computing which is often an intractable problem. This sequence of algebraic approximations is called a *spectral sequence* associated to the filtration $\mathcal{F}$, and was first introduced by Leray [10] in 1946 (see also the book by Dieudonné [5], page 137, for a comprehensive historical survey).

Spectral sequences are now ubiquitous in mathematics. A typical application which is common in discrete geometry, as well as in quantitative real algebraic geometry, is to use the initial terms of a certain spectral sequence to give upper bounds on the topological complexity (for example, the sum of Betti numbers) of the object of interest X (often a semi-algebraic subset of some $\mathbb{R}^n$) (see for example, [2], [8] for applications of this kind). Spectral sequences also have algorithmic applications in the context of computational geometry (see for example [1]).

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Much more recently the notion of *persistent homology* ([6],[14]) associated to a filtration has become an important tool in various applications. In contrast to spectral sequences discussed in the previous paragraph, the emphasis here is not so much on studying the topology of the final object X, but rather on the intermediate spaces of the filtration. Indeed the final object X in many cases is either contractible or homologically trivial. For example, this is the case for filtrations arising from alpha-complexes introduced by Edelsbrunner et al in [7]. The *persistent homology groups* (see below for a precise definition) are defined such that their dimensions equal the dimensions of spaces of homological cycles that appear at a certain fixed point of the filtration $\mathcal{F}$ and disappear at a certain (later) point. The homological cycles that persist for long intervals often carry important information about the underlying data sets that give rise to the filtration, and this is why computing them is important in practice. We refer the reader to survey articles [4],[9],[14] for details regarding these applications.

While spectral sequences and persistent homology were invented for entirely different purposes as explained above, they are both associated to filtrations of topological spaces and it is natural to wonder about the exact relationship between these two notions, and in particular, whether the dimensions of the groups appearing in the spectral sequence of a filtration carries any more information than the dimensions of the persistent homology groups of the same filtration. One of the results in this paper (Theorem 1 below) shows that this is not the case, and the dimensions of the groups appearing in the spectral sequence of a filtration can be recovered from the dimensions of its persistent homology groups. It has been observed by several authors (see for example, [4],[6],[9]) that there exists a close connection between the spectral sequence of a filtration and its persistent homology groups. The goal of this paper is to make precise this relationship – in particular, to derive formulas which expresses the dimensions of each group appearing in the spectral sequence of a filtration in terms of the persistent Betti numbers and vice versa.

For the rest of the paper we fix a field k of coefficients and all homology groups will be taken with coefficients in k, and we will omit the dependence on k from our notation henceforth. Given a *finite* filtration $\mathcal{F}$, given by $\emptyset = \cdots = X_{-1} = X_0 \subset X_1 \subset \cdots \subset X_s \subset X_{s+1} \subset \cdots \subset X_N = X_{N+1} = \cdots = X$ of finite CW-complexes, there is an associated homology spectral sequence $\left(E_{*,*}^{(r)}(\mathcal{F}), d_{*,*}^{(r)} \right)_{r>0}$ (defined below in Section 4), where each $E_{n,s}^{(r)}(\mathcal{F})$, $n, s \geq 0$ is a finite dimensional k-vector space. For each fixed $r > 0$, the set of groups $\left(E_{n,s}^{(r)}(\mathcal{F}) \right)_{n,s}$ is sometimes called the *r-th page* of the spectral sequence of $\mathcal{F}$, and they converge in an appropriate sense explained later to the homology of X as $r \to \infty$. Similarly, the persistent homology groups, $\left(H_n^{s,t}(\mathcal{F}) \right)_{n,s,t}$ (defined precisely below in Section 2) are also finite dimensional k-vector spaces, and correspond to the images under the linear maps induced by all possible inclusions amongst the family $(X_s)_{s \in \mathbb{Z}}$.

It is natural to ask whether the sequence of dimensions $\left(\dim_k E_{n,s}^{(r)}(\mathcal{F}) \right)_{r,n,s \in \mathbb{N}}$ determine the sequence $\left(b_n^{s,t}(\mathcal{F}) = \dim_k H_n^{s,t}(\mathcal{F}) \right)_{n,s,t \in \mathbb{N}, s \leq t}$ and conversely. In the book [6] the authors give a relation between these sets of numbers (see the *Spectral Sequence Theorem*, page 171 in [6]), but this theorem does not produce a simple formula expressing the dimension of each group, $E_{n,s}^{(r)}(\mathcal{F})$, in the spectral sequence, in terms of the persistent Betti numbers $b_n^{s,t}(\mathcal{F})$. The statement of this theorem in [6] has an error and there are some terms missing from the right-hand side of the equality. We fix this error and obtain the corrected version as a corollary to one of our theorems (see Corollary 2 and Remark 3 below).

