

ON A REAL ANALOGUE OF BEZOUT INEQUALITY AND THE NUMBER OF CONNECTED COMPONENTS OF SIGN CONDITIONS

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ABSTRACT. Let \mathbb{R} be a real closed field and $Q_1, \dots, Q_\ell \in \mathbb{R}[X_1, \dots, X_k]$ such that for each $i, 1 \leq i \leq \ell$, $\deg(Q_i) \leq d_i$. For $1 \leq i \leq \ell$, denote by $\mathcal{Q}_i = \{Q_1, \dots, Q_i\}$, V_i the real variety defined by \mathcal{Q}_i , and k_i an upper bound on the real dimension of V_i (by convention $V_0 = \mathbb{R}^k$ and $k_0 = k$). Suppose also that

$$2 \leq d_1 \leq d_2 \leq \frac{1}{k+1}d_3 \leq \frac{1}{(k+1)^2}d_4 \leq \dots \leq \frac{1}{(k+1)^{\ell-3}}d_{\ell-1} \leq \frac{1}{(k+1)^{\ell-2}}d_\ell,$$

and that $\ell \leq k$. We prove that the number of semi-algebraically connected components of V_ℓ is bounded by

$$O(k)^{2k} \left(\prod_{1 \leq j < \ell} d_j^{k_{j-1} - k_j} \right) d_\ell^{k_{\ell-1}}.$$

This bound can be seen as a weak extension of the classical Bezout inequality (which holds only over algebraically closed fields and is provably false over real closed fields) to varieties defined over real closed fields.

Additionally, if $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_k]$ is a finite family of polynomials with $\deg(P) \leq d$ for all $P \in \mathcal{P}$, $\text{card } \mathcal{P} = s$, and $d_\ell \leq \frac{1}{k+1}d$, we prove that the number of semi-algebraically connected components of the realizations of all realizable sign conditions of the family \mathcal{P} restricted to V_ℓ is bounded by

$$O(k)^{2k} (sd)^{k_\ell} \left(\prod_{1 \leq j \leq \ell} d_j^{k_{j-1} - k_j} \right).$$

1. INTRODUCTION

Let \mathbb{R} be a fixed real closed field, and we denote by \mathbb{C} the algebraic closure of \mathbb{R} . Bounds on the number of semi-algebraically connected components, and in fact on all the Betti numbers of real algebraic varieties and of semi-algebraic subsets of \mathbb{R}^k in terms of the number and the degrees of the polynomials used to define them is a well studied problem in quantitative real algebraic geometry. The classical bounds, going back to the work of Oleinik and Petrovsky [21], Thom [25] and Milnor [19], bounded the sum of the Betti numbers of real algebraic varieties, as well as those of basic closed semi-algebraic sets. These and related bounds (see below) are extremely important in real algebraic geometry [10], but have also been used extensively in other areas such as combinatorics [3], discrete and computational geometry [14], and theoretical computer science [20] (the cited references are not by any means exhaustive but only given for illustrative purposes – we refer the reader to [9] for a more extensive survey).

An important application of the bounds mentioned above is in bounding the number of semi-algebraically connected components of the realizations of various sign conditions of a family of polynomials in \mathbb{R}^k or more generally sign conditions restricted to a given real subvariety of \mathbb{R}^k . In order to state these results more precisely, we introduce some notation.

Notation 1. For $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_k]$ a finite family of polynomials, a *sign condition* σ on \mathcal{P} is an element of $\{0, 1, -1\}^{\mathcal{P}}$. The realization $\text{Reali}(\sigma, V)$ of the sign condition σ on another semi-algebraic set V is the semi-algebraic set defined by

$$\text{Reali}(\sigma, V) = \{x \in V \mid \text{sign}(P) = \sigma(P), P \in \mathcal{P}\}.$$

□

Notation 2. For any finite family of polynomials $\mathcal{Q} \subset \mathbb{R}[X_1, \dots, X_k]$ we will denote by $\text{Zer}(\mathcal{Q}, \mathbb{R}^k)$ the set of real zeros of \mathcal{Q} in \mathbb{R}^k . If $\mathcal{Q} = \{Q\}$, then we will use the notation $\text{Zer}(Q, \mathbb{R}^k)$ instead. We will denote by \mathcal{Q}^h (respectively, Q^h) the homogenization of the polynomials in \mathcal{Q} (respectively, the polynomial Q), and denote by $\text{Zer}(\mathcal{Q}^h, \mathbb{P}_{\mathbb{C}}^k)$ (respectively, $\text{Zer}(Q^h, \mathbb{P}_{\mathbb{C}}^k)$) the common zeros of the family \mathcal{Q}^h (respectively, the polynomial Q^h) in the projective space $\mathbb{P}_{\mathbb{C}}^k$. □

Notation 3. For any $Q \in \mathbb{R}[X_1, \dots, X_k]$ we will denote by $\deg(Q)$ the degree of Q . More, generally for a tuple of polynomials $\mathcal{Q} = (Q_1, \dots, Q_{\ell}) \in \mathbb{R}[X_1, \dots, X_k]^{\ell}$ we will denote $\deg(\mathcal{Q}) = (d_1, \dots, d_{\ell})$ where $d_i = \deg(Q_i)$, $1 \leq i \leq \ell$. □

Notation 4. For any semi-algebraic set $S \subset \mathbb{R}^k$, we will denote by $b_i(S)$ the i -th Betti number of S . In particular, $b_0(S)$ is the number of semi-algebraically connected components of S . □

Notation 5. For any semi-algebraic set $S \subset \mathbb{R}^k$, we will denote by $\dim S$ the *real dimension* of S . For any $x \in S$, we denote by $\dim_x S$ the local real dimension of S at x . Note that unlike complex varieties, an irreducible real variety can have different local dimensions at different points. □

Remark 6. We will at times slightly abuse notation and use the same letter to denote a tuple of polynomials as well as the ordered finite set whose elements are the elements of the tuple. This should not cause any confusion. □

The following theorem which generalizes earlier results of Alon [3], Warren [27] and Pollack and Roy [22], appears in [5].

Theorem 7. *Let $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}[X_1, \dots, X_k]$ be finite families of polynomials such that the degrees of the polynomials in \mathcal{P}, \mathcal{Q} are bounded by d , $\text{card } \mathcal{P} = s$, and $\dim_{\mathbb{R}}(V) = k'$, where $V = \text{Zer}(\mathcal{Q}, \mathbb{R}^k)$. Then,*

$$\sum_{\sigma \in \{0, 1, -1\}^{\mathcal{P}}} b_0(\text{Reali}(\sigma, V)) \leq O(1)^k s^{k'} d^k.$$

Notice that in the bound in Theorem 7, while the exponent of s depends on the dimension of the variety V , the exponent of d is that of the ambient space. Moreover, the bound depends only on the maximum degree of the polynomials in \mathcal{P} and \mathcal{Q} . This is a consequence of the fact that the proof involves taking *sums of squares* of the polynomials in \mathcal{P} and \mathcal{Q} , and thus only the maximum degree plays a role in the argument. This feature of taking the sum of squares is something that is common in the proofs of all the bounds mentioned above. As such they all depend on the *maximum* of the degrees of the polynomials used to define the given set or sign conditions.

More recently, a new application of these bounds in discrete and computational geometry, triggered by the work of Guth and Katz [15], raised the question whether even the part of the bound in Theorem 7 that depends only on the degree d could have a finer dependence on the degrees of the polynomials in \mathcal{P} and \mathcal{Q} , in the case when the degrees of the polynomials in \mathcal{Q} and those in \mathcal{P} differ significantly (see [15],[24],[17],[16],[28]). This is one of the primary motivations behind the results proved in the current paper. A second motivation is to prove a version of the Bezout inequality on bounding the number of isolated complex solutions (or more generally the number of connected components) of an affine polynomial system by the product of the degrees, over real closed fields where the original statement of the inequality does not hold (see Example 11 and Remark 14 below).

A first step was taken in this direction in [4] where the authors of the current paper proved the following theorem (actually a more precise statement appears in [4] but the following simplified version is what is important for the present purpose).

Theorem 8. *Let $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}[X_1, \dots, X_k]$ be finite subsets of polynomials such that $\deg(Q) \leq d_1$ for all $Q \in \mathcal{Q}$, $\deg(P) \leq d_2$ for all $P \in \mathcal{P}$. Suppose also that $d_1 \leq d_2$, and the real dimension of $V = \text{Zer}(\mathcal{Q}, \mathbb{R}^k)$ is $k_1 \leq k$, and that $\text{card } \mathcal{P} = s$. Then,*

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_0(\text{Reali}(\sigma, V)) \leq O(1)^k (s d_2)^{k_1} d_1^{k-k_1}.$$

Remark 9. One should compare Theorem 8 with Theorem 7. The new aspect of Theorem 8 is the more refined dependence on the two different degrees, taking into account the dimension of the variety V and the fact that $d_0 \leq d$. \square

Notice that Theorem 8 implies the following corollary about the number of semi-algebraically connected components of real varieties.

Corollary 10. *Let $Q_1, Q_2 \in \mathbb{R}[X_1, \dots, X_k]$ such that $\deg(Q_1) \leq d_1, \deg(Q_2) \leq d_2, d_1 \leq d_2$. Let $V_1 = \text{Zer}(Q_1, \mathbb{R}^k)$ and $\dim V_1 \leq k_1$, and let $V_2 = \text{Zer}(\{Q_1, Q_2\}, \mathbb{R}^k)$. Then,*

$$b_0(V_2) \leq O(1)^k d_1^{k-k_1} d_2^{k_1}.$$

Proof In Theorem 8, take $\mathcal{Q} = \{Q_1\}$, $\mathcal{P} = \{Q_2\}$ and $V = V_1$. \square

While Theorem 8 (in particular, also Corollary 10) has already proved useful in certain applications in discrete and computational geometry (see [2],[24]), some even more recent developments [16],[28] seem to require a more detailed analysis. An important new ingredient in these developments is the so called “polynomial partitioning” result due to Guth and Katz [15], which states that given any set, S , of n points in \mathbb{R}^k , and an auxiliary parameter $r, 0 < r < n$, there exists a polynomial $P \in \mathbb{R}[X_1, \dots, X_k]$ of degree at most $O\left(r^{\frac{1}{k}}\right)$, having the property that each semi-algebraically connected component C of $\mathbb{R}^k \setminus \text{Zer}(P, \mathbb{R}^k)$ contains at most $\frac{n}{r}$ of the points of S . The number of such semi-algebraically connected components C (using for instance Theorem 7) is bounded by $O(r)$, and it is at this point that a quantitative bound on the number of semi-algebraically connected components of a semi-algebraic set or sign conditions enters the proof. The polynomial partitioning theorem is a tool to decompose a given problem involving the set S into sub-problems of smaller size (corresponding to the point sets $C \cap S$ where C is a semi-algebraically connected component of $\mathbb{R}^k \setminus \text{Zer}(P, \mathbb{R}^k)$). However, it might happen that most or even all the points of S are contained in $\text{Zer}(P, \mathbb{R}^k)$ which is a bad case for such a “divide-and-conquer” type argument. In this case, an obvious idea is to try to extend the polynomial partitioning theorem to varieties of lower dimensions, and continue the partitioning recursively. While this approach has been successful till date for partitions of depth at most two, any further advance along these lines would require tight bounds on the number of semi-algebraically connected components of real varieties defined by a sequence polynomials of strictly increasing degrees, and in order to prove the strongest result possible one needs a bound which has separate roles for each of these degrees. The length of degree sequence could be as large as the ambient dimension.

