Refined bounds on the number of connected components of sign conditions on a variety

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Outline

1. History and previous results
2. Main result
    - Statement of main result
    - Idea of the proof
3. Future work

- Let R be a real closed field.
- Let $\mathcal{Q} \subset \mathrm{R}[X_1, \ldots, X_k]$ be a finite set of polynomials with $\deg Q \le d_0$, $Q \in \mathcal{Q}$.
- Denote the common zeros of $Q \in \mathcal{Q}$ by

$$V = \{x \in \mathrm{R}^k | \bigwedge_{Q \in \mathcal{Q}} Q(x) = 0\},$$

and we suppose the real dimension of V in R^k is $k' \le k$.
- Let $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ be a finite set of polynomials with $\deg P \le d$, $P \in \mathcal{P}$, and $\mathrm{card}\, \mathcal{P} = s$.

Definition

A *sign condition* on $\mathcal{P}$ is an element of $\{0, -1, 1\}^{\mathcal{P}}$. The *realization of a sign condition* σ *on* V is the semi-algebraic set

$$\mathcal{R}(\sigma, V) = \{x \in V | \bigwedge_{P \in \mathcal{P}} \mathbf{sign}\, P(x) = \sigma(P)\}.$$

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For sign conditions on a variety:

- (B.-Pollack-Roy, '05) gave a bound for the sum of the i-th Betti numbers over all sign conditions of $\mathcal{P}$ on V, of the form $\binom{s}{k'-i} O(d)^k$ in the case where $d = d_0$.

$$\sum_{\sigma \in \{-1,1,0\}^{\mathcal{P}}} b_i(\mathcal{R}(\sigma, V)) \leq \sum_{j=0}^{k'-i} \binom{s}{j} 4^j d(2d-1)^{k-1}.$$

- As a special case (when $i = 0$) we get a bound on the number of connected components of all realizable sign conditions of $\mathcal{P}$ on the variety V, given by

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Motivated by a new technique in discrete geometry developed by recently by Guth and Katz, it was suggested by J. Matousek that the roles of the degrees d and d_0 in the previous bound might be separated, resulting in a better bound in some cases (when $d_0 \ll d$). This has become very important in applications in discrete geometry.

For example, in a recent paper of Solymosi and Tao, a weaker result of this kind is used to prove a bound on the number of incidences between points and certain k' dimensional varieties of bounded degree.

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Main Theorem

- Let R be a real closed field, and let $\mathcal{Q}, \mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ be finite subsets of polynomials;
- let $\deg(Q) \leq d_0$ for all $Q \in \mathcal{Q}$, $\deg P = d_P$ for all $P \in \mathcal{P}$;
- let the real dimension of $\mathrm{Zer}(\mathcal{Q}, \mathrm{R}^k)$ is $k' \leq k$;
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- for $\mathcal{I} \subseteq \mathcal{P}$ we denote by $d_{\mathcal{I}} = \prod_{P \in \mathcal{I}} d_P$.

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Theorem

With the notation as above,

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_0(\mathcal{R}(\sigma, \mathrm{Z}(\mathcal{Q}, \mathrm{R}^k)))$$

is at most

$$\sum_{\substack{\mathcal{I} \subset \mathcal{P} \\ \#\mathcal{I} \leq k'}} 4^{\#\mathcal{I}} \binom{k+1}{k-k'+\#\mathcal{I}+1} D$$

where

$$D = (2d_0)^{k-k'} d_{\mathcal{I}} \max_{P \in \mathcal{I}} \{2d_0, d_P\}^{k'-\#\mathcal{I}} + 2(k - \#\mathcal{I} + 1).$$

Continued

In particular, if $d_P \leq d$ for all $P \in \mathcal{P}$, we have that

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_0(\mathcal{R}(\sigma, \mathrm{Z}(\mathcal{Q}, \mathrm{R}^k)))$$

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$$\sum_{j=0}^{k'} 4^j \binom{s+1}{j} \left(\binom{k+1}{k-k'+j+1} (2d_0)^{k-k'} d^j \max\{2d_0, d\}^{k'-j} + 2(k-j+1) \right).$$

In the case $2d_0 \leq d$, the bound can be written simply as Red

$$\sum_{\sigma \in \{0,-1,1\}^{\mathcal{P}}} b_0(\mathcal{R}(\sigma, V)) \leq (sd)^{k'} d_0^{k-k'} O(1)^k.$$

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It is instructive to examine the two extreme cases when $k' = k - 1$ and $k' = 0$. The following two examples show that this bound is tight, up to a factor of $O(1)^k$.