We study the relationship between the persistent homology groups of a filtration and the homology spectral sequence via another classical tool in algebraic topology – namely *exact couples*, first introduced by Massey [11]. This gives us a simple way to relate the persistent Betti numbers with the dimensions of the groups occurring in the spectral sequence associated to the filtration. A hidden motivation behind this paper is to clarify the relationship between the spectral sequence groups and the persistent homology groups without getting bogged down in a morass of indices and a large number of intermediate groups of cycles and boundaries. The exact couple formulation is very elegant (and economical) in this respect, and the only extra groups (i.e. groups other than the various $E_{n,s}^{(r)}(\mathcal{F})$) that appear are subgroups of the homology groups of spaces appearing in the filtration. In fact, we will identify these extra groups with the persistent homology groups of the filtration (see Lemma 18 below).

Our main results are the following.

Theorem 1. *Let $\mathcal{F}$ denote the filtration, $\emptyset = \cdots = X_{-1} = X_0 \subset X_1 \subset \cdots \subset X_s \subset X_{s+1} \subset \cdots \subset X_N = X_{N+1} = \cdots = X$ where each X_i is a finite CW-complex. Then, for each $r > 0$, $n, s \geq 0$,*

$$\dim_k E_{n,s}^{(r)}(\mathcal{F}) \;=\; \left(b_n^{s,s+r-1}(\mathcal{F}) - b_n^{s-1,s+r-1}(\mathcal{F})\right) + \left(b_{n-1}^{s-r,s-1}(\mathcal{F}) - b_{n-1}^{s-r,s}(\mathcal{F})\right). \qquad (1)$$

Theorem 1 is a direct consequence of the existence of a long exact sequence linking the the spectral groups $E_{n,s}^{(r)}(\mathcal{F})$ and the persistent homology groups. This long exact sequence which appears in Theorem 19 below encapsulates the exact relationship between these groups.

We also define (following [6]) for $i < j$ the *persistent multiplicities* of the filtration $\mathcal{F}$ by

$$\mu_n^{i,j}(\mathcal{F}) \;=\; \left(b_n^{i,j-1}(\mathcal{F}) - b_n^{i,j}(\mathcal{F})\right) - \left(b_n^{i-1,j-1}(\mathcal{F}) - b_n^{i-1,j}(\mathcal{F})\right). \qquad (2)$$

In the language of persistence theory, the number $\mu_n^{i,j}(\mathcal{F})$ has the following interpretation. It is the dimension of the k-vector space spanned by the n-dimensional homology cycles, which are born at time i and gets killed at time j (see [6]).

We obtain the following corollary to Theorem 1 recovering (and correcting slightly) a result in [6] (page 171, *Spectral Sequence Theorem*) using a slightly different notation (see also Remark 15 below).

Corollary 2. *The following relation holds for every $r > 0$, and all $n \geq 0$.*

$$\sum_s \dim_k E_{n,s}^{(r)}(\mathcal{F}) \;=\; \sum_{j-i \geq r} \left(\mu_n^{i,j}(\mathcal{F}) + \mu_{n-1}^{i,j}(\mathcal{F})\right) + b_n(X).$$

Remark 3. In [6] the statement of the "Spectral Sequence Theorem" is stated incorrectly as

$$\sum_s \dim_k E_{n,s}^{(r)}(\mathcal{F}) \;=\; \sum_{j-i \geq r} \mu_n^{i,j}(\mathcal{F}),$$

and in a previous version of this paper we had a wrong proof of the above erroneous statement. We thank Ana Romero for pointing out a counter-example to this statement which appears in [13] and which directed us to the correct statement. A different corrected version of the statement also appears in [13], where the left hand side is the sum of the dimensions of a different set of groups. To our knowledge the equality in Corollary 2 is new. $\qquad \square$

We next show how to express the persistent Betti numbers in terms of dimensions of the vector spaces appearing in spectral sequences associated to truncations of the filtrations $\mathcal{F}$.