The bound in Theorem 8 depends on just two different degrees. However, as mentioned before methods from discrete geometry motivate the question whether similar bounds can be proved depending on a degree sequence of length greater than two. Before stating our results let us consider what kind of refined bounds are plausible. In the case of a real variety V of \mathbb{R}^k , which is a non-singular complete intersection (even at infinity) and defined by polynomials of degrees $d_1 \leq d_2 \leq \dots \leq d_\ell$, the number of semi-algebraically connected components of V is bounded by (see Proposition 52 as well as Remark 53 below)

$$O(1)^k d_1 \dots d_{\ell-1} d_\ell d_\ell^{k-\ell}.$$

Notice that $k - \ell = \dim V$. It is thus natural to hope that such a bound continues to hold even if the given variety is not a non-singular complete intersection – namely, one might hope that the number of semi-algebraically connected components of a real variety $V \subset \mathbb{R}^k$ defined by a sequence of ℓ polynomials having degrees $d_1 \leq d_2 \leq \dots \leq d_\ell$ is bounded by $O(1)^k d_1 \dots d_\ell d_\ell^{\dim V}$. However, the following well known (counter-)example (that appears in [13]) already shows that this is not the case.

Example 11. Let $k = 3$ and let

$$\begin{aligned} Q_1 &= X_3, \\ Q_2 &= X_3, \\ Q_3 &= \sum_{i=1}^2 \left(\prod_{j=1}^d (X_i - j)^2 \right). \end{aligned}$$

Then the real variety defined by $\{Q_1, Q_2, Q_3\}$ is 0-dimensional, and has d^2 isolated points, whereas the degree sequence $(d_1, d_2, d_3) = (1, 1, 2d)$, and thus the conjectured bound is $d_1 \dots d_\ell d_\ell^{\dim V} = O(d)$. In particular, this example shows that the Bezout inequality which states that the number of isolated complex zeros of a system of polynomial equations is bounded by the product of the degrees of the polynomials appearing in the system, is not true over \mathbb{R} if we replace the complex numbers by a real closed field. \square

While this might seem discouraging at first glance, one way to repair the situation is to formulate a bound that depends not just on the degree sequence and the dimension of the last variety $V = V_3 = \text{Zer}(\{Q_1, Q_2, Q_3\}, \mathbb{R}^k)$, but also takes into account the dimensions of the intermediate varieties $V_1 = \text{Zer}(Q_1, \mathbb{R}^k)$, $V_2 = \text{Zer}(\{Q_1, Q_2\}, \mathbb{R}^k)$ etc. Notice that in Example 11 the dimensions $k_1 = \dim V_1$, and $k_2 = \dim V_2$ are both equal to 2, whereas $k_3 = \dim V_3 = 0$. The number of semi-algebraically connected components in this case is bounded by $O(d_1^{k-k_1} d_2^{k_1-k_2} d_3^{k_2})$, where $d_i = \deg Q_i$. This is the starting point of the formulation of the new results proved in this paper.

We prove the following theorems where the shapes of the bounds should be seen in the light of Example 11.

1.1. Main Results.

Let $Q_1, \dots, Q_\ell \in \mathbb{R}[X_1, \dots, X_k]$ such that for each i , $1 \leq i \leq \ell$, $\deg(Q_i) \leq d_i$. For $1 \leq i \leq \ell$, denote by $\mathcal{Q}_i = \{Q_1, \dots, Q_i\}$, $V_i = \text{Zer}(\mathcal{Q}_i, \mathbb{R}^k)$, and $\dim_{\mathbb{R}}(V_i) \leq k_i$. We set $V_0 = \mathbb{R}^k$, and adopt the convention that $k_i = k$ for $i \leq 0$. It is clear that $k = k_0 \geq k_1 \geq \dots \geq k_\ell$. Suppose also that

$$2 \leq d_1 \leq d_2 \leq \frac{1}{k+1} d_3 \leq \frac{1}{(k+1)^2} d_4 \leq \dots \leq \frac{1}{(k+1)^{\ell-2}} d_\ell.$$

With these assumptions we have the following generalization of Corollary 10.

Theorem 12.

$$b_0(V_\ell) \leq O(1)^k \sum_{\tau=(\tau_0, \tau_1, \dots, \tau_{\ell-1})} F(k, \tau) \left(d_\ell^{\tau_{\ell-1}} \prod_{1 \leq i < \ell} ((k - \tau_{i-1} + 1) d_i)^{\tau_i - 1 - \tau_i} \right)$$

where the sum on the right hand side is taken over all $\tau \in \mathbb{N}^\ell$, with $k = \tau_0 \geq \tau_1 \geq \dots \geq \tau_{\ell-1} \geq 0$, and $\tau_i \leq k_i$, for each i , $1 \leq i < \ell$, and

$$F(k, \tau) = (k - \tau_{\ell-1} + 1) \binom{k - \tau_{\ell-1}}{\tau_0 - \tau_1, \tau_1 - \tau_2, \dots, \tau_{\ell-2} - \tau_{\ell-1}}.$$

This implies that

$$b_0(V_\ell) \leq O(1)^\ell O(k)^{2k} \left(\prod_{1 \leq j < \ell} d_j^{k_{j-1} - k_j} \right) d_\ell^{k_{\ell-1}},$$

and in particular if $\ell \leq k$,

$$b_0(V_\ell) \leq O(k)^{2k} \left(\prod_{1 \leq j < \ell} d_j^{k_{j-1} - k_j} \right) d_\ell^{k_{\ell-1}}.$$

Remark 13. Note that since the real dimension of each variety V_i is at most the complex dimension of V_i , Theorem 12 remains true if we replace real dimension by complex dimension in the statement. \square

Remark 14. In view of Example 11 above, Theorem 12 can be viewed as a weak version of the Bezout inequality over real closed fields. \square

The following slight modification of Example 11 shows that the dependence on the degrees in the bound in Theorem 12 cannot be improved.

Example 15. Let $k = k_0 \geq k_1 \geq k_2 \geq \dots \geq k_\ell = 0$, and d_1, \dots, d_ℓ be even. For $i = 1, \dots, \ell - 1$, let

$$Q_i = \sum_{j=k-k_{i-1}+1}^{k-k_i} \left(\prod_{h=1}^{d_i/2} (X_j - h) \right)^2,$$

and let

$$Q_\ell = \sum_{j=k-k_{\ell-1}+1}^k \left(\prod_{h=1}^{d_\ell/2} (X_j - h) \right)^2.$$

Then, for $0 \leq i \leq \ell$, $\deg(Q_i) = d_i$, the real dimension of the variety $V_i = \text{Zer}(Q_i, \mathbb{R}^k)$ where $Q_i = \{Q_1, \dots, Q_i\}$, is clearly k_i , and

$$b_0(V_\ell) = \frac{1}{2^k} d_1^{k_0-k_1} d_2^{k_1-k_2} \dots d_{\ell-1}^{k_{\ell-2}-k_{\ell-1}} d_\ell^{k_{\ell-1}}.$$

\square

With the same assumptions as in Theorem 12, suppose additionally that $\mathcal{P} \subset \mathbb{R}[X_1, \dots, X_k]$ is a finite family of polynomials with $\deg(P) \leq d$ for all $P \in \mathcal{P}$, and $\text{card } \mathcal{P} = s$, and suppose that $d_\ell \leq \frac{1}{k+1}d$.

Theorem 16.

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_0(\text{Reali}(\sigma, V_\ell)) \leq \sum_{j=0}^{k_\ell} 4^j \binom{s}{j} O(1)^k \Delta \quad (1)$$

where Δ is defined by

$$\Delta = \sum_{\tau = (\tau_0, \tau_1, \dots, \tau_\ell)} F(k, \tau) d^{\tau_\ell} \left(\prod_{1 \leq i \leq \ell} ((k - \tau_{i-1} + 1) d_i)^{\tau_{i-1} - \tau_i} \right),$$

where the sum is taken over all $\tau \in \mathbb{N}^{\ell+1}$, with $k = \tau_0 \geq \tau_1 \geq \dots \geq \tau_\ell \geq 0$, and $\tau_i \leq k_i$, for each $i, 1 \leq i \leq \ell$, and

$$F(k, \tau) = (k - \tau_\ell + 1) \binom{k - \tau_\ell}{\tau_0 - \tau_1, \tau_1 - \tau_2, \dots, \tau_{\ell-1} - \tau_\ell}.$$

This implies that

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_0(\text{Reali}(\sigma, V_\ell)) \leq O(1)^\ell O(k)^{2k} (sd)^{k_\ell} \left(\prod_{1 \leq j \leq \ell} d_j^{k_{j-1} - k_j} \right).$$

In particular, if $\ell \leq k$,

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_0(\text{Reali}(\sigma, V_\ell)) \leq O(k)^{2k} (sd)^{k_\ell} \left(\prod_{1 \leq j \leq \ell} d_j^{k_{j-1} - k_j} \right).$$

Remark 17. Notice that in the case $\ell = 1$, the bound (1) in Theorem 16 implies that of Theorem 8, and hence Theorem 16 is a strict generalization of Theorem 8. \square

With the same assumptions as in Theorem 16, let for $P \in \mathcal{P}$, $d_P = \deg(P)$, and for any subset $\mathcal{I} \subset \mathcal{P}$ let

$$d_{\mathcal{I}} = (k+1)^{\binom{\text{card } \mathcal{I}}{2} + (k_{\ell} - \text{card } \mathcal{I})(\text{card } \mathcal{I} - 1)} \left(\prod_{P \in \mathcal{I}} d_P \right) \left(\max_{P \in \mathcal{I}} d_P \right)^{k_{\ell} - \text{card } \mathcal{I}}.$$

We have the following variant of Theorem 16 (the extra precision with respect to the degrees of polynomials in \mathcal{P} might be useful in applications in incidence geometry).

Theorem 18.

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_0(\text{Reali}(\sigma, V_{\ell})) \leq \sum_{\mathcal{I} \subset \mathcal{P}, j = \text{card } \mathcal{I} \leq k_{\ell}} 4^j O(1)^{\ell} O(k)^{2k} d_{\mathcal{I}} \left(\prod_{1 \leq j \leq \ell} d_j^{k_{j-1} - k_j} \right).$$

Remark 19. The condition on the degrees in Theorems 12 and 16 might look unnatural at first glance but is forced on us by the method of the proof, which involves taking minors of matrices of size at most $(k+1) \times (k+1)$ with entries which are polynomials of degree d_i , $1 \leq i \leq \ell$. We want at each step, the degree d_i to majorize the degree of the polynomial obtained as a minor in the previous step whose entries have degree at most d_j , where $j < i$. Notice that in the case $\ell = 2$, the condition on the degree sequence is just $d_1 \leq d_2$, and this allows us to recover Theorem 8 from Theorem 16. \square

Remark 20. We also note that in [23] the authors define the “complexification” of a semi-algebraic set as the smallest complex variety containing it, and prove an effective bound on the geometric degree of this complexification which depend amongst other quantities on the real dimension of the given set. This degree could be thought of as the “real degree” of the semi-algebraic set. It is possible that Theorem 16 could serve as an alternative basis for a good definition of the “real degree” of a real variety – in the sense that the “real degree” of a real variety V should control the number of semi-algebraically connected components of the intersection of V with any real hypersurface of sufficiently large degree. We do not pursue this idea further in the current paper. \square

Finally, we conjecture that the bounds in Theorems 12 and 16 extend to the sum of all the Betti numbers (instead of just the zero-th one). The techniques developed in this paper are not sufficient to prove this conjecture.

