Isotypic decomposition of cohomology modules of symmetric semi-algebraic sets: Polynomial bounds on the multiplicities

Saugata Basu

Department of Mathematics
Purdue University, West Lafayette, IN

Dagstuhl Seminar, Jun 9, 2015
(joint work with Cordian Riener, Aalto University)
Basic definitions

- Throughout, $\mathbb{R}$ will denote a real closed field.
- Given $P \in \mathbb{R}[X_1, \ldots, X_k]$ we denote by $Z(P, \mathbb{R}^k)$ the set of zeros of $P$ in $\mathbb{R}^k$.
- Given any semi-algebraic subset $S \subset \mathbb{R}^k$ we will denote by $b_i(S, F) = \dim_{\mathbb{F}}(H^i(S, F))$ (i.e. the dimension of the $i$-th cohomology group of $S$ with coefficients in $\mathbb{F}$ assumed to be of characteristic $0$), and we will denote by $b(S, F) = \sum_{i \geq 0} b_i(S, F)$.
- $b(S, F)$ is an important measure of the “complexity” of a semi-algebraic set $S$.
- Upper bounds on Betti numbers of a semi-algebraic set translate into lower bounds for the membership in that set in certain models of computations.
- Knowing very tight bounds on certain Betti numbers (for example, the 0-th Betti numbers) have become important for solving some hard problems in discrete geometry (for example, bounding incidences).
Basic definitions

Throughout, $\mathbb{R}$ will denote a real closed field.

Given $P \in \mathbb{R}[X_1, \ldots, X_k]$ we denote by $Z(P, \mathbb{R}^k)$ the set of zeros of $P$ in $\mathbb{R}^k$.

Given any semi-algebraic subset $S \subset \mathbb{R}^k$ we will denote by $b_i(S, \mathbb{F}) = \dim_{\mathbb{F}}(H^i(S, \mathbb{F}))$ (i.e. the dimension of the $i$-th cohomology group of $S$ with coefficients in $\mathbb{F}$ assumed to be of characteristic 0), and we will denote by $b(S, \mathbb{F}) = \sum_{i \geq 0} b_i(S, \mathbb{F})$.

$b(S, \mathbb{F})$ is an important measure of the “complexity” of a semi-algebraic set $S$.

Upper bounds on Betti numbers of a semi-algebraic set translate into lower bounds for the membership in that set in certain models of computations.

Knowing very tight bounds on certain Betti numbers (for example, the 0-th Betti numbers) have become important for solving some hard problems in discrete geometry (for example, bounding incidences).
Basic definitions

- Throughout, $\mathbb{R}$ will denote a real closed field.
- Given $P \in \mathbb{R}[X_1, \ldots, X_k]$ we denote by $Z(P, \mathbb{R}^k)$ the set of zeros of $P$ in $\mathbb{R}^k$.
- Given any semi-algebraic subset $S \subset \mathbb{R}^k$ we will denote by $b_i(S, \mathbb{F}) = \dim_{\mathbb{F}}(H^i(S, \mathbb{F})$ (i.e. the dimension of the $i$-th cohomology group of $S$ with coefficients in $\mathbb{F}$ assumed to be of characteristic 0), and we will denote by $b(S, \mathbb{F}) = \sum_{i \geq 0} b_i(S, \mathbb{F})$.
- $b(S, \mathbb{F})$ is an important measure of the “complexity” of a semi-algebraic set $S$.
- Upper bounds on Betti numbers of a semi-algebraic set translate into lower bounds for the membership in that set in certain models of computations.
- Knowing very tight bounds on certain Betti numbers (for example, the 0-th Betti numbers) have become important for solving some hard problems in discrete geometry (for example, bounding incidences).
Basic definitions

- Throughout, $\mathbb{R}$ will denote a real closed field.
- Given $P \in \mathbb{R}[X_1, \ldots, X_k]$ we denote by $Z(P, \mathbb{R}^k)$ the set of zeros of $P$ in $\mathbb{R}^k$.
- Given any semi-algebraic subset $S \subset \mathbb{R}^k$ we will denote by $b_i(S, \mathbb{F}) = \dim_{\mathbb{F}}(H^i(S, \mathbb{F}))$ (i.e. the dimension of the $i$-th cohomology group of $S$ with coefficients in $\mathbb{F}$ assumed to be of characteristic 0), and we will denote by $b(S, \mathbb{F}) = \sum_{i \geq 0} b_i(S, \mathbb{F})$.
- $b(S, \mathbb{F})$ is an important measure of the “complexity” of a semi-algebraic set $S$.
- Upper bounds on Betti numbers of a semi-algebraic set translate into lower bounds for the membership in that set in certain models of computations.
- Knowing very tight bounds on certain Betti numbers (for example, the 0-th Betti numbers) have become important for solving some hard problems in discrete geometry (for example, bounding incidences).
Basic definitions

▶ Throughout, \( \mathbb{R} \) will denote a real closed field.
▶ Given \( P \in \mathbb{R}[X_1, \ldots, X_k] \) we denote by \( Z(P, \mathbb{R}^k) \) the set of zeros of \( P \) in \( \mathbb{R}^k \).
▶ Given any semi-algebraic subset \( S \subset \mathbb{R}^k \) we will denote by \( b_i(S, \mathbb{F}) = \dim_{\mathbb{F}}(H^i(S, \mathbb{F})) \) (i.e. the dimension of the \( i \)-th cohomology group of \( S \) with coefficients in \( \mathbb{F} \) assumed to be of characteristic 0), and we will denote by \( b(S, \mathbb{F}) = \sum_{i \geq 0} b_i(S, \mathbb{F}) \).
▶ \( b(S, \mathbb{F}) \) is an important measure of the “complexity” of a semi-algebraic set \( S \).
▶ Upper bounds on Betti numbers of a semi-algebraic set translate into lower bounds for the membership in that set in certain models of computations.
▶ Knowing very tight bounds on certain Betti numbers (for example, the 0-th Betti numbers) have become important for solving some hard problems in discrete geometry (for example, bounding incidences).
Basic definitions

- Throughout, \( \mathbb{R} \) will denote a **real closed field**.
- Given \( P \in \mathbb{R}[X_1, \ldots, X_k] \) we denote by \( Z(P, \mathbb{R}^k) \) the set of zeros of \( P \) in \( \mathbb{R}^k \).
- Given any semi-algebraic subset \( S \subset \mathbb{R}^k \) we will denote by \( b_i(S, \mathbb{F}) = \dim_{\mathbb{F}}(H^i(S, \mathbb{F})) \) (i.e. the dimension of the \( i \)-th cohomology group of \( S \) with coefficients in \( \mathbb{F} \) assumed to be of characteristic 0), and we will denote by \( b(S, \mathbb{F}) = \sum_{i \geq 0} b_i(S, \mathbb{F}) \).
- \( b(S, \mathbb{F}) \) is an important measure of the “complexity” of a semi-algebraic set \( S \).
- Upper bounds on Betti numbers of a semi-algebraic set translate into lower bounds for the membership in that set in certain models of computations.
- Knowing very tight bounds on certain Betti numbers (for example, the 0-th Betti numbers) have become important for solving some hard problems in discrete geometry (for example, bounding incidences).
Doubly exponential (in $k$) bounds on $b(S, \mathbb{F})$ follow from results on effective triangulation of semi-algebraic sets which in turn uses cylindrical algebraic decomposition.

Singly exponential (in $k$) bounds: Long history – Oleĭnik and Petrovskiĭ (1949), Thom, Milnor (1960s) – for real algebraic varieties and basic closed semi-algebraic sets.

More precisely, if $P \in \mathbb{R} [X_1, \ldots, X_k]$ with $\text{deg}(P) \leq d$, then $b(Z(P, \mathbb{R}^k), \mathbb{F}) \leq d(2d - 1)^{k-1}$.

Main idea was to use Morse theory and counting critical points.

Generalized to more general semi-algebraic sets (B-Pollack-Roy, Gabrielov-Vorobjov).

Generalization uses additional tricks such as generalized Mayer-Vietoris inequalities, homotopic approximations by compact sets (Gabrielov-Vorobjov) etc.
Upper bounds on the Betti numbers

- **Doubly exponential (in \( k \)) bounds** on \( b(S, \mathbb{F}) \) follow from results on effective triangulation of semi-algebraic sets which in turn uses cylindrical algebraic decomposition.

- **Singly exponential (in \( k \)) bounds**: Long history – Oleĭnik and Petrovskii (1949), Thom, Milnor (1960s) – for real algebraic varieties and basic closed semi-algebraic sets.

  More precisely, if \( P \in \mathbb{R}[X_1, \ldots, X_k] \) with \( \deg(P) \leq d \), then
  \[
  b(Z(P, \mathbb{R}^k), \mathbb{F}) \leq d(2d - 1)^{k-1}.
  \]

- Main idea was to use Morse theory and counting critical points.

- Generalized to more general semi-algebraic sets (B-Pollack-Roy, Gabrielov-Vorobjov).

- Generalization uses additional tricks such as generalized Mayer-Vietoris inequalities, homotopic approximations by compact sets (Gabrielov-Vorobjov) etc.
Upper bounds on the Betti numbers

- **Doubly exponential (in k)** bounds on $b(S, F)$ follow from results on effective triangulation of semi-algebraic sets which in turn uses cylindrical algebraic decomposition.

- **Singly exponential (in k)** bounds: Long history – Oleĭnik and Petrovskii (1949), Thom, Milnor (1960s) – for real algebraic varieties and basic closed semi-algebraic sets.