Notation 4. Given a finite filtration $\mathcal{F}$, given by $\emptyset = \cdots = X_{-1} = X_0 \subset X_1 \subset \cdots \subset X_s \subset X_{s+1} \subset \cdots \subset X_N = X_{N+1} = \cdots$ of finite CW-complexes, let for $0 \leq t \leq N$, $\mathcal{F}_{\leq t}$ denote the truncated filtration

$$\emptyset = \cdots = X_{-1} = X_0 \subset X_1 \subset \cdots \subset X_t = X_{t+1} = \cdots \qquad \square$$

We prove

Theorem 5. *For each, $n, s, t \geq 0$, $s \leq t$,*

$$b_n^{s,t}(\mathcal{F}) \; = \; \sum_{0 \leq i \leq s} \dim_k \left(E_{n,i}^{(\max(i, t-i+1))}(\mathcal{F}_{\leq t}) \right).$$

The rest of the paper is organized as follows. In Section 2, we recall the definitions of persistent homology groups of a filtration and the persistent Betti numbers. In Section 3, we recall the definitions of exact couples and prove their basic properties. In Section 4, we recall how to associate an exact couple to a filtration via the long exact homology sequence of a pair. We also establish the relationships between the groups in the derived couples and the persistent homology groups of the same filtration. This allows us to prove (rather easily) the main theorems in Section 5.

We do not assume any prior knowledge on spectral sequences and only assume basic linear and only a slight familiarity with homological algebra as prerequisites.

2. PERSISTENT HOMOLOGY

We now recall the precise definition of the persistent homology groups associated to a filtration ([6],[14]).

Let $\mathcal{F}$ denote the filtration of spaces $\emptyset = \cdots = X_{-1} = X_0 \subset X_1 \subset \cdots \subset X_s \subset X_{s+1} \subset \cdots \subset X_N = X_{N+1} = \cdots = X$ of spaces as in the last section.

Notation 6. For $0 \leq s \leq t \leq N$, we let $i_n^{s,t} \colon H_n(X_s) \longrightarrow H_n(X_t)$, denote the homomorphism induced by the inclusion $X_s \hookrightarrow X_t$. $\qquad \square$

With the same notation as in the previous section we define:

Definition 7. [6] For each triple (n, s, t) with $0 \leq s \leq t \leq N$ the corresponding *persistent homology group*, $H_n^{s,t}(\mathcal{F})$ is defined by

$$H_n^{s,t}(\mathcal{F}) \; = \; \mathrm{Im}\big(i_n^{s,t}\big).$$

Note that $H_n^{s,t}(\mathcal{F}) \subset H_n(X_t)$, and $H_n^{s,s}(\mathcal{F}) = H_n(X_s)$. $\qquad \square$

Notation 8. We denote by $b_n^{s,t}(\mathcal{F}) = \dim_k H_n^{s,t}(\mathcal{F})$. $\qquad \square$

3. ALGEBRA OF EXACT SEQUENCES AND COUPLES

There are several ways of defining the homology spectral sequence associated to a filtration $\mathcal{F}$. As mentioned in the introduction we prefer the approach due to Massey [11] (see also [12]) using *exact couples* (defined below) since it avoids defining various intermediate groups of cycles and boundaries and clarifies at a top level the close relationship between the groups appearing in the spectral sequence and the persistent homology groups. The exact relationships between the dimensions of the groups in the spectral sequence and the persistent Betti numbers can then be read off with minimal extra effort.

We begin by recalling a few basic notions.

Recall that a sequence of linear maps between k-vector spaces

$$\cdots \longrightarrow V_{i+1} \xrightarrow{f_{i+1}} V_i \xrightarrow{f_i} V_{i-1} \xrightarrow{f_{i-1}} V_{i-2} \xrightarrow{f_{i-2}} \cdots$$

is said to be *exact* if for each i, $\ker f_i = \operatorname{Im} f_{i+1}$.

The following lemma is an easy consequence of the exactness property.

Lemma 9. *Given an exact sequence*

$$V_2 \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \xrightarrow{f_0} V_{-1} \xrightarrow{f_{-1}} V_{-2}$$

where each V_i is a finite dimensional k-vector space

$$\dim_k (V_0) \;=\; (\dim_k (V_1) - \dim_k (\operatorname{Im}(f_2))) + (\dim_k (V_{-1}) - \dim_k (\operatorname{Im}(f_{-1}))).$$

Proof The lemma follows from the following sequence of inequalities of which the first one is from basic linear algebra, and the remaining consequences of the exactness of the given sequence.

$$
\begin{aligned}
\dim_k (V_0) \;&=\; \dim_k (\ker (f_0)) + \dim_k (\operatorname{Im}(f_0)) \\
&=\; \dim_k (\operatorname{Im}(f_1)) + \dim_k (\ker (f_{-1})) \\
&=\; (\dim_k (V_1) - \dim_k (\ker (f_1))) + (\dim_k (V_{-1}) - \dim_k (\operatorname{Im}(f_{-1}))) \\
&=\; (\dim_k (V_1) - \dim_k (\operatorname{Im}(f_2))) + (\dim_k (V_{-1}) - \dim_k (\operatorname{Im}(f_{-1}))).
\end{aligned}
$$

$\square$

We now define following Massey [11] (see also [12], page 37):