- **Case $k' = 0$:** When $k' = 0$, the variety $\mathrm{Z}(\mathcal{Q}, \mathrm{R}^k)$ is zero dimensional and is the union of at most $O(d_0)^k$ isolated points. The bound of the theorem reduces to $O(d_0)^k$ in this case, and is thus tight.

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- **Case $k' = k-1$:** When $k' = k-1$ and $2d_0 \leq d$, the bound of the theorem is equal to

$$2(2^k+1)d_0 \sum_{j=0}^{k-1} \binom{s+1}{j} d^{k-1} = d_0 O(sd)^{k-1}.$$

Let $\mathcal{P}$ be the set of s polynomials in $X_1, \ldots, X_k$ each of which is the product of d generic linear forms. Let $\mathcal{Q} = \{Q\}$, where

$$Q = \prod_{1 \leq i \leq d_0} (X_k - i).$$

Since the intersection of $\cup_{P \in \mathcal{P}} \mathrm{Z}(P, \mathrm{R}^k)$ with the hyperplane defined by $X_k = i$ is a union of sd generic hyperplanes in R^{k-1}, the number of semi-algebraically connected components of all realizable *strict* sign conditions of $\mathcal{P}$ on $\mathrm{Z}(\mathcal{Q}, \mathrm{R}^k)$ is equal to

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Remark

The number of semi-algebraically connected components (indeed even isolated zeros) of a set of polynomials $\{P_1, \ldots, P_m\} \subset \mathrm{R}[X_1, \ldots, X_k, X_{k+1}]$ with degrees $d_1, \ldots, d_m$ can be greater than the product $d_1 \cdots d_m$. Let $\mathcal{P} = \{P_1, \ldots, P_m\} \subset \mathrm{R}[X_1, \ldots, X_{k+1}]$ be defined as follows.

$$P_1 = \sum_{i=1}^{k} \prod_{j=1}^{d} (X_i - j)^2$$

$$P_j = \prod_{i=1}^{m-j+2} (X_{k+1} - i), 2 \leq j \leq m.$$

Let $\mathcal{P}_i = \{P_1, \ldots, P_i\}$. Notice that for each $i, 1 \leq i < m$, $\mathrm{Z}(\mathcal{P}_i, \mathrm{R}^{k+1})$ strictly contains $\mathrm{Z}(\mathcal{P}_{i+1}, \mathrm{R}^{k+1})$. Moreover, $b_0(\mathrm{Z}(\mathcal{P}, \mathrm{R}^{k+1})) = 2d^k$, while the product of the degrees of the polynomials in $\mathcal{P}$ is $2dm!$. Clearly, for d large enough $2d^k > 2dm!$.

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Remark about proof techniques

- Most of the previously known bounds on the Betti numbers of realizations of sign conditions relied ultimately on the Oleĭnik-Petrovskiĭ-Thom-Milnor bounds on the Betti numbers of real varieties.
- (Oleinĭk-Petrovskiĭ, Thom, Milnor) proved a bound of $d(2d-1)^{k-1}$ on the sum of the Betti numbers of a real algebraic variety in R^k defined by polynomials of degree at most d.
- Since in the proofs of these bounds the finite family of polynomials defining a given real variety is replaced by a single polynomial by taking a sum of squares, it is not possible to separate out the different roles played by the degrees of the polynomials in $\mathcal{P}$ and those in $\mathcal{Q}$.

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- We avoid using the Oleĭnik-Petrovskiĭ-Thom-Milnor bounds, but uses directly classically known formulas for the Betti numbers of smooth, complete intersections in complex projective space. The bounds obtained from these formulas depend more delicately on the individual degrees of the polynomials involved, and this allows us to separate the roles of d and d_0 in our proof.

In the proof of the refined bound, we require the use of certain non-archimedean real closed field extensions of R, namely $\mathrm{R}\langle\varepsilon\rangle$ the field of *algebraic Puiseux series in* ε with coefficients in R.

- An element $x \in \mathrm{R}\langle\varepsilon\rangle$,

$$x = \sum_{i \geq -p} a_i \varepsilon^{i/q},$$

  is bounded over R if $|x| \leq r$ for some $0 \leq r \in \mathrm{R}$, and coincides with those algebraic Puiseux series without negative exponents.