- More precisely, if $P \in \mathbb{R}[X_1, \ldots, X_k]$ with $\deg(P) \leq d$, then $b(Z(P, \mathbb{R}^k), \mathbb{F}) \leq d(2d − 1)^{k−1}$.

- Main idea was to use Morse theory and counting critical points.

- Generalized to more general semi-algebraic sets (B-Pollack-Roy, Gabrielov-Vorobjov).

- Generalization uses additional tricks such as generalized Mayer-Vietoris inequalities, homotopic approximations by compact sets (Gabrielov-Vorobjov) etc.
Upper bounds on the Betti numbers

- Doubly exponential (in $k$) bounds on $b(S, \mathbb{F})$ follow from results on effective triangulation of semi-algebraic sets which in turn uses cylindrical algebraic decomposition.

- Singly exponential (in $k$) bounds: Long history – Oleĭnik and Petrovskiĭ (1949), Thom, Milnor (1960s) – for real algebraic varieties and basic closed semi-algebraic sets.

- More precisely, if $P \in \mathbb{R}[X_1, \ldots, X_k]$ with $\deg(P) \leq d$, then $b(Z(P, \mathbb{R}^k), \mathbb{F}) \leq d(2d - 1)^{k-1}$.

- Main idea was to use Morse theory and counting critical points.

- Generalized to more general semi-algebraic sets (B-Pollack-Roy, Gabrielov-Vorobjov).

- Generalization uses additional tricks such as generalized Mayer-Vietoris inequalities, homotopic approximations by compact sets (Gabrielov-Vorobjov) etc.
Upper bounds on the Betti numbers

- **Doubly exponential (in \( k \)) bounds** on \( b(S, \mathbb{F}) \) follow from results on effective triangulation of semi-algebraic sets which in turn uses **cylindrical algebraic decomposition**.

- **Singly exponential (in \( k \)) bounds**: Long history – Oleĭnik and Petrovskiĭ (1949), Thom, Milnor (1960s) – for real algebraic varieties and basic closed semi-algebraic sets.

- More precisely, if \( P \in \mathbb{R}[X_1, \ldots, X_k] \) with \( \deg(P) \leq d \), then \( b(Z(P, \mathbb{R}^k), \mathbb{F}) \leq d(2d - 1)^{k-1} \).

- Main idea was to use Morse theory and counting critical points.

- Generalized to more general semi-algebraic sets (B-Pollack-Roy, Gabrielov-Vorobjov).

- Generalization uses additional tricks such as generalized Mayer-Vietoris inequalities, homotopic approximations by compact sets (Gabrielov-Vorobjov) etc.
Upper bounds on the Betti numbers

- **Doubly exponential (in \(k\)) bounds** on \( b(S, \mathbb{F}) \) follow from results on effective triangulation of semi-algebraic sets which in turn uses **cylindrical algebraic decomposition**.

- **Singly exponential (in \(k\)) bounds**: Long history – Oleĭnik and Petrovskii (1949), Thom, Milnor (1960s) – for real algebraic varieties and basic closed semi-algebraic sets.

- More precisely, if \( P \in \mathbb{R}[X_1, \ldots, X_k] \) with \( \text{deg}(P) \leq d \), then \( b(Z(P, \mathbb{R}^k), \mathbb{F}) \leq d(2d - 1)^{k-1} \).

- Main idea was to use Morse theory and counting critical points.

- Generalized to more general semi-algebraic sets (B-Pollack-Roy, Gabrielov-Vorobjov).

- Generalization uses additional tricks such as generalized Mayer-Vietoris inequalities, homotopic approximations by compact sets (Gabrielov-Vorobjov) etc.
Lower bounds on the Betti numbers

- For any fixed $d \geq 3$, we have singly exponential lower bound.
- Let $F_{d,k} = \sum_{i=1}^{k} \left( \prod_{j=1}^{d} (X_i - j) \right)^2 - \varepsilon$, and $V_{d,k} = Z(F_{k,d}, R\langle \varepsilon \rangle^k)$.
- $b_0(V_{d,k}, \mathbb{F}) = b_{k-1}(V_{d,k}, \mathbb{F}) = d^k$, which is singly exponential in $k$.
- Notice moreover that each $F_{d,k}$ is a symmetric polynomial.
- Symmetric varieties defined by polynomials of bounded degrees are “simple”. For example, for every fixed degree $d$ there is a polynomial-time algorithm to test whether such a variety is empty (Timofte, Riener).
- But clearly from the topological point of view they are not so simple.
- For fixed degree symmetric polynomials, the Betti numbers of the quotient of the variety (by the symmetric group) are polynomially bounded (B., Riener (2013)).
- For example, $b_0(V_{d,k}/S_k, \mathbb{F}) = \binom{k+d-1}{d-1} = O(k)^d$.
Lower bounds on the Betti numbers

For any fixed \( d \geq 3 \), we have singly exponential lower bound.

Let \( F_{d,k} = \sum_{i=1}^{k} \left( \prod_{j=1}^{d} (X_i - j) \right)^2 - \varepsilon \), and
\( V_{d,k} = \mathbb{Z}(F_{k,d}, R\langle \varepsilon \rangle^k) \).

\[ b_0(V_{d,k}, \mathbb{F}) = b_{k-1}(V_{d,k}, \mathbb{F}) = d^k \], which is singly exponential in \( k \).

Notice moreover that each \( F_{d,k} \) is a symmetric polynomial.

Symmetric varieties defined by polynomials of bounded degrees are “simple”. For example, for every fixed degree \( d \) there is a polynomial-time algorithm to test whether such a variety is empty (Timofte, Riener).

But clearly from the topological point of view they are not so simple.

For fixed degree symmetric polynomials, the Betti numbers of the quotient of the variety (by the symmetric group) are polynomially bounded (B., Riener (2013)).

For example, \( b_0(V_{d,k}/\mathfrak{S}_k, \mathbb{F}) = \binom{k+d-1}{d-1} = O(k)^d \).
Lower bounds on the Betti numbers

- For any fixed $d \geq 3$, we have singly exponential lower bound.
- Let $F_{d,k} = \sum_{i=1}^{k} \left( \prod_{j=1}^{d} (X_i - j) \right)^2 - \varepsilon$, and $V_{d,k} = \mathbb{Z}(F_{k,d}, \mathbb{R}[\varepsilon]^k)$.
- $b_0(V_{d,k}, \mathbb{F}) = b_{k-1}(V_{d,k}, \mathbb{F}) = d^k$, which is singly exponential in $k$.
- Notice moreover that each $F_{d,k}$ is a symmetric polynomial.
- Symmetric varieties defined by polynomials of bounded degrees are “simple”. For example, for every fixed degree $d$ there is a polynomial-time algorithm to test whether such a variety is empty (Timofte, Riener).
- But clearly from the topological point of view they are not so simple.
- For fixed degree symmetric polynomials, the Betti numbers of the quotient of the variety (by the symmetric group) are polynomially bounded (B., Riener (2013)).
- For example, $b_0(V_{d,k}/S_k, \mathbb{F}) = \binom{k+d-1}{d-1} = O(k)^d$. 
Lower bounds on the Betti numbers

- For any fixed $d \geq 3$, we have singly exponential lower bound.
- Let $F_{d,k} = \sum_{i=1}^{k} \left( \prod_{j=1}^{d} (X_i - j) \right)^2 - \varepsilon$, and
  $V_{d,k} = Z(F_{k,d}, R^{\langle \varepsilon \rangle^k})$.
- $b_0(V_{d,k}, \mathbb{F}) = b_{k-1}(V_{d,k}, \mathbb{F}) = d^k$, which is singly exponential in $k$.
- Notice moreover that each $F_{d,k}$ is a symmetric polynomial.
- Symmetric varieties defined by polynomials of bounded degrees are “simple”. For example, for every fixed degree $d$ there is a polynomial-time algorithm to test whether such a variety is empty (Timofte, Riener).
- But clearly from the topological point of view they are not so simple.
- For fixed degree symmetric polynomials, the Betti numbers of the quotient of the variety (by the symmetric group) are polynomially bounded (B., Riener (2013)).
- For example, $b_0(V_{d,k}/\mathfrak{S}_k, \mathbb{F}) = \binom{k+d-1}{d-1} = O(k)^d$. 
Lower bounds on the Betti numbers

- For any fixed $d \geq 3$, we have singly exponential lower bound.
- Let $F_{d,k} = \sum_{i=1}^{k} \left( \prod_{j=1}^{d} (X_i - j) \right)^2 - \varepsilon$, and $V_{d,k} = Z(F_{k,d}, R^{\langle \varepsilon \rangle k})$.
- $b_0(V_{d,k}, \mathbb{F}) = b_{k-1}(V_{d,k}, \mathbb{F}) = d^k$, which is singly exponential in $k$.
- Notice moreover that each $F_{d,k}$ is a symmetric polynomial.
- Symmetric varieties defined by polynomials of bounded degrees are “simple”. For example, for every fixed degree $d$ there is a polynomial-time algorithm to test whether such a variety is empty (Timofte, Riener).
- But clearly from the topological point of view they are not so simple.
- For fixed degree symmetric polynomials, the Betti numbers of the quotient of the variety (by the symmetric group) are polynomially bounded (B., Riener (2013)).
- For example, $b_0(V_{d,k}/\mathfrak{S}_k, \mathbb{F}) = \binom{k+d-1}{d-1} = O(k)^d$. 
Lower bounds on the Betti numbers

- For any fixed $d \geq 3$, we have singly exponential lower bound.
- Let $F_{d,k} = \sum_{i=1}^{k} \left( \prod_{j=1}^{d} (X_i - j) \right)^2 - \varepsilon$, and
  $V_{d,k} = \mathbb{Z}(F_{k,d}, R\langle \varepsilon \rangle^k)$.
- $b_0(V_{d,k}, \mathbb{F}) = b_{k-1}(V_{d,k}, \mathbb{F}) = d^k$, which is singly exponential in $k$.
- Notice moreover that each $F_{d,k}$ is a symmetric polynomial.
- Symmetric varieties defined by polynomials of bounded degrees are “simple”. For example, for every fixed degree $d$ there is a polynomial-time algorithm to test whether such a variety is empty (Timofte, Riener).
- But clearly from the topological point of view they are not so simple.
- For fixed degree symmetric polynomials, the Betti numbers of the quotient of the variety (by the symmetric group) are polynomially bounded (B., Riener (2013)).
- For example, $b_0(V_{d,k}/\mathbb{S}_k, \mathbb{F}) = \binom{k+d-1}{d-1} = O(k)^d$. 
Lower bounds on the Betti numbers

For any fixed $d \geq 3$, we have singly exponential lower bound.