Definition 10. An *exact couple*, $\mathcal{C}$, consists of two finite dimensional k-vector spaces E, D and linear maps ∂, i and p such that the following triangular diagram is exact at each vertex.

$$
\begin{array}{ccc}
 & E & \\
p \nearrow & & \partial \searrow \\
D & \xleftarrow{\;i\;} & D
\end{array}
$$

$\square$

Given an exact couple

$$
\mathcal{C} \;=\;
\begin{array}{ccc}
 & E & \\
p \nearrow & & \partial \searrow \\
D & \xleftarrow{\;i\;} & D
\end{array}
$$

the exactness at each vertex implies that the map $d = p \circ \partial$ is a *differential* i.e. $d^2 = p \circ \partial \circ p \circ \partial = p \circ (\partial \circ p) \circ \partial = 0$.

Let $E' = H(E, d) = \ker d / \operatorname{Im} d$, and $D' = \operatorname{Im} i$.

Then, there exists *induced linear maps* $\partial' : E' \longrightarrow D'$, $i' : D' \longrightarrow D'$, $p' : D' \longrightarrow E'$ defined as follows.

First notice that for any element $x + dE \in E'$, where $x \in \ker (d)$, we have by the exactness at E of the couple $\mathcal{C}$, that $\partial(x) \in \operatorname{Im}(i)$ (since $x \in \ker (d)$ is equivalent to $p \circ \partial(x) = 0$, which implies that $\partial(x) \in \ker (p) = \operatorname{Im}(i)$).
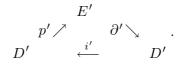
Now define

$$
\begin{aligned}
\partial'(x + dE) &= \partial(x) \in \mathrm{Im}(i) = D', \text{for all } x \in \ker(d), \\
i' &= i|_{\mathrm{Im}\, i}, \\
p'(i(x)) &= p(x) + dE, \text{for all } i(x) \in \mathrm{Im}\, i.
\end{aligned}
\tag{3}
$$

It follows easily from these definitions that the k-vector spaces E', D', and the linear maps ∂', i', p' also form an exact couple. In other words the following diagram is exact.

$$
\begin{array}{ccc}
& E' & \\
p' \nearrow & & \partial' \searrow \\
D' & \xleftarrow{\;i'\;} & D'
\end{array}.
$$

Definition 11. The exact couple

$$
\mathcal{C}' = \begin{array}{ccc}
& E' & \\
p' \nearrow & & \partial' \searrow \\
D' & \xleftarrow{\;i'\;} & D'
\end{array}
$$

is called the (first) *derived couple* of the exact couple $\mathcal{C}$. $\square$

Notation 12. Given an exact couple $\mathcal{C} = \mathcal{C}^{(1)}$, we denote for each $r \geq 1$, $\mathcal{C}^{(r+1)} = \left(\mathcal{C}^{(r)}\right)'$. $\square$

4. Homology Spectral sequence associated to a filtration

We now use the notion of an exact couple introduced in the last section to define the homology spectral sequence of a filtration $\mathcal{F}$ given by $\emptyset = \cdots = X_{-1} = X_0 \subset X_1 \subset \cdots \subset X_s \subset X_{s+1} \subset \cdots \subset X_N = X_{N+1} = \cdots = X$ where each X_i is a finite CW-complex.

Notation 13. For $0 \leq s \leq N$, denoting $F_s H_n(X) = \mathrm{Im}\left(i_n^{s,N}\right)$ we have a filtration of the vector space $H_n(X) = H_n(X, k)$ given by

$$
0 = F_0 H_n(X) \subset F_1 H_n(X) \subset \cdots F_s H_n(X) \subset F_{s+1} H_n(X) \subset \cdots \subset F_N H_n(X) = H_n(X) \quad \square
$$

Defining

$$
\begin{aligned}
\mathrm{Gr}_{n,s}(H_n(X)) &= F_s H_n(X)/F_{s-1} H_n(X), \\
&= \mathrm{Im}\left(i_n^{s,N}\right)/\mathrm{Im}\left(i_n^{s-1,N}\right)
\end{aligned}
\tag{4}
$$

we have

$$
\dim_k H_n(X) = \sum_{s \geq 0} \dim_k \mathrm{Gr}_{n,s}(H_n(X)).
$$

Recall also the homology exact sequence of the pair (X_s, X_{s-1}):

$$
\cdots \longrightarrow H_n(X_{s-1}) \xrightarrow{i_{n,s-1}} H_n(X_s) \xrightarrow{p_{n,s}} H_n(X_s, X_{s-1}) \xrightarrow{\partial_{n,s}} H_{n-1}(X_{s-1}) \longrightarrow \cdots
$$