2. OUTLINE OF THE PROOFS OF THEOREMS 12 AND 16

The main difficulty that one faces in order to prove bounds having the shapes of Theorems 12 and Theorem 16 is that in order to respect the degree sequence one has to be careful about taking “sums of squares” which spoil the dependence on the degrees. The crucial idea is to use the notion of “approximating” varieties. An approximating variety is a variety which is infinitesimally close to the given variety of the same dimension, but having good algebraic properties which allow one to give a precise bound on the number of its semi-algebraically connected components in terms of the sequence of degrees of polynomials defining it (rather than just the maximum degree). If the given variety can be covered (in a technical sense made precise later) by a small number of such approximating varieties, then the problem of bounding the number of semi-algebraically connected components of the given varieties reduces to the problem of bounding the total number of semi-algebraically connected components of these approximating varieties.

The idea of using approximating varieties originates in algorithmic semi-algebraic geometry and it was used in [7] to give efficient algorithms for computing sample points on varieties and in [8] to compute roadmaps of semi-algebraic sets. The combinatorial part of the complexities of these algorithms depends on the dimension of the given variety rather than that of the ambient space, and this is where the approximating varieties play an important role in those papers. In quantitative semi-algebraic geometry, the notion of approximating varieties was used in [4] in order to prove Theorem 8.

The approximation scheme that we use, which is a generalization of the one used in [4] is described in Section 3.1 below. One difficulty in generalizing the scheme in [4] is that the non-singularity of polar varieties of smooth hypersurfaces with respect to generic projections that is used in that paper no longer holds for smooth varieties of higher codimension. A second difficulty is that the sequence of local (real) dimensions at a point $x \in V_\ell$ of the varieties V_1, \dots, V_ℓ is not globally constant, but is only a local invariant. Thus, one cannot expect to have a single global approximating variety with good properties. We overcome the latter problem by taking into account all possible sequences of local dimensions whether they actually occur or not (indexed by the set A below), and construct approximating varieties with acceptable degree sequences to approximate each of them.

Consider the subset of points of U_i of V_ℓ having local dimension $i \leq k_\ell$. At each point $x \in U_i$ the dimension of $V_{\ell-1}$ is between i and $k_{\ell-1}$. Suppose we have already constructed approximations of subsets of $V_{\ell-1}$ consisting points having some fixed local dimension at $V_{\ell-1}$. Using these approximations and adding appropriately many equations in each case we construct a set of approximations of U_i . Taking all these approximating varieties, for all $i, 0 \leq i \leq k_\ell$, and noticing that V_ℓ is the union of the U_i 's we obtain a global approximation of V_ℓ .

More precisely, we construct a family of basic semi-algebraic sets each of the form,

$$S_{\mathcal{P}, \mathcal{Q}} := \{x \in \mathbb{R}^{k_\ell} \mid P(x) = 0, Q(x) \leq 0, P \in \mathcal{P}, Q \in \mathcal{Q}\},$$

where \mathbb{R}' is some real closed extension of \mathbb{R} depending on the particular approximating set. The family of pairs $\{(\mathcal{P}_{\tau, \ell}^\alpha, \mathcal{Q}_{\tau, \ell}^\alpha)\}_{\tau \in A \subset \mathbb{N}^\ell, \alpha \in I(\tau)}$ defining these approximating varieties are indexed by a pair of indices τ, α coming from two finite set of indices $A \subset \mathbb{N}^\ell$ and $I_\ell(\tau)$. While the definition of the second, $I_\ell(\tau)$, is a bit technical and which we defer for later, the definition of the index set A is the following.

$$A = \{\tau = (\tau_1, \dots, \tau_\ell) \in \mathbb{N}^\ell \mid k \geq \tau_1 \geq \tau_2 \geq \dots \geq \tau_\ell, \tau_i \leq k_i, 1 \leq i < \ell\}.$$

For any given $\tau \in A$, let $V_\tau \subset V_\ell$ denote the closure of the set of points $x \in V_\ell$ such that the local real dimension of V_i at x is equal to τ_i , for each $i, 1 \leq i \leq \ell$. The union of the approximating sets $V_{\sigma, \ell}^\alpha = S_{\mathcal{P}_\sigma^\alpha, \mathcal{Q}_\sigma^\alpha}$ with $\sigma \leq \tau$, ‘‘approximates’’ V_τ in a certain precise sense (see Proposition 46 below), and since clearly $V_\ell = \bigcup_{\tau \in A} V_\tau$, the union of all the approximating sets $\{V_{\tau, \ell}^\alpha\}_{\tau \in A \subset \mathbb{N}^\ell, \alpha \in I(\tau)}$ approximate the whole variety V_ℓ . Because of the approximating property, in order to bound the number of semi-algebraically connected components of V_ℓ it suffices to bound the sum of the number of semi-connected components of each one of the approximating sets $V_{\tau, \ell}^\alpha$. The tuples $\mathcal{P}_{\tau, \ell}^\alpha, \mathcal{Q}_{\tau, \ell}^\alpha$ have the following properties that enable us to obtain good bounds on the number of semi-algebraically connected components of $V_{\tau, \ell}^\alpha$ (see Proposition 33 below).

- a) The tuple of polynomials $\mathcal{P}_{\tau, \ell}^\alpha$ define a non-singular, bounded complete intersection of dimension $\tau_\ell \leq k_\ell$. In particular, this means that the cardinality of $\mathcal{P}_{\tau, \ell}^\alpha$ is equal to $k - \tau_\ell$. Suppose that $\mathcal{P}_{\tau, \ell}^\alpha = (P_1, \dots, P_{k - \tau_\ell})$. Let for $1 \leq i \leq \ell$, $l_i = \tau_{i-1} - \tau_i$, with the convention that $\tau_0 = k$, and $L_i = \sum_{h=1}^i l_h$. Then for each $i, 1 \leq i \leq \ell$, the degrees of the polynomials $P_{L_{i-1}+1}, \dots, P_{L_i}$ are bounded by $O(kd_i)$.

- b) $\mathcal{Q}_{\tau,\ell}^\alpha$ is either empty or contains one polynomial, $Q_{\tau,\ell}^\alpha$, with $\deg(Q_{\tau,\ell}^\alpha) = O(d_\ell)$, and $\mathcal{P}', Q_{\tau,\ell}^\alpha$, where \mathcal{P}' is any subset of $\mathcal{P}_{\tau,\ell}^\alpha$, defines a non-singular complete intersection.

It remains to bound the number of semi-algebraically connected components of each $V_{\tau,\ell}^\alpha$ and take the sum of these bounds, for which we use the same result as in [4] where a bound is derived using a classical formula for the Betti numbers of complex non-singular complete intersections and the Smith inequality (see Proposition 52 below). The number of approximating varieties (which is independent of the given degree sequence) and the bounds on the degree sequences of their defining polynomials as stated in Properties a) and b) above are good enough to give us the bound in Theorem 12.

Theorem 16 follows from Theorem 12 using standard techniques already used in [6] and no fundamentally new ingredients.

The rest of the paper is organized as follows. In Section 3, we recall some basic facts about real closed fields of Puiseux series that we need for making deformation arguments. We also recall some results proved in [4] on the choice of generic coordinates. Finally, in Section 4 we prove the main theorems. We define the approximating semi-algebraic sets in Section 4.1.1 and prove their basic properties, including their approximating property (Proposition 46) which is the main technical result needed in the proofs of the main theorems.

3. PRELIMINARY RESULTS

3.1. Deformation of several equations to general position.

We will need some properties of Puiseux series with coefficients in a real closed field. We refer the reader to [10] for further detail.

Notation 21. For \mathbb{R} a real closed field we denote by $\mathbb{R}\langle\varepsilon\rangle$ the real closed field of algebraic Puiseux series in ε with coefficients in \mathbb{R} . We use the notation $\mathbb{R}\langle\varepsilon_1, \dots, \varepsilon_m\rangle$ to denote the real closed field $\mathbb{R}\langle\varepsilon_1\rangle\langle\varepsilon_2\rangle\cdots\langle\varepsilon_m\rangle$. Note that in the unique ordering of the field $\mathbb{R}\langle\varepsilon_1, \dots, \varepsilon_m\rangle$, $0 < \varepsilon_m \ll \varepsilon_{m-1} \ll \cdots \ll \varepsilon_1 \ll 1$. Also, note that both fields $\mathbb{R}\langle\varepsilon\rangle, \mathbb{R}\langle\delta\rangle$ are sub-fields in a natural way of $\mathbb{R}\langle\varepsilon, \delta\rangle$. \square

Notation 22. If \mathbb{R}' is a real closed extension of a real closed field \mathbb{R} , and $S \subset \mathbb{R}^k$ is a semi-algebraic set defined by a first-order formula with coefficients in \mathbb{R} , then we will denote by $\text{Ext}(S, \mathbb{R}') \subset \mathbb{R}'^k$ the semi-algebraic subset of \mathbb{R}'^k defined by the same formula. It is well-known that $\text{Ext}(S, \mathbb{R}')$ does not depend on the choice of the formula defining S [10]. \square

Notation 23. For $x \in \mathbb{R}^k$ and $r \in \mathbb{R}, r > 0$, we will denote by $B_k(x, r)$ the open Euclidean ball centered at x of radius r . If \mathbb{R}' is a real closed extension of the real closed field \mathbb{R} and when the context is clear, we will continue to denote by $B_k(x, r)$ the extension $\text{Ext}(B_k(x, r), \mathbb{R}')$. This should not cause any confusion. \square

Notation 24. For elements $x \in \mathbb{R}\langle\varepsilon\rangle$ which are bounded over \mathbb{R} we denote by $\lim_\varepsilon x$ to be the image in \mathbb{R} under the usual map that sets ε to 0 in the Puiseux series x . \square

Notation 25. Let $Q \in \mathbb{R}[X_1, \dots, X_k], 0 \leq q \leq k$, and $H \in \mathbb{R}[X_{q+1}, \dots, X_k]$. Let ζ be a new variable. We denote

$$\text{Def}(Q, \zeta, q, H) = (1 - \zeta)Q - \zeta H. \quad (2)$$

\square

Notation 26. For $\mathcal{P} = (P_1, \dots, P_m)$, with each $P_i \in \mathbb{R}[X_1, \dots, X_k], 1 \leq i \leq m$, and $\mathcal{H} = (H_1, \dots, H_m)$ with each $H_i \in \mathbb{R}[X_{q+1}, \dots, X_k]$, and ζ a new variable, we denote by $\text{Def}(\mathcal{P}, \zeta, q, \mathcal{H})$ the tuple

$$(\text{Def}(P_1, \zeta, q, H_1), \dots, \text{Def}(P_m, \zeta, q, H_m)),$$

and by $\text{Def}(\mathcal{P}, \zeta, q, \mathcal{H})^h$ the corresponding tuple of homogenized polynomials

$$(\text{Def}(P_1, \zeta, q, H_1)^h, \dots, \text{Def}(P_m, \zeta, q, H_m)^h). \quad \square$$

Notation 27. For $\mathcal{F} = (F_1, \dots, F_{k-p})$, $q \leq p \leq k$, we denote the jacobian matrix

$$\text{Jac}(\mathcal{F}, p, q) := \begin{pmatrix} \frac{\partial F_1}{\partial X_{q+1}} & \cdots & \frac{\partial F_{k-p}}{\partial X_{q+1}} \\ \vdots & & \vdots \\ \frac{\partial F_1}{\partial X_k} & \cdots & \frac{\partial F_{k-p}}{\partial X_k} \end{pmatrix}$$

whose rows are indexed by $[q+1, k]$ and columns by $[1, k-p]$.