Let $F_{d,k} = \sum_{i=1}^{k} \left( \prod_{j=1}^{d} (X_i - j) \right)^2 - \varepsilon$, and $V_{d,k} = Z(F_{k,d}, R\langle \varepsilon \rangle^k)$.

$b_0(V_{d,k}, \mathbb{F}) = b_{k-1}(V_{d,k}, \mathbb{F}) = d^k$, which is singly exponential in $k$.

Notice moreover that each $F_{d,k}$ is a symmetric polynomial.

Symmetric varieties defined by polynomials of bounded degrees are “simple”. For example, for every fixed degree $d$ there is a polynomial-time algorithm to test whether such a variety is empty (Timofte, Riener).

But clearly from the topological point of view they are not so simple.

For fixed degree symmetric polynomials, the Betti numbers of the quotient of the variety (by the symmetric group) are polynomially bounded (B., Riener (2013)).

For example, $b_0(V_{d,k}/\mathfrak{S}_k, \mathbb{F}) = \binom{k+d-1}{d-1} = O(k)^d$. 
For any fixed $d \geq 3$, we have singly exponential lower bound.

Let $F_{d,k} = \sum_{i=1}^{k} \left( \prod_{j=1}^{d} (X_i - j) \right)^2 - \varepsilon$, and $V_{d,k} = Z(F_{k,d}, R\langle \varepsilon \rangle^k)$.

$b_0(V_{d,k}, \mathbb{F}) = b_{k-1}(V_{d,k}, \mathbb{F}) = d^k$, which is singly exponential in $k$.

Notice moreover that each $F_{d,k}$ is a symmetric polynomial.

Symmetric varieties defined by polynomials of bounded degrees are “simple”. For example, for every fixed degree $d$ there is a polynomial-time algorithm to test whether such a variety is empty (Timofte, Riener).

But clearly from the topological point of view they are not so simple.

For fixed degree symmetric polynomials, the Betti numbers of the quotient of the variety (by the symmetric group) are polynomially bounded (B., Riener (2013)).

For example, $b_0(V_{d,k}/\mathcal{S}_k, \mathbb{F}) = \binom{k+d-1}{d-1} = O(k^d)$. 

Where $\mathcal{S}_k$ denotes the symmetric group of degree $k$. 

Notice that $b_0$ is the number of connected components, which for symmetric varieties is $1$. The Betti numbers represent the number of connected components and their rank in each dimension.
Representations of finite groups

- A representation of $G$ over a field $\mathbb{F}$ (assumed to be of characteristic 0) is a homomorphism $\rho : G \to \text{GL}(V)$ for some $\mathbb{F}$-vector space $V$. It is usual to refer to the representation $\rho$ by $V$.
- A representation $\rho : G \to \text{GL}(V)$ is said to be irreducible iff the only $G$-invariant subspaces are 0 and $V$.
- The set, $\text{Irred}(G, \mathbb{F})$, of (equivalence classes of) irreducible representations of $G$ over $\mathbb{F}$, is finite.
- Every finite dimensional representation $V$ of $G$ admits a canonical direct sum decomposition

$$V = \bigoplus_{W \in \text{Irred}(G, \mathbb{F})} V_W,$$

where $V_W \cong_G m_W W$. The components $V_W$ are called the isotopic components, and $m_W$ the multiplicity of the irreducible $W$ in $V$.
- Clearly, $\dim_{\mathbb{F}}(V) = \sum_{W \in \text{Irred}(G, \mathbb{F})} m_W \dim_{\mathbb{F}}(W)$. 
Representations of finite groups

- A representation of $G$ over a field $\mathbb{F}$ (assumed to be of characteristic 0) is a homomorphism $\rho : G \to GL(V)$ for some $\mathbb{F}$-vector space $V$. It is usual to refer to the representation $\rho$ by $V$.
- A representation $\rho : G \to GL(V)$ is said to be **irreducible** iff the only $G$-invariant subspaces are 0 and $V$.
- The set, $\text{Irred}(G, \mathbb{F})$, of (equivalence classes of) irreducible representations of $G$ over $\mathbb{F}$, is finite.
- Every finite dimensional representation $V$ of $G$ admits a canonical direct sum decomposition

$$V = \bigoplus_{W \in \text{Irred}(G, \mathbb{F})} V_W,$$

where $V_W \cong_G m_W W$. The components $V_W$ are called the **isotypic components**, and $m_W$ the **multiplicity** of the irreducible $W$ in $V$.
- Clearly, $\dim_{\mathbb{F}}(V) = \sum_{W \in \text{Irred}(G, \mathbb{F})} m_W \dim_{\mathbb{F}}(W)$. 
Representations of finite groups

- A representation of $G$ over a field $\mathbb{F}$ (assumed to be of characteristic 0) is a homomorphism $\rho : G \rightarrow \text{GL}(V)$ for some $\mathbb{F}$-vector space $V$. It is usual to refer to the representation $\rho$ by $V$.
- A representation $\rho : G \rightarrow \text{GL}(V)$ is said to be irreducible iff the only $G$-invariant subspaces are 0 and $V$.
- The set, $\text{Irred}(G, \mathbb{F})$, of (equivalence classes of) irreducible representations of $G$ over $\mathbb{F}$, is finite.
- Every finite dimensional representation $V$ of $G$ admits a canonical direct sum decomposition

$$V = \bigoplus_{W \in \text{Irred}(G, \mathbb{F})} V_W,$$

where $V_W \cong_G m_W W$. The components $V_W$ are called the isotypic components, and $m_W$ the multiplicity of the irreducible $W$ in $V$.
- Clearly, $\dim_\mathbb{F}(V) = \sum_{W \in \text{Irred}(G, \mathbb{F})} m_W \dim_\mathbb{F}(W)$. 
Representations of finite groups

- A representation of $G$ over a field $\mathbb{F}$ (assumed to be of characteristic 0) is a homomorphism $\rho : G \to \text{GL}(V)$ for some $\mathbb{F}$-vector space $V$. It is usual to refer to the representation $\rho$ by $V$.
- A representation $\rho : G \to \text{GL}(V)$ is said to be **irreducible** iff the only $G$-invariant subspaces are 0 and $V$.
- The set, $\text{Irred}(G, \mathbb{F})$, of (equivalence classes of) irreducible representations of $G$ over $\mathbb{F}$, is finite.
- Every finite dimensional representation $V$ of $G$ admits a canonical direct sum decomposition

$$V = \bigoplus_{W \in \text{Irred}(G, \mathbb{F})} V_W,$$

where $V_W \cong_G m_W W$. The components $V_W$ are called the **isotypic components**, and $m_W$ the **multiplicity** of the irreducible $W$ in $V$.

- Clearly, $\dim_{\mathbb{F}}(V) = \sum_{W \in \text{Irred}(G, \mathbb{F})} m_W \dim_{\mathbb{F}}(W)$. 
Representations of finite groups

- A representation of $G$ over a field $\mathbb{F}$ (assumed to be of characteristic 0) is a homomorphism $\rho : G \rightarrow \text{GL}(V)$ for some $\mathbb{F}$-vector space $V$. It is usual to refer to the representation $\rho$ by $V$.

- A representation $\rho : G \rightarrow \text{GL}(V)$ is said to be irreducible iff the only $G$-invariant subspaces are 0 and $V$.

- The set, $\text{Irred}(G, \mathbb{F})$, of (equivalence classes of) irreducible representations of $G$ over $\mathbb{F}$, is finite.

- Every finite dimensional representation $V$ of $G$ admits a canonical direct sum decomposition

$$V = \bigoplus_{W \in \text{Irred}(G, \mathbb{F})} V_W,$$

where $V_W \cong_G m_W W$. The components $V_W$ are called the isotypic components, and $m_W$ the multiplicity of the irreducible $W$ in $V$.

- Clearly, $\dim_{\mathbb{F}}(V) = \sum_{W \in \text{Irred}(G, \mathbb{F})} m_W \dim_{\mathbb{F}}(W)$. 
Partitions, Young diagrams and dominance ordering

- A partition $\lambda$ of $k$ (denoted $\lambda \vdash k$) is a tuple $(\lambda_1, \ldots, \lambda_\ell)$, $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ with $\lambda_1 + \cdots + \lambda_\ell = k$.
- We denote by $\text{Par}(k)$ the set of partitions of $k$.
- We denote by $\text{Young}(\lambda)$ the Young diagram associated with $\lambda$.
- For example, $\text{Young}((4, 2, 1))$ is given by

```
+---+---+---+
|   |   |   |
|   |   |   |
|   |   |   |
```

- For any two partitions $\mu = (\mu_1, \mu_2, \ldots), \lambda = (\lambda_1, \lambda_2, \ldots) \in \text{Par}(k)$, we say that $\mu \geq \lambda$, if for each $i \geq 0$, $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$. This is a partial order (called the dominance order).
Partitions, Young diagrams and dominance ordering

- A partition $\lambda$ of $k$ (denoted $\lambda \vdash k$) is a tuple $(\lambda_1, \ldots, \lambda_\ell)$, $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ with $\lambda_1 + \cdots + \lambda_\ell = k$.