Now, let

$$
\begin{aligned}
E_{n,s}^1(\mathcal{F}) &= H_n(X_s, X_{s-1}), \tag{5} \\
D_{n,s}^1(\mathcal{F}) &= H_n(X_s). \tag{6}
\end{aligned}
$$

We now bundle together the various $E^1_{n,s}(\mathcal{F})$ (respectively, $D^1_{n,s}(\mathcal{F})$) into one bi-graded vector space $E^1_{*,*}(\mathcal{F})$ (respectively, $D^1_{*,*}(\mathcal{F})$) by defining

$$E^1_{*,*}(\mathcal{F}) \;=\; \bigoplus_{n,s} E^1_{n,s}(\mathcal{F}),$$
$$D^1_{*,*}(\mathcal{F}) \;=\; \bigoplus_{n,s} D^1_{n,s}(\mathcal{F}).$$

Note that the k-vector spaces $E^1_{*,*}(\mathcal{F})$ and $D^1_{*,*}(\mathcal{F})$ are bi-graded and each of them is a direct sum of homogeneous subspaces indexed by the pairs (n,s). We refer the reader to [3] (Chapter 2, §11.2) for background on graded vector spaces and graded homomorphisms (linear maps) between them.

In particular, a bi-graded homomorphism $\phi\colon \bigoplus_{n,s} V_{n,s} \longrightarrow \bigoplus_{n,s} W_{n,s}$ between two bi-graded vetor spaces is said to have bi-degree (i,j) if $\phi(V_{n,s}) \subset W_{n+i,s+j}$.

We denote by $\partial_{*,*}$ (respectively, $i_{*,*}, p_{*,*}$) the bi-graded linear maps $\bigoplus_{n,s} \partial_{n,s}$ (respectively, $\bigoplus_{n,s} i_{n,s}, \bigoplus_{n,s} p_{n,s}$), and get an exact couple

$$\mathcal{C}(\mathcal{F}) \;=\; \begin{array}{ccc} & E^1_{*,*}(\mathcal{F}) & \\ p_{*,*}\nearrow & & \partial_{*,*}\searrow \\ D^1_{*,*}(\mathcal{F}) & \xleftarrow{\;i_{*,*}\;} & D^1_{*,*}(\mathcal{F}) \end{array}.$$

Note that the bi-degrees of the various linear maps $\partial_{*,*}, i_{*,*}, p_{*,*}$, can be read off from the following table.

$$\partial_{n,s}\colon E^1_{n,s}(\mathcal{F}) \longrightarrow D^1_{n-1,s-1}(\mathcal{F}),$$

$$i_{n,s}\colon D^1_{n,s}(\mathcal{F}) \longrightarrow D^1_{n,s+1}(\mathcal{F}),$$

$$p_{n,s}\colon D^1_{n,s}(\mathcal{F}) \longrightarrow E^1_{n,s}(\mathcal{F}).$$

More precisely,

$$\begin{aligned} \mathrm{bidegree}(\partial_{*,*}) &= (-1,-1), \\ \mathrm{bidegree}(i_{*,*}) &= (0,1), \\ \mathrm{bidegree}(p_{*,*}) &= (0,0). \end{aligned}$$

On deriving the exact couple $\mathcal{C}(\mathcal{F})$ r times we obtain for each $r>0$, the couple

$$\mathcal{C}^{(r)}(\mathcal{F}) \;=\; \begin{array}{ccc} & E^{(r)}_{*,*}(\mathcal{F}) & \\ p^{(r)}_{*,*}\nearrow & & \partial^{(r)}_{*,*}\searrow \\ D^{(r)}_{*,*}(\mathcal{F}) & \xleftarrow{\;i^{(r)}_{*,*}\;} & D^{(r)}_{*,*}(\mathcal{F}) \end{array}.$$

Definition 14. The sequence of pairs $\left(E^r_{*,*}(\mathcal{F}),\, d^{(r)}_{*,*}(\mathcal{F}) \right)_{r>0}$ is called the (homology) *spectral sequence associated to the filtration $\mathcal{F}$.* $\qquad\square$

Notice , $E^{(r+1)}_{*,*}(\mathcal{F}) = \left(E^{(r)}_{*,*}(\mathcal{F}) \right)' = H\left(E^{(r)}_{*,*}(\mathcal{F}), d^{(r)}_{*,*} \right)$ and $D^{(r+1)}_{*,*}(\mathcal{F}) = \left(D^{(r)}_{*,*}(\mathcal{F}) \right)' = i^{(r)}_{*,*}\left(D^{(r)}_{*,*}(\mathcal{F}) \right)$.