For $J \subset [q+1, k]$, $\text{card } J = k-p$ and $k \in J$, let Jac_J denote the $(k-p) \times (k-p)$ matrix extracted from the matrix $\text{Jac}(\mathcal{F}, p, q)$ by extracting the rows whose index are in J , and let

$$\text{jac}_J = \det(\text{Jac}_J).$$

Let

$$\mathcal{F}_J := \mathcal{F} \cup \bigcup_{i \in [q+1, k] \setminus J} \{\text{jac}_{J \cup \{i\} \setminus \{k\}}\},$$

and the finite constructible set

$$C_J(\mathcal{F}) := \{x \in \text{Zer}(\mathcal{F}_J, \mathbb{R}^k \mid \text{jac}_J(x) \neq 0\}.$$

\square

Proposition 28. *Let $\mathcal{F} = (F_1, \dots, F_{k-p})$, each $F_i \in \mathbb{R}[X_1, \dots, X_k]$, and such that the variety $\text{Zer}(\mathcal{F}^h, \mathbb{P}_{\mathbb{C}}^k)$ is a non-singular complete intersection. Let $x \in \mathbb{R}^k$ be a non-generate critical point of the projection map to the X_k -coordinate restricted to the variety $V = \text{Zer}(\mathcal{F}, \mathbb{R}^k)$. Then, there exists a subset $J \subset [1, k]$, $\text{card } J = k-p$, $k \in J$, satisfying the following two conditions.*

1. *The $(k-p) \times (k-p)$ matrix, Jac_J , extracted from the matrix $\text{Jac}(\mathcal{F}, p, 0)$ by extracting the rows whose index are in J , evaluated at x is non-singular.*
2. *The point x is a simple zero of the system*

$$\mathcal{F}_J := \mathcal{F} \cup \bigcup_{i \in [1, k] \setminus J} \{\text{jac}_{J \cup \{i\} \setminus \{k\}}\},$$

where $\text{jac}_{J'} = \det(\text{Jac}_{J'})$ for any $J' \subset [1, k]$ with $\text{card } J' = k-p$.

Proof Since the variety V is a p -dimensional, non-singular and x is a critical point of the projection map to the X_k coordinate restricted to V , by the inverse function theorem we can choose p coordinates (not including X_k) such that the remaining $k-p$ co-ordinates of points of V in a small enough neighborhood U of x are smooth functions of these chosen p co-ordinates. Without loss of generality let these p coordinates be X_1, \dots, X_p . We will denote the remaining co-ordinate functions on U by $X_{p+1}(X_1, \dots, X_p), \dots, X_k(X_1, \dots, X_p)$ noting that they are smooth semi-algebraic functions of X_1, \dots, X_p .

We use that

1. $\text{Jac}(\mathcal{F}, p, 0)(x)$ has full rank since x is a non-singular point of V , and
2. $\text{Hess}(X_k(X_1, \dots, X_p))(x)$ is non-singular since x is a non-degenerate critical point with respect to X_k .

Let $J = [p+1, k]$, and consider the Jacobian matrix $\text{Jac}(\mathcal{F}_J, 0, 0)$.

$$\text{Jac}(\mathcal{F}_J, 0, 0) = \begin{pmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_{k-p}}{\partial X_1} & \frac{\partial \text{jac}_{J \cup \{1\} \setminus \{k\}}}{\partial X_1} & \cdots & \frac{\partial \text{jac}_{J \cup \{p\} \setminus \{k\}}}{\partial X_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_1}{\partial X_k} & \cdots & \frac{\partial F_{k-p}}{\partial X_k} & \frac{\partial \text{jac}_{J \cup \{1\} \setminus \{k\}}}{\partial X_k} & \cdots & \frac{\partial \text{jac}_{J \cup \{p\} \setminus \{k\}}}{\partial X_k} \end{pmatrix}$$

Since, by definition of the functions $X_{p+1}(X_1, \dots, X_p), \dots, X_k(X_1, \dots, X_p)$

$$F_i(X_1, \dots, X_p, X_{p+1}(X_1, \dots, X_p), \dots, X_k(X_1, \dots, X_p)) \equiv 0,$$

for $1 \leq i \leq k-p$, by the chain rule for $1 \leq j \leq p$,

$$0 = \frac{\partial F_1}{\partial X_j} + \frac{\partial F_1}{\partial X_{p+1}} \frac{\partial X_{p+1}}{\partial X_j} + \cdots + \frac{\partial F_1}{\partial X_k} \frac{\partial X_k}{\partial X_j}$$

$$\vdots$$

(3)

$$0 = \frac{\partial F_{k-p}}{\partial X_j} + \frac{\partial F_{k-p}}{\partial X_{p+1}} \frac{\partial X_{p+1}}{\partial X_j} + \cdots + \frac{\partial F_{k-p}}{\partial X_k} \frac{\partial X_k}{\partial X_j}.$$

Let $\Delta = \det \text{Jac}(\mathcal{F}, p, p)$. Notice that in the sub-matrix $\text{Jac}(\mathcal{F}, p, 0)$ of $\text{Jac}(\mathcal{F}_J, 0, 0)$, for each $1 \leq j \leq p$, adding

$$\sum_{i=p+1}^k \frac{\partial X_i}{\partial X_j} \cdot \text{row}_i(\text{Jac}(\mathcal{F}, p, 0))$$

to the j -th row and using (2) we can clear out the first p rows. Since, $\text{rank}(\text{Jac}(\mathcal{F}, p, 0)(x)) = k-p$, this implies that $\Delta(x) \neq 0$.

From Cramer's Rule, we have

$$\frac{\partial X_k}{\partial X_1} = \frac{-\text{jac}_{J \cup \{1\} \setminus \{k\}}}{\Delta}$$

$$\vdots$$

$$\frac{\partial X_k}{\partial X_p} = \frac{-\text{jac}_{J \cup \{p\} \setminus \{k\}}}{\Delta}.$$

Let for $1 \leq i \leq p$,

$$G_i(X_1, \dots, X_p) = -\text{jac}_{J \cup \{i\} \setminus \{k\}}(X_1, \dots, X_p, X_{p+1}(X_1, \dots, X_p), \dots, X_k(X_1, \dots, X_p)).$$

Substituting above we get that

$$\frac{\partial X_k}{\partial X_1} = \frac{G_1(X_1, \dots, X_p)}{\Delta},$$

$$\vdots$$

$$\frac{\partial X_k}{\partial X_p} = \frac{G_p(X_1, \dots, X_p)}{\Delta}.$$

From the quotient rule,

$$\frac{\partial^2 X_k}{\partial X_i \partial X_j} = \frac{\frac{\partial G_1}{\partial X_i} \Delta - G_1 \frac{\partial \Delta}{\partial X_j}}{\Delta^2},$$

and in particular

$$\text{Hess}(X_k)(x) = \left(\frac{\partial^2 X_k}{\partial X_i \partial X_j}(x) \right)_{1 \leq i, j \leq p} = \left(\frac{\frac{\partial G_i}{\partial X_j}(x)}{\Delta(x)} \right)_{1 \leq i, j \leq p}$$

noticing that since x is a critical point of the function X_k restricted to V , $G_1(x) = \dots = G_p(x) = 0$.

Again from the chain rule we have that for $1 \leq i, j \leq p$,

$$\frac{\partial G_i}{\partial X_j} = -\frac{\partial \text{jac}_{J \cup \{i\} \setminus \{k\}}}{\partial X_j} - \frac{\partial \text{jac}_{J \cup \{i\} \setminus \{k\}}}{\partial X_{p+1}} \frac{\partial X_{p+1}}{\partial X_j} - \dots - \frac{\partial \text{jac}_{J \cup \{i\} \setminus \{k\}}}{\partial X_k} \frac{\partial X_k}{\partial X_j} \quad (4)$$

Finally, for each $1 \leq j \leq p$, adding

$$\sum_{i=p+1}^k \frac{\partial X_i}{\partial X_j} \cdot \text{row}_i(\text{Jac}(\mathcal{F}_J, 0, 0))$$

to the j -th row and using (2) and (4) we see that $\text{Jac}(\mathcal{F}_J, 0, 0)(x)$ is row equivalent to the matrix

$$\begin{pmatrix} \mathbf{0} & -\frac{\text{Hess}(X_k)}{\Delta}(x) \\ \mathbf{I}_{k-p} & * \end{pmatrix}$$

which is clearly non-singular, since x is a non-degenerate critical point of X_k which implies that the $\text{Hess}(X_k)(x)$ is non-singular, and we have already observed that $\Delta(x) \neq 0$. \square

Definition 29. Let $X \subset \mathbb{P}_{\mathbb{C}}^k$ be a non-singular variety, and $(H_{\mu})_{\mu=(\mu_0:\mu_1)}$ a pencil of hyperplanes. We call the pencil of varieties $(X_{\mu} = X \cap H_{\mu})_{\mu}$ a Lefschetz pencil if it satisfies the two following conditions.

1. The base locus B is smooth of co-dimension two in X .
2. Each member X_{μ} of the pencil has at most one ordinary double point as a singularity.

\square

The main result about Lefschetz pencil we will require is the following which appears as Corollary 2.10 in [26].

Proposition 30. *If $X \subset \mathbb{P}_{\mathbb{C}}^k$ is a non-singular variety, then any generic pencil of hyperplane sections of X is Lefschetz.*

Remark 31. Observe that a generic tuple of polynomials $\mathcal{H} = (H_1, \dots, H_{k-p})$ where each $H_i \in \mathbb{R}[X_1, \dots, X_k]$ with $\deg(H_i) = d_i$ and is chosen generically, will have the property that the variety $W = \text{Zer}(\mathcal{H}, \mathbb{P}_{\mathbb{C}}^k)$ is non-singular and the pencil of hyperplane sections $(W_{\mu} = W \cap H_{\mu})_{\mu}$ indexed by $\mu = (\mu_0:\mu_1)$, where $H_{\mu} \subset \mathbb{P}_{\mathbb{C}}^k$ is defined by the equation $\mu_0 X_0 + \mu_1 X_k = 0$, is a Lefschetz pencil for the variety W by Proposition 30 above. \square

Let $0 \leq p \leq k$, and $\mathcal{P} = (P_1, \dots, P_{k-p})$, $P_i \in \mathbb{R}[X_1, \dots, X_k]$ with $\deg P_i \leq d_i$, and $P \in \mathbb{R}[X_1, \dots, X_k]$, $\deg P \leq d$. Let $0 \leq q < p \leq k$ and $\mathcal{H} = (H_1, \dots, H_{k-p})$ be a tuple of polynomials with $H_i \in \mathbb{R}[X_{q+1}, \dots, X_k]$ with $\deg(H_i) = d_i$, and $H \in \mathbb{R}[X_{q+1}, \dots, X_k]$ be another polynomial with $\deg(H) = d$, such that

1. The variety $W = \text{Zer}(\mathcal{H} \cup H, \mathbb{P}_{\mathbb{C}}^{k-q})$ is a non-singular complete intersection.
2. The pencil of hyperplane sections $(W_{\mu} = W \cap H_{\mu})_{\mu}$ indexed by $\mu = (\mu_0:\mu_1)$, where $H_{\mu} \subset \mathbb{P}_{\mathbb{C}}^{k-q}$ is defined by the equation $\mu_0 X_0 + \mu_1 X_k = 0$, is a Lefschetz pencil for the variety W .

Also, let for every $y \in \mathbb{R}^q$,

$$\mathcal{F}_y = \text{Def}(\mathcal{P}, \zeta, q, \mathcal{H})(y, \cdot), \text{Def}(P, \delta, q, H)(y, \cdot).$$

We also need the following notation.