- We denote by $\text{Par}(k)$ the set of partitions of $k$.

- We denote by $\text{Young}(\lambda)$ the Young diagram associated with $\lambda$.

- For example, $\text{Young}((4, 2, 1))$ is given by

```
\begin{array}{|c|c|c|c|c|}
\hline
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
\hline
\end{array}
```

- For any two partitions $\mu = (\mu_1, \mu_2, \ldots), \lambda = (\lambda_1, \lambda_2, \ldots) \in \text{Par}(k)$, we say that $\mu \succeq \lambda$, if for each $i \geq 0$, $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$. This is a partial order (called the dominance order).
Partitions, Young diagrams and dominance ordering

- A partition $\lambda$ of $k$ (denoted $\lambda \vdash k$) is a tuple $(\lambda_1, \ldots, \lambda_\ell)$, $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ with $\lambda_1 + \cdots + \lambda_\ell = k$.
- We denote by $\text{Par}(k)$ the set of partitions of $k$.
- We denote by $\text{Young}(\lambda)$ the Young diagram associated with $\lambda$.
- For example, $\text{Young}((4,2,1))$ is given by

```
  +---+---+---+
  |   |   |   |
  +---+---+---+
  |   |   |
  +---+---+---+
  |       |
  +-------+
```

- For any two partitions $\mu = (\mu_1, \mu_2, \ldots), \lambda = (\lambda_1, \lambda_2, \ldots) \in \text{Par}(k)$, we say that $\mu \succeq \lambda$, if for each $i \geq 0$, $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$. This is a partial order (called the dominance order).
Partitions, Young diagrams and dominance ordering

- A partition $\lambda$ of $k$ (denoted $\lambda \vdash k$) is a tuple $(\lambda_1, \ldots, \lambda_\ell)$, $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ with $\lambda_1 + \cdots + \lambda_\ell = k$.
- We denote by $\text{Par}(k)$ the set of partitions of $k$.
- We denote by $\text{Young}(\lambda)$ the Young diagram associated with $\lambda$.
- For example, $\text{Young}((4, 2, 1))$ is given by

```
\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\end{array}
\]
```

- For any two partitions $\mu = (\mu_1, \mu_2, \ldots), \lambda = (\lambda_1, \lambda_2, \ldots) \in \text{Par}(k)$, we say that $\mu \succeq \lambda$, if for each $i \geq 0$, $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$. This is a partial order (called the dominance order).
Partitions, Young diagrams and dominance ordering

- A partition \( \lambda \) of \( k \) (denoted \( \lambda \vdash k \)) is a tuple \( (\lambda_1, \ldots, \lambda_\ell) \), \( \lambda_1 \geq \cdots \geq \lambda_\ell > 0 \) with \( \lambda_1 + \cdots + \lambda_\ell = k \).
- We denote by \( \text{Par}(k) \) the set of partitions of \( k \).
- We denote by \( \text{Young}(\lambda) \) the Young diagram associated with \( \lambda \).
- For example, \( \text{Young}((4,2,1)) \) is given by

```
   /
  / 
/
```

- For any two partitions \( \mu = (\mu_1, \mu_2, \ldots), \lambda = (\lambda_1, \lambda_2, \ldots) \in \text{Par}(k) \), we say that \( \mu \triangleright= \lambda \), if for each \( i \geq 0 \), \( \mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i \). This is a partial order (called the dominance order).
Dominance order on $\text{Par}(6)$
Semi-standard tableau, Kostka numbers

- Given partitions $\mu, \lambda = (\lambda_1, \lambda_2, \ldots,) \vdash k$, a semi-standard tableau of shape $\mu$ and content $\lambda$ is a Young diagram in $\text{Young}(\mu)$ with entries in the boxes which are non-decreasing along rows and increasing along columns – and for each $i > 0$, the number of $i$’s is equal to $\lambda_i$.

- For example,

```
    1 1 1 2
   2 2
   3
```

is a semi-standard of shape $(4, 2, 1)$ and content $(3, 3, 1)$.

- For $\lambda, \mu \vdash k$, the Kostka number $K(\mu, \lambda)$ is the number of semi-standard Young tableaux of shape $\mu$ and content $\lambda$.

- Fact: for all $\mu, \lambda \vdash k$, $K(\mu, \mu) = K((k), \mu) = 1$, and $K(\mu, \lambda) \neq 0$ iff $\mu \succeq \lambda$. 
Semi-standard tableau, Kostka numbers

Given partitions $\mu, \lambda = (\lambda_1, \lambda_2, \ldots, ) \vdash k$, a **semi-standard tableau** of shape $\mu$ and content $\lambda$ is a Young diagram in $\text{Young}(\mu)$ with entries in the boxes which are non-decreasing along rows and increasing along columns – and for each $i > 0$, the number of $i$’s is equal to $\lambda_i$.

For example,

\[
\begin{array}{cccc}
1 & 1 & 1 & 2 \\
2 & 2 \\
3
\end{array}
\]

is a semi-standard of shape $(4, 2, 1)$ and content $(3, 3, 1)$.

For $\lambda, \mu \vdash k$, the **Kostka number** $K(\mu, \lambda)$ is the number of semi-standard Young tableux of shape $\mu$ and content $\lambda$.

Fact: for all $\mu, \lambda \vdash k$, $K(\mu, \mu) = K((k), \mu) = 1$, and $K(\mu, \lambda) \neq 0$ iff $\mu \trianglerighteq \lambda$. 


Semi-standard tableau, Kostka numbers

- Given partitions $\mu, \lambda = (\lambda_1, \lambda_2, \ldots, ) \vdash k$, a semi-standard tableau of shape $\mu$ and content $\lambda$ is a Young diagram in $\text{Young}(\mu)$ with entries in the boxes which are non-decreasing along rows and increasing along columns – and for each $i > 0$, the number of $i$’s is equal to $\lambda_i$.
- For example,

\[
\begin{array}{cccc}
1 & 1 & 1 & 2 \\
2 & 2 \\
3
\end{array}
\]

is a semi-standard of shape $(4, 2, 1)$ and content $(3, 3, 1)$.
- For $\lambda, \mu \vdash k$, the Kostka number $K(\mu, \lambda)$ is the number of semi-standard Young tableaux of shape $\mu$ and content $\lambda$.
- Fact: for all $\mu, \lambda \vdash k$, $K(\mu, \mu) = K((k), \mu) = 1$, and $K(\mu, \lambda) \neq 0$ iff $\mu \succeq \lambda$. 
Given partitions $\mu, \lambda = (\lambda_1, \lambda_2, \ldots, ) \vdash k$, a semi-standard tableau of shape $\mu$ and content $\lambda$ is a Young diagram in $\text{Young}(\mu)$ with entries in the boxes which are non-decreasing along rows and increasing along columns -- and for each $i > 0$, the number of $i$’s is equal to $\lambda_i$.

For example,

```
  1 1 1 2
  2 2
  3
```

is a semi-standard of shape $(4, 2, 1)$ and content $(3, 3, 1)$.

For $\lambda, \mu \vdash k$, the Kostka number $K(\mu, \lambda)$ is the number of semi-standard Young tableaux of shape $\mu$ and content $\lambda$.

Fact: for all $\mu, \lambda \vdash k$, $K(\mu, \mu) = K((k), \mu) = 1$, and $K(\mu, \lambda) \neq 0$ iff $\mu \succeq \lambda$. 
Irreducible representations of $\mathcal{S}_k$

- The irreducible representations (also called Specht modules) of $\mathcal{S}_k$ are in 1-1 correspondence with the set, $\text{Par}(k)$, of partitions of $k$.
- Given a partition $\lambda = (\lambda_1, \ldots, \lambda_p) \in \text{Par}(\lambda)$, we denote by $S^\lambda$ the corresponding Specht module.
- In particular, $S^{(k)} = 1_{\mathcal{S}_k}$, $S^{(1^k)} = \text{sign}_{\mathcal{S}_k}$.
- The dimension of $S^\lambda$ equals the number of standard of Young tableau of shape $\lambda$. Its also give by the hook length formula below.
- For a box $b$ in the Young diagram, $\text{Young}(\lambda)$, of a partition $\lambda$, let $h_b$ denote the length of the the hook of $b$ i.e. $h_b$ is the number of boxes in $\text{Young}(\lambda)$ strictly to the right and below $b$ plus 1.
- Hook length formula:
  \[
  \dim_{\mathbb{F}} S^\lambda = \frac{k!}{\prod_{b \in \text{Young}(\lambda)} h_b}
  \]
- $\dim_{\mathbb{F}} S^{(k)} = \dim_{\mathbb{F}} S^{1^k} = 1$. 