Remark 15. Note that in the literature there are several different conventions used for the pair of indices that appear in the subscript, and it is also common to use the pair $(n-s, s)$ to index the group that is being indexed by (n, s) above (for example, in the book [6] the alternative indexing scheme is used). This should cause no confusion.   □

It follows by induction on r that the linear maps $\partial_{*,*}^{(r)}$, $i_{*,*}^{(r)}$, $p_{*,*}^{(r)}$ are graded homomorphisms and their bi-degrees are displayed below.

$$\partial_{n,s}^{(r)}\colon E_{n,s}^{(r)}(\mathcal{F}) \longrightarrow D_{n-1,s-r}^{(r)}(\mathcal{F}),$$

$$i_{n,s}^{(r)}\colon D_{n,s}^{(r)}(\mathcal{F}) \longrightarrow D_{n,s+1}^{(r)}(\mathcal{F}), \tag{7}$$

$$p_{n,s}^{(r)}\colon D_{n,s}^{(r)}(\mathcal{F}) \longrightarrow E_{n,s}^{(r)}(\mathcal{F}).$$

More precisely,

$$\begin{aligned}
\mathrm{bidegree}\left(\partial_{*,*}^{(r)}\right) &= (-1,-r),\\
\mathrm{bidegree}\left(i_{*,*}^{(r)}\right) &= (0,1),\\
\mathrm{bidegree}\left(p_{*,*}^{(r)}\right) &= (0,0).
\end{aligned}$$

It then follows that

$$d_{n,s}^{(r)}\colon E_{n,s}^{(r)}(\mathcal{F}) \longrightarrow E_{n-1,s-r}^{(r)}(\mathcal{F}),$$

and thus,

$$\mathrm{bidegree}\left(d_{n,s}^{(r)}\right) = (-1,-r).$$

Theorem 16. *The spectral sequence defined above converges after a finite number of terms with*

$$E_{n,s}^{\infty}(\mathcal{F}) \;\cong\; \mathrm{Gr}_{n,s}(H_n(X)).$$

Proof Since clearly the groups $E_{n,s}^1(\mathcal{F}) = 0$ for all but finite number of pairs (n, s), it is clear that $d_r^{*,*} = 0$ for r large enough and this implies that the spectral sequence $\left(E_{*,*}^r(\mathcal{F}), d_{*,*}^{(r)}(\mathcal{F})\right)_{r>0}$ converges unconditionally. The remaining part is a standard result (see for example [12], Chapter 2).   □

Remark 17. Notice that since in the filtration $\mathcal{F}$, $X_i = \emptyset$ for $i \leq 0$, we have that $d_{n,s}^{(r)} = 0$ for all $r \geq s$. Also, it follows from (5) that $E_{n,s}^1(\mathcal{F}) = 0$ for all $s > N$. This implies that $E_{n,s}^{(r)}(\mathcal{F}) = 0$ for all $s > N$ and $r > 0$. Thus, $d_{n,s}^{(r)}$ and $d_{n,s-r}^{(r)}$ are both 0 for $r \geq \max(s, N-s+1)$, and this implies that $E_{n,s}^{\infty}(\mathcal{F}) \cong E_{n,s}^{(\max(s, N-s+1))}(\mathcal{F})$.   □

The crucial link between the exact couples $\mathcal{C}^{(r)}(\mathcal{F})$ and the persistent homology groups is captured in the following observation.

Lemma 18. *For each $r > 0, n, s \geq 0$*

$$D_{n,s}^{(r)}(\mathcal{F}) \;=\; \mathrm{Im}\big(i_n^{s,s+r-1}\big) = H_n^{s,s+r-1}(\mathcal{F}),$$

and

$$i_{n,s}^{(r)} \;=\; i_n^{s+r-1,s+r}\big|_{D_{n,s}^{(r)}(\mathcal{F})}.$$

Proof For $r=1$ both claims follow directly from the definition of $D^1_{n,s}(\mathcal{F})$ (see Eqn. (6) above) and the definition of the derived couple (see Eqn. (3)). For $r>1$, the claim follows by induction. First notice that $D^{(r)}_{n,s}=i^{(r-1)}_{n,s}\big(D^{(r-1)}_{n,s-1}\big)$ (Eqn. (7)). By induction hypothesis we have

$$D^{(r-1)}_{n,s}(\mathcal{F}) \;=\; \mathrm{Im}\big(i^{s,s+r-2}_n\big)=H^{s,s+r-2}_n(\mathcal{F})\;,$$

and

$$i^{(r-1)}_{n,s} \;=\; i^{s+r-2,s+r-1}_n\big|_{\mathrm{Im}(i^{s,s+r-2}_n)}\;.$$

It then follows from the definition of the derived couple (Definition 11) that

$$D^{(r)}_{n,s}(\mathcal{F}) \;=\; \mathrm{Im}\big(i^{s,s+r-1}_n\big)=H^{s,s+r-1}_n(\mathcal{F}),$$

and

$$i^{(r)}_{n,s} \;=\; i^{s+r-1,s+r}_n\big|_{D^{(r)}_{n,s}(\mathcal{F})}\;. \qquad\square$$