Notation 32. For $1 \leq p \leq q \leq k$, we denote by $\pi_{[p,q]}$ the projection map on the coordinates X_p, \dots, X_q , and also denote by $\mathbb{R}^{[p,q]}$ the subspace spanned by these coordinates. For any set $S \subset \mathbb{R}^k$, and $z \in \mathbb{R}^{[1,p]}$ we will denote by S_z the fiber $S \cap \pi_{[1,p]}^{-1}(z)$. \square

Proposition 33. *Then, for every $y \in \mathbb{R}^q$, the following holds.*

1. $\text{Def}(\mathcal{P}, \zeta, q, \mathcal{H})(y, \cdot)^h, \text{Def}(P, \delta, q, H)(y, \cdot)^h$ defines a non-singular complete intersection $V_y \subset \mathbb{P}_{\mathbb{C}(\delta, \zeta)}^{k-q}$ of dimension $p - q - 1$.
2. The pencil of hyperplane sections $(V_{y, \mu} = V_y \cap H_\mu)_\mu$ indexed by $\mu = (\mu_0 : \mu_1)$, where $H_\mu \subset \mathbb{P}_{\mathbb{C}(\delta, \zeta)}^{k-q}$ is defined by the equation $\mu_0 X_0 + \mu_1 X_k = 0$, is a Lefschetz pencil for the variety V_y .
3. For each singular point $x \in C^k$ of the pencil $(V_{y, \mu})_\mu$, there exists $J \subset [k - q + 1, k]$, $\text{card } J = k - p$ and $k \in J$, such that $x \in C_J(\mathcal{F}_y)$, and x is a simple zero of the system $\mathcal{F}_{y, J}$.

Proof Replacing ζ and δ by new variables s and t (respectively), and setting $s = t = 1$ we have that $(\text{Def}(\mathcal{P}, 1, q, \mathcal{H})(y, \cdot)^h, \text{Def}(P, 1, q, H)(y, \cdot)^h) = (\mathcal{H}^h, H^h)$ define a non-singular complete intersection in $W \subset \mathbb{P}_{\mathbb{C}}^{k-q}$ (by hypothesis). Moreover, the pencil of hyperplane sections $(W_\mu = W \cap H_\mu)_\mu$ is Lefschetz by hypothesis. Since the property of being a non-singular complete intersection as well as a fixed pencil of hyperplane section being Lefschetz is stable, it also holds for an open neighborhood of the point $(s, t) = (1, 1)$. The set of pairs (s, t) for which any of these two properties is violated is Zariski closed and is not the whole of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$, and in particular its complement contains the subset where $0 < s \ll t \ll 1$. This proves parts 1) and 2) of the proposition. Part 3) follows from Proposition 28. \square

We also need the following proposition.

Proposition 34. *Let C be a bounded s.a. connected component of $S_{\mathcal{P}, \mathcal{Q}}$. Then, there exists a subset $\mathcal{Q}' \subset \mathcal{Q}$ and a semi-algebraically connected component D of $\text{Zer}(\mathcal{P} \cup \mathcal{Q}', \mathbb{R}^k)$ such that $D \subset C$.*

Proof See Proposition 13.1 in [10]. \square

Proposition 35. *Let $\mathcal{F} = (F_1, \dots, F_k)$ be a tuple of polynomials with $F_i \in \mathbb{R}[X_1, \dots, X_k]$ and let $\mathcal{H} = (H_1, \dots, H_k)$, with $H_i \in \mathbb{R}[X_1, \dots, X_k]$, be a generic tuple of polynomials with $\deg(\mathcal{H}) \leq \deg(\mathcal{F})$. Let $x \in \mathbb{R}^k$ a simple zero of \mathcal{F} . Then, there exists a simple zero $\tilde{x} \in \text{Zer}(\text{Def}(\mathcal{F}, \zeta, 0, \mathcal{H}), \mathbb{R}(\zeta)^k)$, such that $\lim_{\zeta} \tilde{x} = x$.*

Proof It follows from the fact that x is a simple zero of the family \mathcal{F} that any infinitesimal perturbation of the family \mathcal{F} will have a simple zero, $\tilde{x} \in C(\zeta)^k$, in an infinitesimal neighborhood of x . Moreover, \tilde{x} must belong to $\mathbb{R}(\zeta)^k$ as long as the perturbed polynomials also have real coefficients. Otherwise, since complex zeros must occur in conjugate pairs, if $\tilde{x} \notin \mathbb{R}(\zeta)^k$, then $\tilde{x} \neq \bar{\tilde{x}}$, while $\lim_{\zeta} \tilde{x} = \lim_{\zeta} \bar{\tilde{x}} = x$, and this implies that x is not a simple zero of \mathcal{F} . \square

3.2. Generic Coordinates.

We recall in this section a result proved in [4] that we will require.

Notation 36. For a real algebraic set $V = \text{Zer}(Q, \mathbb{R}^k)$ we let $\text{reg}(V)$ denote the non-singular points in dimension $\dim V$ of V (Definition 3.3.9 in [11]). \square

Definition 37. Let $V = \text{Zer}(Q, \mathbb{R}^k)$ be a real algebraic set. Define $V^k = V$, and for $0 \leq i \leq k-1$ define

$$V^{(i)} = V^{(i+1)} \setminus \text{reg}(V^{(i+1)}).$$

Let $d_V(i)$ denote the dimension of $V^{(i)}$. \square

Definition 38. Let $V = \text{Zer}(Q, \mathbb{R}^k)$ be a real algebraic set, $1 \leq j \leq k$, and $\ell \in \text{Gr}(k, k-j)$. We say that the linear space ℓ is j -good with respect to V if either:

- $j \notin d_V([0, k])$,
- or $d_V(i) = j$ and

$$A_\ell := \{x \in \text{reg}(V^{(i)}) \mid \dim T_x V^{(i)} \cap \ell = 0\}$$

is a non-empty dense Zariski open subset of $V^{(i)}$. \square

Definition 39. Let $V = \text{Zer}(Q, \mathbb{R}^k)$ and $B = \{v_1, \dots, v_k\}$ be a basis of \mathbb{R}^k . We say that the basis B is *generic* with respect to V if for each $j, 1 \leq j \leq k$, the linear space $\text{span}(v_1, \dots, v_{k-j})$ is j -good with respect to V . \square

The following proposition appears in [4].

Proposition 40. *Let $V = \text{Zer}(Q, \mathbb{R}^k)$ and $\{v_1, \dots, v_k\}$ be a basis of \mathbb{R}^k . Then, there exists a non-empty open semi-algebraic subset U of linear transformations $\text{GL}(k, \mathbb{R})$ such that for every $T \in U$ the basis $\{T(v_1), \dots, T(v_k)\}$ is generic with respect to V .*

4. PROOFS OF THE MAIN THEOREMS

We now fix polynomials Q_1, \dots, Q_ℓ and the varieties V_1, \dots, V_ℓ as in Theorem 12. We will assume if necessary by initially squaring each polynomial that each Q_i is non-negative over \mathbb{R}^k . since this increases each degree by a multiplicative factor of 2, this does not affect the asymptotics of the bound.

4.1. Proof of Theorem 12.

We first introduce some necessary notation and then in Section 4.1.1 below we describe the construction of certain semi-algebraic sets approximating the varieties V_j . The main properties of these sets is then proved in Section 4.1.2. The approximating properties of these sets is proved in Proposition 46 and the quantitative estimates on the degrees of the polynomials appearing in the description of these approximating sets is proved in Proposition 51.

Notation 41. For any semi-algebraic set S and $x \in S$, we denote by $\dim_x S$ the local dimension of S at x . For $0 \leq j \leq \ell$ and $x \in V_j$, we denote

$$\dim^{(j)}(x) = (\dim_x V_1, \dots, \dim_x V_j). \quad \square$$

Notation 42. For $0 \leq j \leq \ell$ we call $\tau = (\tau_1, \dots, \tau_j) \in \mathbb{N}^j$ *admissible* if it satisfies the following two conditions.

1. $\tau_1 \geq \dots \geq \tau_j$,

2. for $1 \leq i < j$, $\tau_i \leq k_i$.

We denote the subset of admissible tuples of \mathbb{N}^j by A_j , and denote by A the set A_ℓ . For $\sigma = (\sigma_1, \dots, \sigma_j), \tau = (\tau_1, \dots, \tau_j) \in A_j$, we say $\sigma \leq \tau$, if $\sigma_i \leq \tau_i$ for each $i, 1 \leq i \leq j$. \square

Notation 43. For each $j, 1 \leq j \leq \ell$, we denote by \mathbb{R}_j the real closed field $\mathbb{R}\langle \delta_j, \dots, \delta_1, \zeta_1, \eta_1, \dots, \zeta_j, \eta_j \rangle$. Notice that \mathbb{R}_j is a real closed extension of the field \mathbb{R}_{j-1} . For any semi-algebraic subset $S \subset \mathbb{R}_j$, we will denote by S_b the union of semi-algebraically connected components of S which are bounded over \mathbb{R} . \square

Remark 44. For readers familiar with arguments in real algebraic geometry involving multiple infinitesimals, this ordering of the infinitesimals in Notation 43 might seem somewhat counter-intuitive, since we will consider the varieties V_i 's in the order V_1, V_2 , etc., and the infinitesimal δ_i will be used to perturb the variety V_i , one would expect that the infinitesimals δ_i 's to be ordered the other way round. The reason behind this ordering of the infinitesimals will become clear in the proof of Proposition 46 below. \square

4.1.1. Construction of families of approximating semi-algebraic sets.

We now describe the construction of certain semi-algebraic sets approximating the varieties V_j . We first assume that V_1 , and hence each V_j , are bounded over \mathbb{R} .

For any $\tau \in A_j$ we define an index set $I_j(\tau)$, and a family $(V_{\tau,j}^\alpha \subset \mathbb{R}_j^k)_{\alpha \in I_j(\tau)}$ as follows. Each $V_{\tau,j}^\alpha = (S_{\mathcal{P}_{\tau,j}^\alpha, \mathcal{Q}_{\tau,j}^\alpha})_b$, where $\mathcal{P}_{\tau,j}^\alpha, \mathcal{Q}_{\tau,j}^\alpha$ are ordered tuples of polynomials defined inductively as follows.

1. If $j=0$, then for $\tau = ()$, define $I_0(\tau) = \{-1\}$, and $\mathcal{P}_{\tau,0}^{(-1)} = (0), \mathcal{Q}_{\tau,0}^{(-1)} = ()$.
2. Otherwise, we denote by $\tau' = (\tau_1, \dots, \tau_{j-1})$ and let $p = \tau_{j-1}, q = \tau_j$. Let H be a generic polynomial in $\mathbb{R}[X_{q+1}, \dots, X_k]$ strictly positive over \mathbb{R}^{k-q} with $\deg(H) = d$,

$$\begin{aligned} \bar{P}_j &= Q_j + \sum_{1 \leq i \leq k-p} Q_i \in \mathbb{R}[X_1, \dots, X_k], \\ \tilde{P}_j &= \text{Def}(\bar{P}_j, \delta_j, q, H) \in \mathbb{R}\langle \delta_j \rangle[X_{q+1}, \dots, X_k]. \end{aligned}$$

- 3.

$$\begin{aligned} I_j(\tau) &= (I_{j-1}(\tau'), -1), \text{ if } \tau_{j-1} = \tau_j, \\ &= I_{j-1}(\tau') \times \binom{[\tau_j + 1, k]}{k - \tau_{j-1} + 1}, \text{ else.} \end{aligned}$$

4. For each triple $(\alpha \in I_{j-1}(\tau'), \mathcal{P}_{\tau',j-1}^\alpha, \mathcal{Q}_{\tau',j-1}^\alpha)$

- if $\tau_{j-1} = \tau_j$, then denoting $\beta = (\alpha, -1)$ let

$$\begin{aligned} \mathcal{P}_{\tau,j}^\beta &= \mathcal{P}_{\tau',j-1}^\alpha, \\ \mathcal{Q}_{\tau,j}^\beta &= (\tilde{P}_j). \end{aligned}$$

- otherwise, suppose that

$$\mathcal{P} = \mathcal{P}_{\tau',j-1}^\alpha = (P_1, \dots, P_{k-p}) \subset \mathbb{R}_{j-1}[X_1, \dots, X_k]^{k-p},$$

with $\deg(P_i) = d'_i$, for $1 \leq i \leq k-p$, and $\bar{d}' = (d'_1, \dots, d'_{k-p})$. Let

$$\mathcal{H} = (H_1, \dots, H_{k-p})$$

be generic polynomials in $\mathbb{R}[X_{q+1}, \dots, X_k]$ with $\deg(H_i) = d'_i$ and strictly positive over \mathbb{R}^{k-q} , $1 \leq i \leq k-p$.