Irreducible representations of $\mathfrak{S}_k$

- The irreducible representations (also called Specht modules) of $\mathfrak{S}_k$ are in 1-1 correspondence with the set, $\text{Par}(k)$, of partitions of $k$.
- Given a partition $\lambda = (\lambda_1, \ldots, \lambda_p) \in \text{Par}(\lambda)$, we denote by $S^\lambda$ the corresponding Specht module.
- In particular, $S^{(k)} = \mathbf{1}_{\mathfrak{S}_k}$, $S^{(1^k)} = \text{sign}_{\mathfrak{S}_k}$.
- The dimension of $S^\lambda$ equals the number of standard of Young tableau of shape $\lambda$. Its also given by the hook length formula below.
- For a box $b$ in the Young diagram, $\text{Young}(\lambda)$, of a partition $\lambda$, let $h_b$ denote the length of the hook of $b$ i.e. $h_b$ is the number of boxes in $\text{Young}(\lambda)$ strictly to the right and below $b$ plus 1.
- Hook length formula:

$$\dim_F S^\lambda = \frac{k!}{\prod_{b \in \text{Young}(\lambda)} h_b}$$

- $\dim_F S^{(k)} = \dim_F S^{1^k} = 1$. 


Irreducible representations of $\mathfrak{S}_k$

- The irreducible representations (also called *Specht modules*) of $\mathfrak{S}_k$ are in 1-1 correspondence with the set, $\text{Par}(k)$, of partitions of $k$.
- Given a partition $\lambda = (\lambda_1, \ldots, \lambda_p) \in \text{Par}(\lambda)$, we denote by $S^\lambda$ the corresponding Specht module.
- In particular, $S^{(k)} = 1_{\mathfrak{S}_k}$, $S^{(1^k)} = \text{sign}_{\mathfrak{S}_k}$.
- The dimension of $S^\lambda$ equals the number of standard of Young tableau of shape $\lambda$. Its also give by the *hook length formula* below.
- For a box $b$ in the Young diagram, $\text{Young}(\lambda)$, of a partition $\lambda$, let $h_b$ denote the length of the *the hook of b* i.e. $h_b$ is the number of boxes in $\text{Young}(\lambda)$ strictly to the right and below $b$ plus 1.

Hook length formula:

$$\dim_F S^\lambda = \frac{k!}{\prod_{b \in \text{Young}(\lambda)} h_b}$$

- $\dim_F S^{(k)} = \dim_F S^{1^k} = 1$. 
Irreducible representations of $\mathfrak{S}_k$

- The irreducible representations (also called *Specht modules*) of $\mathfrak{S}_k$ are in 1-1 correspondence with the set, $\text{Par}(k)$, of partitions of $k$.
- Given a partition $\lambda = (\lambda_1, \ldots, \lambda_p) \in \text{Par}(\lambda)$, we denote by $S^\lambda$ the corresponding Specht module.
- In particular, $S^{(k)} = 1_{\mathfrak{S}_k}$, $S^{(1^k)} = \text{sign}_{\mathfrak{S}_k}$.
- The dimension of $S^\lambda$ equals the number of standard of Young tableau of shape $\lambda$. Its also give by the *hook length formula* below.
- For a box $b$ in the Young diagram, $\text{Young}(\lambda)$, of a partition $\lambda$, let $h_b$ denote the length of the *the hook of b* i.e. $h_b$ is the number of boxes in $\text{Young}(\lambda)$ strictly to the right and below $b$ plus 1.
- Hook length formula:

$$\dim_F S^\lambda = \frac{k!}{\prod_{b \in \text{Young}(\lambda)} h_b}$$

- $\dim_F S^{(k)} = \dim_F S^{1^k} = 1$. 
Irreducible representations of $\mathcal{S}_k$

- The irreducible representations (also called *Specht modules*) of $\mathcal{S}_k$ are in 1-1 correspondence with the set, $\text{Par}(k)$, of partitions of $k$.

- Given a partition $\lambda = (\lambda_1, \ldots, \lambda_p) \in \text{Par}(\lambda)$, we denote by $\mathbb{S}_\lambda$ the corresponding Specht module.

- In particular, $\mathbb{S}^{(k)} = 1_{\mathcal{S}_k}, \mathbb{S}^{(1^k)} = \text{sign}_{\mathcal{S}_k}$.

- The dimension of $\mathbb{S}_\lambda$ equals the number of standard of Young tableau of shape $\lambda$. It's also given by the *hook length formula* below.

- For a box $b$ in the Young diagram, $\text{Young}(\lambda)$, of a partition $\lambda$, let $h_b$ denote the length of the *the hook of $b$* i.e. $h_b$ is the number of boxes in $\text{Young}(\lambda)$ strictly to the right and below $b$ plus 1.

- **Hook length formula:**

  $$\dim_F \mathbb{S}_\lambda = \frac{k!}{\prod_{b \in \text{Young}(\lambda)} h_b}$$

- $\dim_F \mathbb{S}^{(k)} = \dim_F \mathbb{S}^{1^k} = 1$. 
Irreducible representations of $\mathfrak{S}_k$

- The irreducible representations (also called *Specht modules*) of $\mathfrak{S}_k$ are in 1-1 correspondence with the set, $\text{Par}(k)$, of partitions of $k$.
- Given a partition $\lambda = (\lambda_1, \ldots, \lambda_p) \in \text{Par}(\lambda)$, we denote by $S^\lambda$ the corresponding Specht module.
- In particular, $S^{(k)} = 1_{\mathfrak{S}_k}$, $S^{(1^k)} = \text{sign}_{\mathfrak{S}_k}$.
- The dimension of $S^\lambda$ equals the number of standard of Young tableau of shape $\lambda$. Its also give by the *hook length formula* below.
- For a box $b$ in the Young diagram, $\text{Young}(\lambda)$, of a partition $\lambda$, let $h_b$ denote the length of the *the hook of $b$* i.e. $h_b$ is the number of boxes in $\text{Young}(\lambda)$ strictly to the right and below $b$ plus 1.
- Hook length formula:

\[
\dim_{\mathbb{F}} S^\lambda = \frac{k!}{\prod_{b \in \text{Young}(\lambda)} h_b}
\]

\[\dim_{\mathbb{F}} S^{(k)} = \dim_{\mathbb{F}} S^{1^k} = 1.\]
Irreducible representations of $\mathfrak{S}_k$

- The irreducible representations (also called Specht modules) of $\mathfrak{S}_k$ are in 1-1 correspondence with the set, $\text{Par}(k)$, of partitions of $k$.
- Given a partition $\lambda = (\lambda_1, \ldots, \lambda_p) \in \text{Par}(\lambda)$, we denote by $\mathbb{S}^\lambda$ the corresponding Specht module.
- In particular, $\mathbb{S}^{(k)} = 1_{\mathfrak{S}_k}$, $\mathbb{S}^{(1^k)} = \text{sign}_{\mathfrak{S}_k}$.
- The dimension of $\mathbb{S}^\lambda$ equals the number of standard of Young tableau of shape $\lambda$. Its also give by the hook length formula below.
- For a box $b$ in the Young diagram, $\text{Young}(\lambda)$, of a partition $\lambda$, let $h_b$ denote the length of the hook of $b$ i.e. $h_b$ is the number of boxes in $\text{Young}(\lambda)$ strictly to the right and below $b$ plus 1.

Hook length formula:

$$\dim_\mathbb{F} \mathbb{S}^\lambda = \frac{k!}{\prod_{b \in \text{Young}(\lambda)} h_b}$$

- $\dim_\mathbb{F} \mathbb{S}^{(k)} = \dim_\mathbb{F} \mathbb{S}^{1^k} = 1$. 
Young modules and Specht modules

For $\lambda \vdash k$, we will denote

$$M^\lambda = \text{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_k}(1_{\mathfrak{S}_\lambda})$$

(the Young module of $\lambda$). It is isomorphic to the permutation representation of $\mathfrak{S}_k$ on the set of cosets in $\mathfrak{S}_k$ of the subgroup $\mathfrak{S}_\lambda$.

Clearly, $\dim_F M^\lambda = \binom{k}{\lambda}$.

(Young’s theorem)

$$M^\lambda \cong_{\mathfrak{S}_k} \bigoplus_{\mu \trianglerighteq \lambda} K(\mu, \lambda)S^\mu.$$

For example:

$$M^{(k)} \cong_{\mathfrak{S}_k} S^{(k)} \cong_{\mathfrak{S}_k} 1_{\mathfrak{S}_k},$$

$$M^{1^k} \cong_{\mathfrak{S}_k} \bigoplus_{\mu \vdash k} \dim_F(S^\mu)S^\mu \cong_{\mathfrak{S}_k} F[\mathfrak{S}_k].$$
Young modules and Specht modules

- For $\lambda \vdash k$, we will denote
  \[ M^\lambda = \text{Ind}_{\mathcal{S}_\lambda}^{\mathcal{S}_k}(1_{\mathcal{S}_\lambda}) \]
  (the Young module of $\lambda$). It is isomorphic to the permutation representation of $\mathcal{S}_k$ on the set of cosets in $\mathcal{S}_k$ of the subgroup $\mathcal{S}_\lambda$.

- Clearly, $\dim_{\mathbb{F}} M^\lambda = \binom{k}{\lambda}$.

- (Young’s theorem)
  \[ M^\lambda \cong_{\mathcal{S}_k} \bigoplus_{\mu \succeq \lambda} K(\mu, \lambda) S^\mu. \]

- For example:
  \[ M^{(k)} \cong_{\mathcal{S}_k} S^{(k)} \cong_{\mathcal{S}_k} 1_{\mathcal{S}_k}, \]
  \[ M^{1_k} \cong_{\mathcal{S}_k} \bigoplus_{\mu \vdash k} \dim_{\mathbb{F}}(S^\mu) S^\mu \cong_{\mathcal{S}_k} \mathbb{F}[\mathcal{S}_k]. \]
Young modules and Specht modules

- For $\lambda \vdash k$, we will denote $M^\lambda = \text{Ind}_{\mathcal{S}_\lambda}^{\mathcal{S}_k}(1_{\mathcal{S}_\lambda})$

  (the Young module of $\lambda$). It is isomorphic to the permutation representation of $\mathcal{S}_k$ on the set of cosets in $\mathcal{S}_k$ of the subgroup $\mathcal{S}_\lambda$.