Theorem 19. *For each $r>0$, the groups $E^{(r)}_{p,q}(\mathcal{F})$ and the persistent homology groups $H^{s,t}_n(\mathcal{F})$ are related by the following long exact sequence.*

$$\cdots\longrightarrow H^{s,s+r-1}_n(\mathcal{F}) \xrightarrow{p^{(r)}_{n,s}} E^{(r)}_{n,s}(\mathcal{F}) \xrightarrow{\partial^{(r)}_{n,s}} H^{s-r,s-1}_{n-1}(\mathcal{F}) \xrightarrow{i^{(r)}_{n-1,s-r}} H^{s-r+1,s-r}_{n-1}(\mathcal{F}) \longrightarrow \cdots$$

Moreover, for each r,n,s, $\mathrm{Im}\big(i^{(r)}_{n,s}\big)=H^{s,s+r}_n(\mathcal{F})$.

Proof Unravel the exact couple $\mathcal{C}^{(r)}(\mathcal{F})$ and use Lemma 18. $\qquad\square$

5. Relations between $\dim_k E^r_{n,s}(\mathcal{F})$ and $b^{s,t}_n(\mathcal{F})$

In this section we prove the main theorems.

Proof (of Theorem 1). Using Theorem 19, Lemma 9 and Lemma 18 we obtain that for each r,n,s with $r>0,n,s\geq 0$,

$$\begin{aligned}
\dim_k E^{(r)}_{n,s}(\mathcal{F}) \;&=\; \Big(\dim_k D^{(r)}_{n,s} - \dim_k\mathrm{Im}\big(i^{(r)}_{n,s-1}\big)\Big)+ \\
&\quad\; \Big(\dim_k D^{(r)}_{n-1,s-r} - \dim_k\mathrm{Im}\big(i^{(r)}_{n-1,s-r}\big)\Big) \\
&=\; \big(b^{s,s+r-1}_n(\mathcal{F})-b^{s-1,s+r-1}_n(\mathcal{F})\big)+\big(b^{s-r,s-1}_{n-1}(\mathcal{F})-b^{s-r,s}_{n-1}(\mathcal{F})\big).
\end{aligned}$$

$$\square$$

Remark 20. Notice that for $r\geq \max(s,N-s+1)$ we have

$$\begin{aligned}
b^{s,s+r-1}_n(\mathcal{F}) \;&=\; b^{s,N}_n(\mathcal{F}), \\
b^{s-1,s+r-1}_n(\mathcal{F}) \;&=\; b^{s-1,N}_n(\mathcal{F}), \\
b^{s-r,s-1}_{n-1}(\mathcal{F}) \;&=\; 0, \\
b^{s-r,s}_{n-1}(\mathcal{F}) \;&=\; 0.
\end{aligned}$$

Using Remark 17 we verify that

$$\begin{aligned}
\dim_k E^{(\infty)}_{n,s}(\mathcal{F}) \;&=\; \dim_k E^{(\max(s,N-s+1))}_{n,s}(\mathcal{F}) \\
&=\; b^{s,N}_n(\mathcal{F}) - b^{s-1,N}_n(\mathcal{F}) \\
&=\; \dim_k\mathrm{Im}\big(i^{s,N}_n\big) - \dim_k\mathrm{Im}\big(i^{s-1,N}_n\big) \\
&=\; \dim_k\big(\mathrm{Im}\big(i^{s,N}_n\big)/\mathrm{Im}\big(i^{s-1,N}_n\big)\big) \\
&=\; \dim_k \mathrm{Gr}_{n,s}(H_n(X)).
\end{aligned}$$

$$\square$$

Proof (of Corollary 2).

Denote for $n \geq 0$, and $s, t \in \mathbb{Z}$

$$
\begin{aligned}
\gamma_n^{s,t}(\mathcal{F}) &= \left(b_n^{s-t,s-1}(\mathcal{F}) - b_n^{s-t,s}(\mathcal{F}) \right), \\
\nu_n^{s,t}(\mathcal{F}) &= \left(b_n^{s,s+t-1}(\mathcal{F}) - b_n^{s-1,s+t-1}(\mathcal{F}) \right).
\end{aligned}
$$

It follows from Eqn. (1) that

$$
\sum_s \dim_k E_r^{n,s}(\mathcal{F}) = \sum_{0 \leq s \leq N+1} \left(\nu_n^{s,t}(\mathcal{F}) + \gamma_{n-1}^{s,t}(\mathcal{F}) \right).
$$

Also, summing both sides of Eqn (2) we obtain two different expressions for $\sum_{j-i \geq r} \mu_n^{i,j}(\mathcal{F})$, one in terms of the $\nu_n^{s,t}$, and the other in terms of the $\gamma_n^{s,t}$.