We define (using Notation 26)

$$\begin{aligned}\tilde{\mathcal{P}} &= \text{Def}(\mathcal{P}, \eta_j, q, \mathcal{H}) \\ \mathcal{F} &= (\tilde{\mathcal{P}}, \tilde{P}_j).\end{aligned}\tag{5}$$

Finally, for each $J \in \binom{[\tau_j+1, k]}{k-\tau_{j-1}+1}$, denoting $\beta = (\alpha, J)$, and following the notation introduced above (and using Notation 27)

$$\begin{aligned}\mathcal{P}_{\tau,j}^\beta &= \text{Def}(\mathcal{F}_J, \zeta_j, k, \mathcal{H}'), \\ \mathcal{Q}_{\tau,j}^\beta &= \mathcal{Q}_{\tau',j-1}^\alpha,\end{aligned}\tag{6}$$

where $\mathcal{H}' = (H'_1, \dots, H'_{k-q})$ is another tuple of generic polynomials strictly positive over \mathbb{R}^k with $\deg(\mathcal{H}') = (\bar{d}_\alpha, d_j, d', \dots, d')$, where $d' = (k-p+1)d_j$ and $\bar{d}_\alpha = \deg(\mathcal{P})$.

For each $j, 0 \leq j \leq \ell$, $\tau \in A_j$, let (cf. Notation 41)

$$\begin{aligned}\tilde{V}_\tau &= \bigcup_{\alpha \in I_j(\tau)} V_{\tau,j}^\alpha, \\ V_\tau &= \overline{\{x \in V_j \mid \dim^{(j)}(x) = \tau\}}.\end{aligned}$$

Proposition 45. *For each $j, 1 \leq j \leq \ell$,*

$$V_j = \bigcup_{\tau = (\tau_1, \dots, \tau_j) \in A_j, \tau_j \leq k_j} V_\tau.$$

Proof This is immediate from the definition of A_j and the various $V_\tau, \tau = (\tau_1, \dots, \tau_j) \in A_j, \tau_j \leq k_j$, and the fact that $\dim V_i \leq k_i$ for $0 \leq i \leq j$. \square

4.1.2. Properties of the approximating sets.

The following proposition and its corollary guarantees the approximating properties of the sets $V_{\tau,j}^\alpha$ defined above and is the main technical proposition of the paper.

Assume that the given system of coordinates is generic with respect to the finite number of varieties V_τ (cf. Proposition 40).

Proposition 46. *For all $\tau = (\tau_1, \dots, \tau_j) \in A_j$, with $\tau_j \leq k_j$,*

$$V_\tau \subset W_\tau \subset V_j,$$

where

$$W_\tau = \bigcup_{\sigma} \lim_{\delta_j} \tilde{V}_\sigma$$

and the union is taken over all $\sigma \in A_j$ with $\sigma_j = \tau_j$, and $\sigma_i \leq \tau_i$ for all $1 \leq i < j$.

In the proof of Proposition 46 we need the following technical lemma that we prove first. We draw the attention of the reader to the ordering of the infinitesimals in this lemma, which is particularly delicate and plays a very important role in the proof of the lemma.

Lemma 47. *Let $P, H \in \mathbb{R}[X_1, \dots, X_k]$, P non-negative, and H strictly positive at all points of \mathbb{R}^k . Let $V \subset \mathbb{R}\langle \bar{\varepsilon} \rangle^k$ be a semi-algebraic set bounded over \mathbb{R} , where $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m)$. Let $\tilde{P} = (1 - \delta)P - \delta H$, and W a semi-algebraically connected component of $\text{Zer}(\tilde{P}, \mathbb{R}\langle \delta \rangle^k)$, such that $W = \text{Zer}(\tilde{P}, \mathbb{R}\langle \delta \rangle^k) \cap B_k(x, r)$, for some $x \in \mathbb{R}^k$ and $r > 0$, $r \in \mathbb{R}$. Suppose that $\lim_{\varepsilon_1}(\text{Ext}(V, \mathbb{R}\langle \delta, \bar{\varepsilon} \rangle)) \cap W \neq \emptyset$. Then, $\text{Ext}(V, \mathbb{R}\langle \delta, \bar{\varepsilon} \rangle) \cap \text{Ext}(W, \mathbb{R}\langle \delta, \bar{\varepsilon} \rangle) \neq \emptyset$.*

Proof Let $G \in \mathbb{R}(X_1, \dots, X_k)$ denote the rational function $\frac{P}{P+H}$ which is continuous, and takes non-negative values at all points of \mathbb{R}^k by hypothesis. Let $y \in \text{Ext}(V, \mathbb{R}\langle \delta, \bar{\varepsilon} \rangle)$ be such that $z = \lim_{\varepsilon_1} y \in W$. Since, W is contained in $\overline{B_k(x, r)}$, and $y \in \text{Ext}(V, \mathbb{R}\langle \delta, \bar{\varepsilon} \rangle)$ is ε_1 -infinitesimally close to $z \in W$, it is clear that $\text{Ext}(V, \mathbb{R}\langle \delta, \bar{\varepsilon} \rangle) \cap \overline{B_k(x, r)}$ contains y and in particular is not empty. Let C be the semi-algebraically connected component of $\text{Ext}(V, \mathbb{R}\langle \delta, \bar{\varepsilon} \rangle) \cap \overline{B_k(x, r)}$ which contains y .

We prove that $\text{Ext}(C, \mathbb{R}\langle \delta, \bar{\varepsilon} \rangle) \cap \text{Ext}(W, \mathbb{R}\langle \delta, \bar{\varepsilon} \rangle) \neq \emptyset$. Suppose otherwise. Then, $G(y) \neq \delta$. Suppose without loss of generality that $G(y) - \delta > 0$. Since, $z = \lim_{\varepsilon_1} y \in \text{Zer}(\tilde{P}, \mathbb{R}\langle \delta \rangle^k)$, it is clear that $\lim_{\varepsilon_1}(G(y) - \delta) = 0$. Let $h = \inf_{y \in C} G(y)$. Since, C is a semi-algebraic set defined over $\mathbb{R}\langle \bar{\varepsilon} \rangle$, and G is a continuous rational function defined over \mathbb{R} , it follows that $h \in \mathbb{R}\langle \bar{\varepsilon} \rangle$. Moreover, since $\text{Ext}(C, \mathbb{R}\langle \delta, \bar{\varepsilon} \rangle) \cap \text{Zer}(\tilde{P}, \mathbb{R}\langle \delta, \bar{\varepsilon} \rangle^k) = \emptyset$, $G(y) - \delta > 0$, and C is closed and bounded, the infimum of G over C is achieved at a point, and hence $h > \delta$. On the other hand, from the fact that $\lim_{\varepsilon_1}(G(y) - \delta) = 0$, it follows that $\lim_{\varepsilon_1} h = \delta$. This is impossible. \square

Proof (of Proposition 46) We first prove the inclusion $V_\tau \subset W_\tau$.

Let $x \in V_\tau$ with $\dim^{(j)}(x) = \tau$. We will prove that $x \in W_\tau$ which suffices to prove the inclusion $V_\tau \subset W_\tau$, since W_τ is closed and V_τ is the closure of the set of points y with $\dim^{(j)}(y) = \tau$. The proof of the claim that $x \in W_\tau$ is by induction on j . Suppose the claim holds for $j - 1$. There are two cases to consider.

1. $\tau_j = \tau_{j-1}$: The induction hypothesis implies that $x \in \lim_{\delta_{j-1}} \tilde{V}_{\sigma'}$, where $\sigma' \in A_{j-1}$ with $\sigma'_{j-1} = \tau_{j-1} = \tau_j$, and $\sigma'_i \leq \tau_i$ for $1 \leq i < j - 1$. Let $\alpha \in I_{j-1}(\sigma')$ be such that $x \in \lim_{\delta_{j-1}} (V_{\sigma', j-1}^\alpha)$. Hence, there exists $x' \in V_{\sigma', j-1}^\alpha$ such that $\lim_{\delta_{j-1}} x' = x$. Moreover, since, $\tilde{P}_j(x) = 0$, we have that $\lim_{\delta_{j-1}} \tilde{P}_j(x') = 0$. From the definition of \tilde{P}_j and the fact that $\delta_j \gg \delta_{j-1} > 0$, we obtain that $\tilde{P}_j(x') \leq 0$, and hence $x' \in V_{\sigma', j}^\beta$, and $x \in \lim_{\delta_j} V_{\sigma', j}^\beta$ where $\sigma = (\sigma', \tau_j)$, and $\beta = (\alpha, -1)$.
2. $q = \tau_j < \tau_{j-1}$: We prove that every neighborhood, U , of x in V_j contains a point of W_τ . Let U be a small enough neighborhood of x in V_j . Then there exists a non-empty open subset $U' \subset U$ such that each $x' \in U'$ is a regular point of V_j of dimension q . For each $x' \in U'$, clearly $q \leq \dim_{x'} V_{j-1} \leq \dim_x V_{j-1} = \tau_{j-1}$, the second inequality coming from upper semi-continuity property of the local dimension function. Now if there exists $x' \in U'$, with $\dim_{x'} V_{j-1} = q = \dim_x V_j = q$, we are reduced to case 1. So we can assume that $q < \dim_{x'} V_{j-1} \leq \tau_{j-1}$ for each $x' \in U'$. Using the genericity of the given co-ordinates and shrinking U' if necessary by subtracting a Zariski closed set of co-dimension at least one we can assume that the tangent space $T_{x'} V_j$ is transversal to $\pi_{[1, q]}^{-1}(z')$ (recall Notation 32), where $z' = \pi_{[1, q]}(x')$, and hence in particular x' is an isolated point of $(V_j)_{z'}$ for all $x' \in U'$. Shrinking U' further we can also assume that x' is not an isolated point of $(V_{j-1})_{z'}$ where $z' = \pi_{[1, q]}(x')$ for all $x' \in U'$. To see this suppose that there exists a non-empty open subset U'' of U' such that for all $x'' \in U''$, x'' is an isolated point of $(V_{j-1})_{z''}$ where $z'' = \pi_{[1, q]}(x'')$. Then, there exists for any $x'' \in U''$ an open neighborhood W of x'' in V_{j-1} contained in $(V_{j-1})_{\pi_{[1, q]}(U'')}$ such that the dimension of W is $\leq q$, which is contrary to our assumption.

Now for each $x' \in U'$, since x' is an isolated point of $(V_j)_{z'}$, and $\lim_{\delta_j} \left(\left(S_{\emptyset, \tilde{P}_j} \right)_{z'} \right)_b = (V_j)_{z'}$, there exists a semi-algebraically connected component of $\left(\left(S_{\emptyset, \tilde{P}_j} \right)_{z'} \right)_b$, and hence that of $\text{Zer}(\tilde{P}_j, \mathbb{R}\langle \delta_j \rangle^k)_{z'}$ (say W') such that $\lim_{\delta_j} W' = x'$. Since x' is not an isolated point of $(V_{j-1})_{z'}$, $\text{Ext}((V_{j-1})_{z'}, \mathbb{R}\langle \delta_j \rangle) \cap W' \neq \emptyset$, and the local dimension of $\text{Ext}((V_i)_{z'}, \mathbb{R}\langle \delta_j \rangle)$ at each point of $\text{Ext}((V_i)_{z'}, \mathbb{R}\langle \delta_j \rangle)$ is at most τ_i for $1 \leq i \leq j-1$.