- The $\lim_\varepsilon$ map from the subring of bounded elements $\mathrm{R}\langle\varepsilon\rangle_b$ to R takes ε to 0. For $S \subset \mathrm{R}\langle\varepsilon\rangle_b^k$ we use the notation $\lim_\varepsilon(S)$ for the coordinate-wise map.

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The main steps of the proof of the refined bound:

- The main idea behind our improved bound is to reduce the problem of bounding the number of semi-algebraically connected components of all sign conditions on a variety to the problem of bounding the sum of the $\mathbb{Z}_2$-Betti numbers of certain smooth complete intersections in complex projective space.
- We consider the real closed extension of R to $\mathrm{R}\langle 1/\Omega\rangle$, and from now on we only consider subsets of $B_k(0,\Omega)$, where $B_k(0,\Omega)$ is the semi-algebraic set defined by $|x|^2 < \Omega$.

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- We set $Q = \sum_{F \in \mathcal{Q}} F^2$.
- We consider an infinitesimal perturbation of Q, the polynomial

$$\mathrm{Def}(Q, \zeta) := (1 - \zeta)Q - \zeta H,$$

where H is a generic non-negative real polynomial of degree $2d_0$, and prove

Proposition

Let C be a semi-algebraically connected component of

$$\mathrm{Z}(Q, \mathrm{R}\langle 1/\Omega\rangle^k) \cap B_k(0, \Omega).$$

Then, there exists a semi-algebraically connected component, $D \subset \mathrm{R}\langle 1/\Omega, \zeta\rangle^k$ of the semi-algebraic set

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- We then consider the *critical system of* $\mathrm{Def}(Q,\zeta)$,

$$\mathrm{Cr}_{k-k'-1}(\mathrm{Def}(Q,\zeta)) := \left\{ \mathrm{Def}(Q,\zeta), \frac{\partial \mathrm{Def}(Q,\zeta)}{\partial X_1}, \ldots, \frac{\partial \mathrm{Def}(Q,\zeta)}{\partial X_{k-k'-1}} \right\},$$

  to *cut out* a subvariety W,

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  - $\lim_\zeta W = \mathrm{Z}(Q, \mathrm{R}\langle 1/\Omega\rangle^k) \cap \overline{B_k(0,\Omega)}$, and
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- We prove that these intersections X satisfies:
  - $X_{\mathbb{C}'}$ is a non-singular complete intersection in $\mathbb{P}^k_{\mathbb{C}'}$, and
  - there is a correspondence between the semi-algebraically connected components of $\mathcal{R}(\tau, W)$ and those of X.

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- It now suffices to bound the sum of the numbers of connected components of these intersections X. The final bound is obtained by multiplying the combinatorial part (the number of such intersections X) with the algebraic part (a bound on the 0-th Betti number of each such X).
- We bound the number of connected components of the real variety X by bounding, in fact, the sum of the Betti numbers of the complex projective variety defined by the same equations as X and then using the Smith Inequality. (It is also possible to use bi-homogeneous Bezout bounds at this point to directly bound the number of semi-algbraically connected components of X.
- (Smith inequality) Let $\mathcal{Q} \subset \mathrm{R}[X_1, \ldots, X_k]$ be a finite set of homogeneous polynomials, then

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- Once we have a non-singular complex projective variety which is a complete intersection, there is a classical formula (see Benedetti-Loeser-Risler, '91) which expresses the sum of the Betti numbers in terms of the degree sequence of the defining polynomials.

A complex interlude

Definition

A projective variety $X \subset \mathbb{P}^k_{\mathbb{C}}$ of codimension m is a *non-singular complete intersection* if it is the intersection of m non-singular hypersurfaces in $\mathbb{P}^k_{\mathbb{C}}$ that meet transversally at each point of the intersection.

Fix an m-tuple of natural numbers $\bar{d} = (d_1, \ldots, d_m)$. Let $X_{\mathbb{C}} = \mathrm{Zer}(\{Q_1, \ldots, Q_m\}, \mathbb{P}^k_{\mathbb{C}})$, such that the degree of Q_i is d_i, denote a complex projective variety of codimension m which is a non-singular complete intersection. It is a classical fact that the Betti numbers of $X_{\mathbb{C}}$ depend only on the degree sequence and not on the specific $X_{\mathbb{C}}$. In fact, it follows from Lefshetz theorem on hyperplane sections that