- Clearly, $\dim_\mathbb{F} M^\lambda = \binom{k}{\lambda}$.

- (Young’s theorem)

  $M^\lambda \cong_{\mathcal{S}_k} \bigoplus_{\mu \succeq \lambda} K(\mu, \lambda)S^\mu$.

- For example:

  \[
  M^{(k)} \cong_{\mathcal{S}_k} S^{(k)} \cong_{\mathcal{S}_k} 1_{\mathcal{S}_k},
  \]

  \[
  M^{1, k} \cong_{\mathcal{S}_k} \bigoplus_{\mu \vdash k} \dim_\mathbb{F}(S^\mu)S^\mu \cong_{\mathcal{S}_k} \mathbb{F}[\mathcal{S}_k].
  \]
Young modules and Specht modules

- For $\lambda \vdash k$, we will denote
  
  $$M^\lambda = \text{Ind}_{S^\lambda}^{S_k}(1_{\lambda})$$

  (the Young module of $\lambda$). It is isomorphic to the permutation representation of $S_k$ on the set of cosets in $S_k$ of the subgroup $S^\lambda$.

- Clearly, $\text{dim}_F M^\lambda = \binom{k}{\lambda}$.

- (Young's theorem)

  $$M^\lambda \cong_{S_k} \bigoplus_{\mu \geq \lambda} K(\mu, \lambda)S^\mu.$$  

- For example:

  $M^{(k)} \cong_{S_k} S^{(k)} \cong_{S_k} 1_{S_k}$,  

  $M^{1_k} \cong_{S_k} \bigoplus_{\mu \vdash k} \text{dim}_F(S^\mu)S^\mu \cong_{S_k} F[S_k].$
Action of a finite group on a space $X$

- Let a finite group $G$ act on a topological space $X$.
- The action of $G$ on $X$ induces an action of $G$ on the cohomology group $H^*(X, \mathbb{F})$, making $H^*(X, \mathbb{F})$ into a $G$-module.
- If $\text{card}(G)$ is invertible in $\mathbb{F}$ (and so in particular, if $\mathbb{F}$ is a field of characteristic $0$) we have the isomorphisms

$$H^*(X/G, \mathbb{F}) \xrightarrow{\sim} H^*_G(X, \mathbb{F}) \xrightarrow{\sim} H^*(X, \mathbb{F})^G.$$ 

- In particular, if $S \subset \mathbb{R}^k$, is a symmetric semi-algebraic set, $H^*(S, \mathbb{F})$ is a finite dimensional $\mathbb{S}_k$-module, and

$$H^*_\mathbb{S}_k(S, \mathbb{F}) \cong H^*(S, \mathbb{F})^{\mathbb{S}_k}.$$
Action of a finite group on a space $X$

- Let a finite group $G$ act on a topological space $X$.
- The action of $G$ on $X$ induces an action of $G$ on the cohomology group $H^\ast(X, \mathbb{F})$, making $H^\ast(X, \mathbb{F})$ into a $G$-module.
- If $\text{card}(G)$ is invertible in $\mathbb{F}$ (and so in particular, if $\mathbb{F}$ is a field of characteristic 0) we have the isomorphisms
  \[
  H^\ast(X/G, \mathbb{F}) \xrightarrow{\sim} H^\ast_G(X, \mathbb{F}) \xrightarrow{\sim} H^\ast(X, \mathbb{F})^G.
  \]
- In particular, if $S \subset \mathbb{R}^k$, is a symmetric semi-algebraic set, $H^\ast(S, \mathbb{F})$ is a finite dimensional $\mathfrak{S}_k$-module, and
  \[
  H^\ast_{\mathfrak{S}_k}(S, \mathbb{F}) \cong H^\ast(S, \mathbb{F})^{\mathfrak{S}_k}.
  \]
Action of a finite group on a space $X$

- Let a finite group $G$ act on a topological space $X$.
- If $\text{card}(G)$ is invertible in $F$ (and so in particular, if $F$ is a field of characteristic 0) we have the isomorphisms

$$H^*(X/G, F) \xrightarrow{\sim} H^*_G(X, F) \xrightarrow{\sim} H^*(X, F)^G.$$

- In particular, if $S \subset \mathbb{R}^k$, is a symmetric semi-algebraic set, $H^*(S, F)$ is a finite dimensional $\mathbb{S}_k$-module, and

$$H^*_\mathbb{S}_k(S, F) \cong H^*(S, F)^{\mathbb{S}_k}.$$
Action of a finite group on a space $X$

- Let a finite group $G$ act on a topological space $X$.
- If $\text{card}(G)$ is invertible in $F$ (and so in particular, if $F$ is a field of characteristic 0) we have the isomorphisms
  
  \[ H^*(X/G, F) \xrightarrow{\sim} H^*_G(X, F) \xrightarrow{\sim} H^*(X, F)^G. \]

- In particular, if $S \subset \mathbb{R}^k$, is a symmetric semi-algebraic set, $H^*(S, F)$ is a finite dimensional $\mathcal{G}_k$-module, and
  
  \[ H^*_\mathcal{G}_k(S, F) \cong H^*(S, F)^{\mathcal{G}_k}. \]
Key example

Let

\[
F_k = \sum_{i=1}^{k} (X_i(X_i - 1))^2 - \varepsilon,
\]

\[
V_k = Z(F_k, \mathbb{R}^k).
\]

\[
H^0(V_k, \mathbb{F}) \cong \bigoplus_{0 \leq i \leq k} H^0(V_{k,i}, \mathbb{F}),
\]

where for \(0 \leq i \leq k\), \(V_{k,i}\) is the \(\mathfrak{S}_k\)-orbit of the connected component of \(V_k\) infinitesimally close (as a function of \(\varepsilon\)) to the point \(x^i = (0, \ldots, 0, 1, \ldots, 1)\), and \(H^0(V_{k,i}, \mathbb{F})\) is an invariant subspace of \(H^0(V_k, \mathbb{F})\).
Key example

Let

\[ F_k = \sum_{i=1}^{k} (X_i(X_i - 1))^2 - \varepsilon, \]

\[ V_k = Z(F_k, \mathbb{R}^k). \]

\[ H^0(V_k, \mathbb{F}) \cong \bigoplus_{0 \leq i \leq k} H^0(V_{k,i}, \mathbb{F}), \]

where for \( 0 \leq i \leq k \), \( V_{k,i} \) is the \( S_k \)-orbit of the connected component of \( V_k \) infinitesimally close (as a function of \( \varepsilon \)) to the point \( x^i = (0, \ldots, 0, 1, \ldots, 1) \), and \( H^0(\mathbb{V}_{k,i}, \mathbb{F}) \) is an invariant subspace of \( H^0(V_k, \mathbb{F}) \).
The isotropy subgroup of the point $x^i$ under the action of $\mathcal{S}_k$ is $\mathcal{S}_i \times \mathcal{S}_{k-i}$, and $\text{orbit}(x^i)$ is thus in 1-1 correspondence with the cosets of the subgroup $\mathcal{S}_i \times \mathcal{S}_{k-i}$.

It now follows from the definition of Young’s module:

$$H^0(V_{k,i}, \mathbb{F}) \cong \mathcal{S}_k \quad M^{(i,k-i)} \text{ if } i \geq k - i,$$

$$\cong \mathcal{S}_k \quad M^{(k-i,i)} \text{ otherwise.}$$
Key example (cont).

- The isotropy subgroup of the point $x^i$ under the action of $\mathfrak{S}_k$ is $\mathfrak{S}_i \times \mathfrak{S}_{k-i}$, and $\text{orbit}(x^i)$ is thus in 1-1 correspondence with the cosets of the subgroup $\mathfrak{S}_i \times \mathfrak{S}_{k-i}$.

- It now follows from the definition of Young's module:

$$H^0(V_{k,i}, \mathbb{F}) \cong_{\mathfrak{S}_k} M^{(i,k-i)} \text{ if } i \geq k - i,$$
$$\cong_{\mathfrak{S}_k} M^{(k-i,i)} \text{ otherwise.}$$
Key example (cont).

- It follows that for \( k \) odd,

\[
H^0(V_k, \mathbb{F}) \cong \bigoplus_{\ell(\lambda) \leq 2} (M^\lambda \oplus M^\lambda)
\]

\[
\cong \bigoplus_{\ell(\lambda) \leq 2} 2K(\mu, \lambda)S^\mu
\]

\[
\cong \bigoplus_{\ell(\mu) \leq 2} m_{0,\mu}S^\mu,
\]

where for each \( \mu = (\mu_1, \mu_2) \vdash k \),

\[
m_{0,\mu} = 2(\mu_1 - \lfloor k/2 \rfloor)
\]

\[
= 2\mu_1 - k + 1
\]

\[
= \mu_1 - \mu_2 + 1.
\]
Key example (cont).

- For $k$ even:

$$
H^0(V_k, F) \cong \mathcal{G}_k \quad M^{(k/2,k/2)} \oplus \bigoplus_{\lambda \vdash k \atop \ell(\lambda) \leq 2 \atop \lambda \neq (k/2,k/2)} (M^\lambda \oplus M^\lambda)
$$

$$
\cong \mathcal{G}_k \quad \bigoplus_{\mu \vdash k \atop \ell(\mu) \leq 2} m_{0,\mu} S^\mu,
$$

where for each $\mu = (\mu_1, \mu_2) \vdash k$,

$$
m_{0,\mu} = 2(\mu_1 - k/2) + 1 = \mu_1 - \mu_2 + 1.
$$

- We deduce for all $k$,

$$
m_{0,\mu} = \mu_1 - \mu_2 + 1 \leq k + 1.
$$
Key example (cont).