We claim that there exists, $\sigma' \in A_{j-1}, \sigma' \leq \tau'$, $\alpha \in I_{j-1}(\sigma')$, where $\tau' = (\tau_1, \dots, \tau_{j-1})$, $x' \in U'$, $z' = \pi_{[1, q]}(x')$, such that $\lim_{\delta_{j-1}} ((V_{\sigma', j-1}^\alpha)_{z'})_b$ meets $\text{Ext}((V_{\tau'})_{z'}, \mathbb{R}\langle \delta_j \rangle) \cap W'$. If there does not exist such a tuple x', z', σ', α , then there exists some $\tau'' \in A_{j-1}$, with $\tau'' \leq \tau'$, such that $W_{\tau''}$ does not meet the extension to $\mathbb{R}\langle \delta_j \rangle^k$ of an open neighborhood of $\text{Ext}((V_{\tau''})_{z'}, \mathbb{R}\langle \delta_j \rangle) \cap W'$ in $\mathbb{R}\langle \delta_j \rangle^k$ of the form $\bigcup_{z \in \pi_{[1, q]}(U'')} \{z\} \times \mathbb{R}\langle \delta_j \rangle^{[q+1, k]}$ where $U'' \subset U'$ is open in U' . This contradicts the inductive hypothesis that $V_{\tau''} \subset W_{\tau''}$. Fix x', z', σ', α as above.

Notice that since $\lim_{\delta_j} W' = x'$, there exists $r > 0$, such that $W' = \text{Zer}(\tilde{P}_j, \mathbb{R}\langle \delta_j \rangle^k)_{z'} \cap \overline{B_k(x', r)}_{z'}$. It now follows from Lemma 47 that if $\lim_{\delta_{j-1}} ((V_{\sigma', j-1}^\alpha)_{z'})_b$ meets W' , then so does $\text{Ext}((V_{\sigma', j-1}^\alpha)_{z'}, \mathbb{R}\langle \delta_j \rangle)$. Note that the order $\delta_j \gg \delta_{j-1}$ is important here (cf. Remark 44).

It follows that there exists a semi-algebraically connected component C of $\text{Zer}((\mathcal{P}_{\sigma', j-1}^\alpha, \tilde{P}_j), \mathbb{R}_j^k)_{z'}$, such that $x' \in \lim_{\delta_j} C$, and $C \subset (V_{\sigma', j-1}^\alpha)_{z'}$. Moreover, using the fact that $z' \in \mathbb{R}^{[1, q]}$, and Proposition 33 we have that polynomials in $(\mathcal{P}_{\sigma', j-1}^\alpha(z', \cdot), \tilde{P}_j(z', \cdot))$ define a non-singular complete intersection of dimension $p - q - 1$ in $\mathbb{R}_j^{[q+1, k]}$, where $p = \sigma'_{j-1}$. Let $\tilde{\mathcal{P}} = \mathcal{P}_{\sigma', j-1}^\alpha$, and let \mathcal{F} be the tuple of polynomials defined by Eqn. (5). Then, there exists a semi-algebraically connected component \tilde{C} of $\text{Zer}(\mathcal{F}, \mathbb{R}_j^k)_{z'}$, such that $x' \in \lim_{\delta_j} \tilde{C}$. There are a finite number of X_k -critical points (all of which are simple) on \tilde{C} by Remark 31 and Proposition 33. If (z', w') , $w' \in \mathbb{R}_j^{[q+1, k]}$, is one such critical point, then (z', w') is contained in the finite constructible set $C_J(\mathcal{F})$ for some $J \in \binom{[q+1, k]}{k-p+1}$, and such that w' is a simple zero of the system $\mathcal{F}_J(z', \cdot)$. Hence, there exists by Proposition 35 a simple zero, w'' , of the system $\mathcal{P}_{\sigma, j}^\beta(z, \cdot)$ (cf. (6)) where $\sigma = (\sigma', \tau_j)$ and $\beta = (\alpha, J)$, such that $\lim_{\delta_j} w'' = w'$. Clearly, then $x'' = (z', w'') \in V_{\sigma, j}^\beta$, and $x' = \lim_{\delta_j} x''$ and thus $x' \in \lim_{\delta_j} V_{\sigma, j}^\beta$. Notice that $\sigma_j = \tau_j$ and $\sigma \leq \tau$.

The inclusion $\lim_{\delta_j} \tilde{V}_\tau \subset V_j$, from which the second inclusion $V_\tau \subset V_j$ follows immediately, is due to the fact that for each $\beta \in I_j(\tau)$, $V_{\tau, j}^\beta$ is either contained in the part of the semi-algebraic set defined by $\tilde{P}_j \leq 0$ which is bounded over \mathbb{R} or in the algebraic variety $\text{Zer}(\tilde{P}_j, \mathbb{R}_j^k)_b$ depending on whether $\tau_{j-1} = \tau_j$ or $\tau_{j-1} > \tau_j$ respectively. It is clear from definition of \tilde{P}_j , that the images under \lim_{δ_j} of the last two sets are contained in V_j . \square

The following slight refinement of Proposition 46 is required to ensure that the degree of the last polynomial does not enter the bound with a factor of $(k - \tau_{i-1} - 1)$ as is the case of the other degrees d_i , with $i < \ell$, but rather just as d_ℓ . This slight improvement is possible since we do not need to ensure that the dimension of the approximating varieties drops appropriately (to k_ℓ) when we approximate the last variety V_ℓ . If we were not interested in obtaining the tightest possible dependence on k in the multiplicative factor in the bound (the factor that is independent of the degrees), then this refinement would not have been necessary. However, in order to ensure that the results in the current paper properly generalize the results in [4] we need to take this extra care.

Notation 48. For all $\sigma = (\sigma_1, \dots, \sigma_j) \in A_j$, denote by $\sigma' = (\sigma_1, \dots, \sigma_{j-1}, \sigma_{j-1})$. \square

Corollary 49. For all $\tau \in A_j$,

$$V_\tau \subset W'_\tau \subset V_j,$$

where

$$W'_\tau = \bigcup_{\sigma} \lim_{\delta_j} \tilde{V}_{\sigma'}$$

and the union is taken over all $\sigma \in A_j$ with $\sigma_j = \tau_j$, and $\sigma_i \leq \tau_i$ for all $1 \leq i < j$.

Proof It is clear from the definition that for all $\sigma \in A_j$

$$\tilde{V}_\sigma \subset \tilde{V}_{\sigma'},$$

and that $\lim_{\delta_j} \tilde{V}_{\sigma'} \subset V_j$. The corollary now follows from Proposition 46. \square

Corollary 50.

$$b_0(V_\ell) \leq \sum_{\tau \in A_\ell} \sum_{\beta \in I_\ell(\tau')} b_0(V_{\tau', \ell}^\beta).$$

Proof Follows immediately from Corollary 50 after noting that (using Proposition 45)

$$V_\ell = \bigcup_{\tau \in A} V_\tau.$$

\square

Following notation introduced above we have:

Proposition 51. Let $\tau \in A_j$, $\tau_{j-1} = p$, and $\alpha \in I_j(\tau')$.

1. Then $\text{card } \mathcal{P}_{\tau', j}^\alpha = k - p$, and $\text{card } \mathcal{Q}_{\tau', j}^\alpha = 1$.
2. Suppose that $\mathcal{P}_{\tau', j}^\alpha = (P_1, \dots, P_{k-p})$. Let for $1 \leq i \leq j - 1$, $\ell_i = \tau_{i-1} - \tau_i$, with the convention that $\tau_0 = k$, and $L_i = \sum_{h=1}^i \ell_h$. Then for each i , $1 \leq i < j$, the degrees of the polynomials $P_{L_{i-1}+1}, \dots, P_{L_i}$ are bounded by $(k - \tau_{i-1} + 1) d_i \leq (k + 1) d_i$.
3. $\deg Q \leq d_\ell$ for $Q \in \mathcal{Q}_{\tau', j}^\alpha$.
4. $b_0(V_{\tau', j}^\alpha) \leq \sum_{\mathcal{Q}' \subset \mathcal{Q}_{\tau', j}^\alpha} b_0(\text{Zer}((\mathcal{P}_{\tau', j}^\alpha, \mathcal{Q}')_b, \mathbb{R}_j^k))$.

Proof Follows from the definitions of the tuples $\mathcal{P}_{\tau', j}^\alpha$, $\mathcal{Q}_{\tau', j}^\alpha$ and Proposition 34. \square

The following proposition appears in [4] and is a consequence of the classical formula for the Euler-Poincare characteristic of non-singular complex projective intersections and the Smith inequality.

Proposition 52. Let $\mathcal{F} = \{F_1, \dots, F_m\} \subset \mathbb{R}[X_1, \dots, X_k]$ with $\deg(F_i) = d_i$, $d_1 \leq d_2 \leq \dots \leq d_m$. Moreover, assume that $\mathcal{F}^h = \{F_1^h, \dots, F_m^h\}$ defines a non-singular complete intersection in $\mathbb{P}_{\mathbb{C}}^k$. Then,

$$b_0(\text{Zer}(\mathcal{F}, \mathbb{R}^k)_b) \leq \binom{k+1}{m+1} d_1 \cdots d_{m-1} d_m^{k-m+1} + 2(k-m+1).$$

Remark 53. We note that in Proposition 52 if the polynomials in \mathcal{F} do not define a non-singular complete intersection, it is still possible to bound the sum of the Betti numbers of the corresponding complex variety by $O(1)^m O(m d_m)^k$ using a result of Katz [18], which in turn uses previous results of Bombieri [12], and Adolphson and Sperber [1]. These results use the theory of exponential sums over finite fields, and are of a much deeper nature than the classical formula giving the Betti numbers in terms of the degree sequence in the non-singular complete intersection case which is used to prove Proposition 52. However, the results of Katz [18] which do not assume non-singularity and are very general, do not have the finer dependence on the degree sequence (see the bound given above), and this finer dependence on the degree sequence is the key point in Proposition 52 above. \square

Corollary 54. For each $\tau = (\tau_1, \dots, \tau_\ell) \in A_\ell$ and $\alpha \in I_\ell(\tau')$ and $\mathcal{Q} \subset \mathcal{Q}_{\tau', \ell}^\alpha$,

$$b_0(\text{Zer}((\mathcal{P}_{\tau', \ell}^\alpha, \mathcal{Q})_b, \mathbf{R}_\ell^k)) \leq O(1)^k d_\ell^{\tau_\ell - 1} \prod_{1 \leq i < \ell} ((k - \tau_{i-1} + 1) d_i)^{\tau_{i-1} - \tau_i}.$$

It now follows from Corollary 54 and part 4) of Proposition 51 that

Corollary 55. For each $\tau = (\tau_1, \dots, \tau_\ell) \in A_\ell$ and $\alpha \in I_\ell(\tau')$

$$b_0(V_{\tau, \ell}^\alpha) \leq O(1)^k d_\ell^{\tau_\ell - 1} \prod_{1 \leq i < \ell} ((k - \tau_{i-1} + 1) d_i)^{\tau_{i-1} - \tau_i}.$$

Let $\tau \in A_\ell$ and d_1, \dots, d_ℓ satisfy the hypothesis of Theorem 12.