$$b_i(X_{\mathbb{C}}) = b_i(\mathbb{P}^k_{\mathbb{C}}),\ 0 \leq i < k - m.$$

Also, by Poincaré duality we have that,

$$b_i(X_{\mathbb{C}}) = b_{2(k-m)-i}(X_{\mathbb{C}}),\ 0 \leq i \leq k - m.$$

Thus, all the Betti numbers of $X_{\mathbb{C}}$ are determined once we know $b_{k-m}(X_{\mathbb{C}})$ or equivalently the Euler-Poincaŕe characteristic

$$\chi(X_{\mathbb{C}}) = \sum_{i \geq 0} (-1)^i b_i(X_{\mathbb{C}}).$$

Denoting $\chi(X_C)$ by $\chi_m^k(d_1, \ldots, d_m)$ (since it only depends on the degree sequence) we have the following recurrence relation which is classical.

$$\chi_m^k(d_1, \ldots, d_m) = \begin{cases} k+1 \\ d_1 \ldots d_m \\ d_m \chi_{m-1}^{k-1}(d_1, \ldots, d_{m-1}) - (d_m - 1)\chi_m^{k-1}(d_1, \ldots, d_m) \end{cases} \quad (1)$$

We have the following inequality.

Proposition

Suppose $1 \leq d_1 \leq d_2 \leq \cdots \leq d_m$. *The function* $\chi_m^k(d_1, \ldots, d_m)$ *satisfies*

$$|\chi_m^k(d_1, \ldots, d_m)| \leq \binom{k+1}{m+1} d_1 \ldots d_{m-1} d_m^{k-m+1}.$$

Now let $\beta_m^k(d_1, \ldots, d_m)$ denote $\sum_{i \geq 0} b_i(X_{\mathrm{C}})$.
The following corollary is an immediate consequence of Proposition 4 and the remarks preceding it.

Corollary

$$\beta_m^k(d_1, \ldots, d_m) \leq \binom{k+1}{m+1} d_1 \ldots d_{m-1} d_m^{k-m+1} + 2(k-m+1).$$

- The shape of the bound of the main theorem is due to the previous corollary. In the useage of the corollary to obtain the bound, the d_0 will appear $k - k'$ times (the number of polynomials in $\mathrm{Cr}_{k-k'-1}(\mathrm{Def}(Q, \zeta))$) and the d will appear $j \leq k'$ times (corresponding to choosing j of the perturbed polynomials of $\mathcal{P}$).

Final remarks on the proof:

- Note that although our goal is to bound the *zero*-th Betti numbers of a sign condition restricted to a variety, we actually bound the sum of *all* the Betti numbers of a complex projective variety to obtain this bound.
- However, since we only have a correspondence between the semi-algebraically connected components this does not follow immediately from our proof.

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An application – bounding the number of geometric permutations

Goodman,Pollack and Wenger (1996) reduced of bounding the number of geometric permutations of n well separated convex bodies in $\mathbb{R}^d$ induced by k-transversals to bounding the number of semi-algebraically connected components realizable sign conditions of

$$\binom{2^{k+1}-2}{k}\binom{n}{k+1}$$

polynomials in d^2 variables, where each polynomial has degree at most $2k$, on an algebraic variety (the real Grassmannian of k-planes in $\mathbb{R}^d$) in $\mathbb{R}^{d^2}$ defined by polynomials of degree 2. The real Grassmanian has dimension $k(d-k)$.

Geometric permutations (cont).

Applying Theorem 2 we obtain that the number of semi-algebraically connected components of all realizable sign conditions in this case is bounded by

$$\left(k\binom{2^{k+1}-2}{k}\binom{n}{k+1}\right)^{k(d-k)} (O(1))^{d^2},$$

which is a strict improvement of the previous bound of

$$\left(\binom{2^{k+1}-2}{k}\binom{n}{k+1}\right)^{k(d-k)} (O(k))^{d^2},$$

when k is close to d.

A potential application:
The ability to distinguish the roles of the degrees of the polynomials defining the variety versus the sign condition is important in a new method which has been developed for bounding the number of incidences between points and algebraic varieties of constant degree.
This method, which is referred to in the literature as a *ham sandwich decomposition*, has recently been used by Guth-Katz to solve one of the Erdös problems in Discrete Computational Geometry and has received recent attention from several authors.

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Possible future work:

- It is now natural to ask whether the refined bound can be extended to a result on the higher Betti numbers as in the result of (B.-Pollack-Roy, '05).
- Given the refined bound, can one design an algorithm for finding sample points in each of the connected components of sign conditions on a variety whose complexity mirrors the bound ?
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Thank you