- For $k$ even:

$$H^0(V_k, F) \cong \mathbb{S}_k \  M^{(k/2, k/2)} \oplus \bigoplus_{\lambda \vdash k \atop \ell(\lambda) \leq 2 \atop \lambda \neq (k/2, k/2)} (M^\lambda \oplus M^\lambda)$$

$$\cong \mathbb{S}_k \bigoplus_{\mu \vdash k \atop \ell(\mu) \leq 2} m_{0, \mu} S^\mu,$$

where for each $\mu = (\mu_1, \mu_2) \vdash k$,

$$m_{0, \mu} = 2(\mu_1 - k/2) + 1 = \mu_1 - \mu_2 + 1.$$ 

- We deduce for all $k$, 

$$m_{0, \mu} = \mu_1 - \mu_2 + 1 \leq k + 1.$$
\(\mathbb{S}_k\)-equivariant Poincaré duality

What about \(H^{k-1}(V_k, \mathbb{F})\)?

**Theorem**

Let \(V \subset \mathbb{R}^k\) be a bounded smooth compact semi-algebraic oriented hypersurface, which is stable under the standard action of \(\mathbb{S}_k\) on \(\mathbb{R}^k\). Then, for each \(p, 0 \leq p \leq k - 1\), there is a \(\mathbb{S}_k\)-module isomorphism

\[
H^p(V, \mathbb{F}) \cong H^{k-p-1}(V, \mathbb{F}) \otimes \text{sign}_k.
\]

This implies in our example that

\[
H^{k-1}(V_k, \mathbb{F}) \cong \bigoplus_{\mu \vdash k \atop \ell(\mu) \leq 2} m_{0, \mu} \mathbb{S}^{\mu_k}.
\]

What about $H^{k-1}(V_k, \mathbb{F})$?

**Theorem**

Let $V \subset \mathbb{R}^k$ be a bounded smooth compact semi-algebraic oriented hypersurface, which is stable under the standard action of $\mathcal{G}_k$ on $\mathbb{R}^k$. Then, for each $p, 0 \leq p \leq k - 1$, there is a $\mathcal{G}_k$-module isomorphism

$$H^p(V, \mathbb{F}) \xrightarrow{\sim} H^{k-p-1}(V, \mathbb{F}) \otimes \text{sign}_k.$$

This implies in our example that

$$H^{k-1}(V_k, \mathbb{F}) \cong \bigoplus_{\mu \vdash k, \ell(\mu) \leq 2} m_{0, \mu} S^{\bar{\mu}}.$$
$\mathfrak{S}_k$-equivariant Poincaré duality

What about $H^{k-1}(V_k, \mathbb{F})$?

**Theorem**

Let $V \subset \mathbb{R}^k$ be a bounded smooth compact semi-algebraic oriented hypersurface, which is stable under the standard action of $\mathfrak{S}_k$ on $\mathbb{R}^k$. Then, for each $p, 0 \leq p \leq k - 1$, there is a $\mathfrak{S}_k$-module isomorphism

$$H^p(V, \mathbb{F}) \sim H^{k-p-1}(V, \mathbb{F}) \otimes \text{sign}_k.$$ 

This implies in our example that

$$H^{k-1}(V_k, \mathbb{F}) \cong \bigoplus_{\mu \vdash k, \ell(\mu) \leq 2} m_{0, \mu} \mathbb{S}^{\mu}.$$
$\mathcal{S}_k$-equivariant Poincaré duality

What about $H^{k-1}(V_k, \mathbb{F})$?

**Theorem**

*Let $V \subset \mathbb{R}^k$ be a bounded smooth compact semi-algebraic oriented hypersurface, which is stable under the standard action of $\mathcal{S}_k$ on $\mathbb{R}^k$. Then, for each $p, 0 \leq p \leq k - 1$, there is a $\mathcal{S}_k$-module isomorphism*

$$H^p(V, \mathbb{F}) \xrightarrow{\sim} H^{k-p-1}(V, \mathbb{F}) \otimes \text{sign}_k.$$  

This implies in our example that

$$H^{k-1}(V_k, \mathbb{F}) \cong \bigoplus_{0 \leq \ell(\mu) \leq 2} m_{0, \mu} \mathbb{S}^{\tilde{\mu}}.$$
Key example (cont).

In particular for $k = 2, 3$ we have:

\[
\begin{align*}
H^0(V_2, \mathbb{F}) & \cong \mathbb{S}_2 \quad 3\mathbb{S}^{(2)} \oplus \mathbb{S}^{(1,1)}, \\
H^0(V_3, \mathbb{F}) & \cong \mathbb{S}_3 \quad 4\mathbb{S}^{(3)} \oplus 2\mathbb{S}^{(2,1)}, \\
H^1(V_2, \mathbb{F}) & \cong \mathbb{S}_2 \quad 3\mathbb{S}^{(1,1)} \oplus \mathbb{S}^{(2)}, \\
H^2(V_3, \mathbb{F}) & \cong \mathbb{S}_3 \quad 4\mathbb{S}^{(1,1,1)} \oplus 2\mathbb{S}^{(2,1)}.
\end{align*}
\]
Key example (cont).

- For \( \mu = (\mu_1, \mu_2) \vdash k \), by the hook-length formula we have,
  \[
  \dim \mathcal{S}^\mu = \frac{k! \left( \mu_1 - \mu_2 + 1 \right)}{(\mu_1 + 1)! \mu_2!}.
  \]

- Since \( H^0(V_k, \mathbb{F}) \cong \mathcal{S}_k \bigoplus_{\mu = (\mu_1, \mu_2) \vdash k} m_{0, \mu} \mathcal{S}^\mu \), and hence
  \[
  \dim_{\mathbb{F}}(H^0(V_k, \mathbb{F})) = \sum_{\mu = (\mu_1, \mu_2) \vdash k} m_{0, \mu} \dim_{\mathbb{F}}(\mathcal{S}^\mu) = 2^k,
  \]
  we obtain as a consequence the identity
  \[
  k! \left( \sum_{\substack{\mu_1 \geq \mu_2 \geq 0 \\ \mu_1 + \mu_2 = k}} \frac{(\mu_1 - \mu_2 + 1)^2}{(\mu_1 + 1)! \mu_2!} \right) = 2^k.
  \]
Key example (cont).

- For $\mu = (\mu_1, \mu_2) \vdash k$, by the hook-length formula we have,

$$\dim S^\mu = \frac{k! (\mu_1 - \mu_2 + 1)}{(\mu_1 + 1)! \mu_2!}.$$

- Since $H^0(V_k, \mathbb{F}) \cong S_k \bigoplus_{\mu = (\mu_1, \mu_2) \vdash k} m_{0, \mu} S^\mu$, and hence $\dim_{\mathbb{F}}(H^0(V_k, \mathbb{F})) = \sum_{\mu = (\mu_1, \mu_2) \vdash k} m_{0, \mu} \dim_{\mathbb{F}}(S^\mu) = 2^k$, we obtain as a consequence the identity

$$k! \left( \sum_{\mu_1 \geq \mu_2 \geq 0 \atop \mu_1 + \mu_2 = k} \frac{(\mu_1 - \mu_2 + 1)^2}{(\mu_1 + 1)! \mu_2!} \right) = 2^k.$$
Previous Results

Theorem (B., Riener (2013))

Let $P \in \mathbb{R}[X_1, \ldots, X_k]$, be non-negative polynomial of degree bounded by $d$, and and such that $V = Z(P, \mathbb{R}^k)$ is invariant under the action of $\mathfrak{S}_k$. Then,

$$b(V/\mathfrak{S}_k, \mathbb{F}) \leq (k)^{2d}(O(d))^{2d+1}.$$ 

Note that $H^*(V/\mathfrak{S}_k, \mathbb{F})$ is isomorphic to the isotypic component of $H^*(V, \mathbb{F})$ belonging to the trivial representation $1_{\mathfrak{S}_k}$, and $b(V/\mathfrak{S}_k, \mathbb{F})$ is its multiplicity.
Previous Results

Theorem (B., Riener (2013))

Let $P \in \mathbb{R}[X_1, \ldots, X_k]$, be non-negative polynomial of degree bounded by $d$, and such that $V = Z(P, \mathbb{R}^k)$ is invariant under the action of $\mathfrak{S}_k$. Then,

$$b(V/\mathfrak{S}_k, \mathbb{F}) \leq (k)^{2d}(O(d))^{2d+1}.$$ 

Note that $H^*(V/\mathfrak{S}_k, \mathbb{F})$ is isomorphic to the isotypic component of $H^*(V, \mathbb{F})$ belonging to the trivial representation $1_{\mathfrak{S}_k}$, and $b(V/\mathfrak{S}_k, \mathbb{F})$ is its multiplicity.
Previous Results

**Theorem (B., Riener (2013))**

Let $P \in \mathbb{R}[X_1, \ldots, X_k]$, be non-negative polynomial of degree bounded by $d$, and such that $V = \mathbb{Z}(P, \mathbb{R}^k)$ is invariant under the action of $\mathfrak{S}_k$. Then,

$$b(V/\mathfrak{S}_k, \mathbb{F}) \leq (k)^{2d}(O(d))^{2d+1}.$$ 

Note that $H^*(V/\mathfrak{S}_k, \mathbb{F})$ is isomorphic to the isotypic component of $H^*(V, \mathbb{F})$ belonging to the trivial representation $1_{\mathfrak{S}_k}$, and $b(V/\mathfrak{S}_k, \mathbb{F})$ is its multiplicity.
More notation