Lemma 56. Then,

$$\frac{d_\ell^{\tau_\ell - 1} \prod_{1 \leq i < \ell} ((k - \tau_{i-1} + 1) d_i)^{\tau_{i-1} - \tau_i}}{d_\ell^{k_\ell - 1} \prod_{1 \leq i < \ell} ((k - k_{i-1} + 1) d_i)^{k_{i-1} - k_i}} \leq O(k)^k.$$

Proof Using the inequality that for $2 \leq i \leq \ell$,

$$\frac{d_{i-1}}{d_i} \leq \frac{1}{k+1} \leq \frac{1}{k - k_{i-2} + 1}$$

we get that the expression on the left hand side of the proposition is bounded by

$$\frac{\prod_{1 \leq i < \ell} (k - \tau_{i-1} + 1)^{\tau_{i-1} - \tau_i}}{\prod_{1 \leq i < \ell} (k - k_{i-1} + 1)^{k_{i-1} - \tau_i}}.$$

The sum of the various exponents of the numerator is

$$\sum_{i=1}^{\ell-1} (\tau_{i-1} - \tau_i) = \tau_0 - \tau_{\ell-1} \leq k,$$

and for each $i, 1 \leq i < \ell$, $(k - \tau_{i-1} + 1) \leq (k + 1)$. The denominator is a non-zero integer. \square

We next bound the cardinality of the index set A_ℓ .

Lemma 57. The cardinality of A_ℓ is bounded by

$$O(1)^{k+\ell}.$$

Proof The number of tuples $\tau = (\tau_1, \dots, \tau_\ell)$ in which $k \geq \tau_1 > \tau_2 > \dots > \tau_\ell \geq 0$ is bounded by the volume of the corresponding ℓ -dimensional simplex in \mathbb{R}^ℓ which is equal to $\frac{(k+1)^\ell}{\ell!}$. Allowing some of the τ_i 's to be equal, the number of tuples is bounded by

$$\sum_{0 \leq i \leq \ell} \binom{\ell}{i} \frac{(k+1)^{\ell-i}}{(\ell-i)!} \leq 2^\ell \sum_{0 \leq i \leq \ell} \frac{(k+1)^{\ell-i}}{(\ell-i)!} = O(1)^{k+\ell}.$$

□

Lemma 58. For each $\tau = (\tau_1, \dots, \tau_\ell)$ the cardinality of the index set $I_\ell(\tau)$ is bounded by

$$(k - \tau_\ell + 1) \binom{k - \tau_\ell}{\tau_1 - \tau_2, \dots, \tau_{\ell-1} - \tau_\ell}.$$

Proof It is clear from the definition that the cardinality of the index set $I_\ell(\tau)$ is bounded by

$$\begin{aligned} \prod_{1 \leq j \leq \ell} \binom{k - \tau_j + 1}{k - \tau_{j-1} + 1} &= \frac{(k - \tau_\ell + 1)!}{(\tau_0 - \tau_1)! (\tau_1 - \tau_2)! \dots (\tau_{\ell-1} - \tau_\ell)!} \\ &= (k - \tau_\ell + 1) \binom{k - \tau_\ell}{\tau_0 - \tau_1, \tau_1 - \tau_2, \dots, \tau_{\ell-1} - \tau_\ell}. \end{aligned}$$

□

Proof (of Theorem 12). We first prove the theorem in case V_0 is bounded. It follows from Corollary 50 and Corollary 55 that

$$b_0(V_\ell) \leq \sum_{\tau \in A_\ell} \sum_{\alpha \in I_\ell(\tau')} \left(O(1)^k d_\ell^{\tau_\ell - 1} \prod_{1 \leq i < \ell} ((k - \tau_{i-1} + 1) d_i)^{\tau_{i-1} - \tau_i} \right).$$

Using Lemma 58 to bound the cardinality of the index set $I_\ell(\tau')$, we get that the right hand side of the above inequality is bounded by

$$O(1)^k \sum_{\tau \in A_\ell} F(k, \tau) \left(d_\ell^{\tau_\ell - 1} \prod_{1 \leq i < \ell} ((k - \tau_{i-1} + 1) d_i)^{\tau_{i-1} - \tau_i} \right),$$

where

$$F(k, \tau) = (k - \tau_{\ell-1} + 1) \binom{k - \tau_{\ell-1}}{\tau_0 - \tau_1, \tau_1 - \tau_2, \dots, \tau_{\ell-2} - \tau_{\ell-1}}.$$

The theorem in the bounded case now follows from Lemma 56 and Lemma 57.

In the general case, we first replace the given sequence of polynomials Q_1, \dots, Q_ℓ , by a new sequence, Q_0, Q_1, \dots, Q_ℓ , where

$$Q_0 = \sum_{i=1}^{k+1} X_i^2 - \Omega,$$

where Ω is infinitely large and positive over \mathbb{R} . For each $i, 0 \leq i \leq \ell$, defining $\hat{Q}_i = \{Q_0, \dots, Q_i\}$, and $\hat{V}_i = \text{Zer}(\hat{Q}_i, \mathbb{R}\langle 1/\Omega \rangle^{k+1})$ we have that each \hat{V}_i is bounded over $\mathbb{R}\langle 1/\Omega \rangle$, and also that $b_0(V_\ell) \leq b_0(\hat{V}_\ell)$. Applying the same arguments as in the bounded case we obtain that

$$b_0(\hat{V}_\ell) \leq O(1)^k \sum_{\tau = (\tau_{-1}, \tau_0, \dots, \tau_{\ell-1})} F(k, \tau) \left(d_\ell^{\tau_\ell - 1} \prod_{1 \leq i < \ell} ((k - \tau_{i-1} + 1) d_i)^{\tau_{i-1} - \tau_i} \right),$$

where the sum is taken over all $\tau \in \mathbb{N}^\ell$, with $k + 1 = \tau_{-1} > k = \tau_0 \geq \tau_1 \cdots \geq \tau_{\ell-1} \geq 0$, and $\tau_i \leq k_i$, for each $i, 1 \leq i < \ell$, and

$$F(k, \tau) = (k - \tau_{\ell-1} + 1) \binom{k - \tau_{\ell-1}}{\tau_0 - \tau_1, \tau_1 - \tau_2, \dots, \tau_{\ell-2} - \tau_{\ell-1}}.$$

Notice that since the local dimension of the variety \hat{V}_0 is constant, it suffices to fix $\tau_0 = k$ in the sum above, and the contribution of the degree of the polynomial Q_0 gets absorbed into the $O(1)^k$ term. □

4.2. Proof of Theorem 16.

We introduce a new family of polynomials defined as follows.

$$\tilde{\mathcal{P}} = \bigcup_{1 \leq i \leq s} \{P_i \pm \varepsilon \gamma_i, P_i \pm \delta \gamma_i\},$$

where $\varepsilon, \delta, \gamma_1, \dots, \gamma_s$ new variables. For any subset $I = \{(\epsilon_1, \sigma_1, i_1), \dots, (\epsilon_m, \sigma_m, i_m)\} \subset \{+1, -1\} \times \{\varepsilon, \delta\} \times \{1, \dots, s\}$ we denote by $\tilde{\mathcal{P}}_I$ the subset of $\tilde{\mathcal{P}}$ defined by

$$\tilde{\mathcal{P}}_I = \bigcup_{1 \leq j \leq m} \{P_{i_j} + \epsilon_j \sigma_j \gamma_{i_j}\}.$$

Let \mathbb{R}' denote the real closed field $\mathbb{R}\langle \varepsilon, \delta, \gamma_1, \dots, \gamma_s \rangle$.

Proposition 59. *For each $I \subset \{+1, -1\} \times \{\varepsilon, \delta\} \times \{1, \dots, s\}$, the dimension of the variety $\text{Zer}(\tilde{\mathcal{P}}, \mathbb{R}^k) \cap \text{Ext}(V_\ell, \mathbb{R}')$ is at most $k_\ell - \text{card } I$. In particular, $\text{Zer}(\tilde{\mathcal{P}}, \mathbb{R}^k) \cap \text{Ext}(V_\ell, \mathbb{R}')$ is empty if $\text{card } I > k_\ell$.*

Proof It follows immediately from the fact that the various γ_i 's are algebraically independent over \mathbb{R} . □

For any finite family $\mathcal{F} \subset \mathbb{R}[X_1, \dots, X_k]$ we call a formula $\bigwedge_{F \in \mathcal{F}} F \sigma_F 0$, where each $\sigma_F \in \{\geq, \leq\}$ a *weak sign condition* on \mathcal{F} .

Proposition 60. *Let V_1 be bounded, and let $\sigma \in \{-1, 0, 1\}^{\mathcal{P}}$ and C a semi-algebraically connected component of $\text{Reali}(\sigma, V_\ell) \subset \mathbb{R}^k$. Then, there exists a weak sign condition $\tilde{\sigma}$ on $\tilde{\mathcal{P}}$ and a semi-algebraically connected component \tilde{C} of $\text{Reali}(\tilde{\sigma}, \text{Ext}(V_\ell, \mathbb{R}'))$ such that $\lim_{\delta} \tilde{C} \subseteq \text{Ext}(C, \mathbb{R}')$.*

Proof The proof is similar to the proof Proposition 4 in [6] and omitted. □

The following proposition occurs in [10] (Proposition 13.1).

Proposition 61. *Let V_1 be bounded and let $\mathcal{F} \subset \mathbb{R}[X_1, \dots, X_k]$ be a finite set of polynomials and $\tilde{\sigma}$ a weak sign condition on \mathcal{F} . Let C be a semi-algebraically connected component of $\text{Reali}(\tilde{\sigma}, \text{Ext}(V_\ell, \mathbb{R}'))$. Then there exists a subset $\mathcal{F}' \subset \mathcal{F}$ and a semi-algebraically connected component D of $\text{Zer}(\mathcal{F}', \text{Ext}(V_\ell, \mathbb{R}'))$, such that $D \subset C$.*

Proof (of Theorem 16). In the case V_1 is bounded, using successively Propositions 60 and 61 it suffices to bound the total number of semi-algebraically connected components of the real algebraic sets

$$\text{Zer}\left(\mathcal{Q}_\ell \cup \tilde{\mathcal{P}}_I, \mathbb{R}^k\right)$$

for subsets $I \subset \{+1, -1\} \times \{\varepsilon, \delta\} \times \{1, \dots, s\}$. Moreover, using Proposition 59 the set of different subsets I that we need to consider is bounded by

$$\sum_{j=0}^{k_\ell} 4^j \binom{s}{j} = O(s)^{k_\ell}.$$

Notice that each $\text{Zer}\left(\mathcal{Q}_\ell \cup \tilde{\mathcal{P}}_I, \mathbb{R}^{rk}\right) = \text{Zer}\left((Q_1, \dots, Q_\ell, P_I), \mathbb{R}^{rk}\right)$, where $P_I = \sum_{P \in \tilde{\mathcal{P}}_I} P^2$. The sequence of degrees of the polynomials $(Q_1, \dots, Q_\ell, P_I) = (d_1, \dots, d_\ell, 2d)$. Now apply Theorem 12 to finish the proof. In the general case, use the same technique as in the proof of Theorem 12 to reduce to the bounded case. \square

Proof (of Theorem 18). In the proof of Theorem 16 instead of bounding the number of semi-algebraically connected components of the various algebraic sets $\text{Zer}\left(\{Q_1, \dots, Q_\ell, P_I\}, \mathbb{R}^{rk}\right)$ using Theorem 12, apply Theorem 12 directly to the sequence $\mathcal{Q}, \tilde{\mathcal{P}}_I$, noting that its real zeros are the same as $\text{Zer}\left(\{Q_1, \dots, Q_\ell, P_I\}, \mathbb{R}^{rk}\right)$, and also that the degree sequence associated to $\tilde{\mathcal{P}}_I$ can be made to satisfy the requirement of Theorem 12 by multiplying the i -th largest degree in the sequence by $(k+1)^{i-1}$. \square

Acknowledgments. The authors were partially supported by NSF grant CCF-0915954.

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