More precisely:

$$
\begin{aligned}
\sum_{j-i \geq r} \mu_n^{i,j}(\mathcal{F}) &= \sum_{j-i \geq r} \left(\left(b_n^{i,j-1}(\mathcal{F}) - b_n^{i,j}(\mathcal{F}) \right) - \left(b_n^{i-1,j-1}(\mathcal{F}) - b_n^{i-1,j}(\mathcal{F}) \right) \right) \\
&= \sum_{r \leq t \leq N+1} \sum_{0 \leq s \leq N+1} \left(\gamma_n^{s,t}(\mathcal{F}) - \gamma_n^{s,t+1}(\mathcal{F}) \right), \quad (8)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{j-i \geq r} \mu_n^{i,j}(\mathcal{F}) &= \sum_{j-i \geq r} \left(\left(b_n^{i,j-1}(\mathcal{F}) - b_n^{i-1,j-1}(\mathcal{F}) \right) - \left(b_n^{i,j}(\mathcal{F}) - b_n^{i-1,j}(\mathcal{F}) \right) \right) \\
&= \sum_{r \leq t \leq N+1} \sum_{0 \leq s \leq N+1} \left(\nu_n^{s,t}(\mathcal{F}) - \nu_n^{s,t+1}(\mathcal{F}) \right). \quad (9)
\end{aligned}
$$

It follows first by changing the order of summation, and then using telescoping on Eqn. (8) that

$$
\begin{aligned}
\sum_{j-i \geq r} \mu_n^{i,j}(\mathcal{F}) &= \sum_{0 \leq s \leq N+1} \sum_{r \leq t \leq N+1} \left(\gamma_n^{s,t}(\mathcal{F}) - \gamma_n^{s,t+1}(\mathcal{F}) \right) \quad (10) \\
&= \sum_{0 \leq s \leq N+1} \left(\gamma_n^{s,r}(\mathcal{F}) - \gamma_n^{s,N+1} \right) \\
&= \sum_{0 \leq s \leq N+1} \gamma_n^{s,r}(\mathcal{F}) - \sum_{0 \leq s \leq N+1} \left(\left(b_n^{s-N-1,s-1}(\mathcal{F}) - b_n^{s-N-1,s}(\mathcal{F}) \right) \right) \\
&= \sum_s \gamma_n^{s,r}(\mathcal{F}).
\end{aligned}
$$

Similarly from Eqn. (9) we get that

$$
\begin{aligned}
\sum_{j-i \geq r} \mu_n^{i,j}(\mathcal{F}) &= \sum_{0 \leq s \leq N+1} \sum_{r \leq t \leq N+1} \left(\nu_n^{s,t}(\mathcal{F}) - \nu_n^{s,t+1}(\mathcal{F}) \right) \quad (11) \\
&= \sum_{0 \leq s \leq N+1} \left(\nu_n^{s,r}(\mathcal{F}) - \nu_n^{s,N+1} \right) \\
&= \sum_{0 \leq s \leq N+1} \nu_n^{s,r}(\mathcal{F}) - \sum_{0 \leq s \leq N+1} \left(b_n^{s,s+N}(\mathcal{F}) - b_n^{s-1,s+N}(\mathcal{F}) \right) \\
&= \sum_s \nu_n^{s,r}(\mathcal{F}) - b_n(X).
\end{aligned}
$$

The corollary follows from Eqns. (10) and (11).

$\square$

Proof (of Theorem 5). It follows from Theorem 16 and (4) that for each $n \geq 0$ and $i \leq t$,

$$
\dim_k E_{n,i}^{\infty}(\mathcal{F}_{\leq t}) = b_n^{i,t}(\mathcal{F}) - b_n^{i-1,t}(\mathcal{F}).
$$

The theorem follows after taking the sum of both sides over all $i, 0 \leq i \leq s$, and noting that $b_n^{i,t}(\mathcal{F}) = 0$ for $i \leq 0$. Finally, by Remark 17 we have that $E_{n,i}^{\infty}(\mathcal{F}_{\leq t}) \cong E_{n,i}^{(\max(i,t-i+1))}(\mathcal{F}_{\leq t})$.

$\square$

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