For any $\mathcal{G}_k$-symmetric semi-algebraic subset $S \subset \mathbb{R}^k$, and $\lambda \vdash k$, we denote

\[
m_{i,\lambda}(S, \mathbb{F}) = \text{mult}(S^\lambda, H^i(S, \mathbb{F})),
\]

\[
m_\lambda(S, \mathbb{F}) = \sum_{i \geq 0} m_{i,\lambda}(S, \mathbb{F}).
\]
Theorem (B., Riener (2014))

Let $P \in \mathbb{R}[X_1, \ldots, X_k]$ be a $S_k$-symmetric polynomial, with $\deg(P) \leq d$. Let $V = Z(P, R^K)$. Then, for all $\mu = (\mu_1, \mu_2, \ldots) \vdash k$, $m_\mu(V, \mathbb{F}) > 0$ implies that

$$\text{card}(\{i \mid \mu_i \geq 2d\}) \leq 2d, \text{card}(\{j \mid \tilde{\mu}_j \geq 2d\}) \leq 2d.$$  

Moreover, for

$$m_\mu(V, \mathbb{F}) \leq k^{O(d^2)} d^d.$$
Figure: The shaded area contains all Young diagrams of partitions in $\text{Par}(k)$, while the darker area contains the Young diagrams of the partitions which can possibly appear in the $H^*(V, \mathbb{F})$ for fixed $d$ and large $k$. 
Asymptotics

Note that by a famous result of Hardy and Ramanujan (1918)

\[
\text{card(Par}(k)) \sim \frac{1}{4\sqrt{3}k} e^{\frac{\pi}{3} \sqrt{\frac{2k}{3}}}, \ k \to \infty
\]

which is exponential in \( k \);

whereas it follows from the last theorem that

\[
\text{card(\{\mu \vdash k \mid m_\mu(V, F) > 0\})}
\]

is polynomially bounded in \( k \) (for fixed \( d \)).
Asymptotics

- Note that by a famous result of Hardy and Ramanujan (1918)
  \[ \text{card}(\text{Par}(k)) \sim \frac{1}{4\sqrt{3}k} e^{\pi \sqrt{\frac{2k}{3}}}, \ k \to \infty \]
  which is exponential in \( k \);

- whereas it follows from the last theorem that
  \[ \text{card}(\{ \mu \vdash k \mid m_\mu(V, \mathbb{F}) > 0 \}) \]
  is polynomially bounded in \( k \) (for fixed \( d \)).
Proof Ingredients

- Degree principle.
- Equivariant Morse theory, equivariant Mayer-Vietoris sequence.
- Some tableau combinatorics. Pieri’s rule.
Proof Ingredients

- Degree principle.
- Equivariant Morse theory, equivariant Mayer-Vietoris sequence.
- Some tableau combinatorics. Pieri’s rule.
Proof Ingredients

- Degree principle.
- Equivariant Morse theory, equivariant Mayer-Vietoris sequence.
- Some tableau combinatorics. Pieri’s rule.
Similar results bounding multiplicities in the isotypic decomposition of the cohomology modules of:

- More general actions of the symmetric group – permuting blocks of size larger than one.
- Symmetric semi-algebraic sets.
- Symmetric complex varieties.
- Symmetric projective varieties.
More results

Similar results bounding multiplicities in the isotypic decomposition of the cohomology modules of:

- More general actions of the symmetric group – permuting blocks of size larger than one.
- Symmetric semi-algebraic sets.
- Symmetric complex varieties.
- Symmetric projective varieties.
More results

Similar results bounding multiplicities in the isotypic decomposition of the cohomology modules of:

- More general actions of the symmetric group – permuting blocks of size larger than one.
- Symmetric semi-algebraic sets.
- Symmetric complex varieties.
- Symmetric projective varieties.
More results

Similar results bounding multiplicities in the isotypic decomposition of the cohomology modules of:

- More general actions of the symmetric group – permuting blocks of size larger than one.
- Symmetric semi-algebraic sets.
- Symmetric complex varieties.
- Symmetric projective varieties.
Algorithmic conjecture

Conjecture
For any fixed $d > 0$, there is an algorithm that takes as input the description of a symmetric semi-algebraic set $S \subset \mathbb{R}^k$, defined by a $\mathcal{P}$-closed formula, where $\mathcal{P}$ is a set symmetric polynomials of degrees bounded by $d$, and computes $m_{i,\lambda}(S, \mathbb{Q})$, for each $\lambda \vdash k$ with $m_{i,\lambda}(S, \mathbb{Q}) > 0$, as well as all the Betti numbers $b_i(S, \mathbb{Q})$, with complexity which is polynomial in $\text{card}(\mathcal{P})$ and $k$. 
Let $F \in \mathbb{R}[X_1, \ldots, X_d]_{\leq d}$ be a symmetric polynomial of degree at most $d$, and let for $k \geq d$

$F_k = \phi_{d,k}(F) \in \mathbb{R}[X_1, \ldots, X_k]_{\leq d}$ where

$\phi_{d,k} : \mathbb{R}[X_1, \ldots, X_d]_{\leq d} \to \mathbb{R}[X_1, \ldots, X_k]_{\leq d}$ is the canonical injection.

Let $(V_k = Z(F_k, \mathbb{R}^k))_{k \geq d}$ be the corresponding sequence of symmetric real varieties.

Also, let $\mu = (\mu_1, \ldots, \mu_\ell) \vdash k_0$ be any fixed partition, and for all $k \geq k_0 + \mu_1$, let $\{\mu\}_k = (k - k_0, \mu_1, \mu_2, \ldots, \mu_\ell) \vdash k$.

It is a consequence of the hook-length formula that

$$\dim_{\mathbb{F}}(S^{\{\mu\}}_k) = \frac{\dim_{\mathbb{F}}(S^{\mu}_k)}{|\mu|!} P_{\mu}(k),$$

where $P_{\mu}(T)$ is a monic polynomial having distinct integer roots, and $\deg(P_{\mu}) = |\mu|$. 
Representational stability question

- Let $F \in R[X_1, \ldots, X_d]_{\leq d}$ be a symmetric polynomial of degree at most $d$, and let for $k \geq d$
  $F_k = \phi_{d,k}(F) \in R[X_1, \ldots, X_k]_{\leq d}$ where
  $\phi_{d,k} : R[X_1, \ldots, X_d]_{\leq d} \to R[X_1, \ldots, X_k]_{\leq d}$ is the canonical injection.

- Let $(V_k = Z(F_k, R^k))_{k \geq d}$ be the corresponding sequence of symmetric real varieties.

- Also, let $\mu = (\mu_1, \ldots, \mu_\ell) \vdash k_0$ be any fixed partition, and for all $k \geq k_0 + \mu_1$, let $\{\mu\}_k = (k - k_0, \mu_1, \mu_2, \ldots, \mu_\ell) \vdash k$.

- It is a consequence of the hook-length formula that
  $$\dim_{\mathbb{F}}(S^{\{\mu\}_k}) = \frac{\dim_{\mathbb{F}}(S^{\mu})}{|\mu|!} P_{\mu}(k),$$
  where $P_{\mu}(T)$ is a monic polynomial having distinct integer roots, and $\deg(P_{\mu}) = |\mu|$.
Representational stability question

- Let $F \in \mathbb{R}[X_1, \ldots, X_d]^{\mathbb{S}_d}$ be a symmetric polynomial of degree at most $d$, and let for $k \geq d$
  $F_k = \phi_{d,k}(F) \in \mathbb{R}[X_1, \ldots, X_k]^{\mathbb{S}_k}$ where
  $\phi_{d,k} : \mathbb{R}[X_1, \ldots, X_d]^{\mathbb{S}_d} \rightarrow \mathbb{R}[X_1, \ldots, X_k]^{\mathbb{S}_k}$ is the canonical injection.

- Let $(V_k = Z(F_k, \mathbb{R}^k)_{k \geq d}$ be the corresponding sequence of symmetric real varieties.

- Also, let $\mu = (\mu_1, \ldots, \mu_\ell) \vdash k_0$ be any fixed partition, and for all $k \geq k_0 + \mu_1$, let $\{\mu\}_k = (k - k_0, \mu_1, \mu_2, \ldots, \mu_\ell) \vdash k$.

- It is a consequence of the hook-length formula that

  $$\dim_{\mathbb{F}}(S^{\{\mu\}_k}) = \frac{\dim_{\mathbb{F}}(S^{\mu})}{|\mu|!} P_\mu(k),$$

  where $P_\mu(T)$ is a monic polynomial having distinct integer roots, and $\deg(P_\mu) = |\mu|$. 
Representational stability question

- Let $F \in \mathbb{R}[X_1, \ldots, X_d]_{\leq d}^\mathfrak{S}_d$ be a symmetric polynomial of degree at most $d$, and let for $k \geq d$
  $F_k = \phi_{d,k}(F) \in \mathbb{R}[X_1, \ldots, X_k]^\mathfrak{S}_k$ where
  $\phi_{d,k} : \mathbb{R}[X_1, \ldots, X_d]_{\leq d}^\mathfrak{S}_d \rightarrow \mathbb{R}[X_1, \ldots, X_k]^\mathfrak{S}_k$ is the canonical injection.

- Let $(V_k = Z(F_k, R^k))_{k \geq d}$ be the corresponding sequence of symmetric real varieties.

- Also, let $\mu = (\mu_1, \ldots, \mu_\ell) \vdash k_0$ be any fixed partition, and for all $k \geq k_0 + \mu_1$, let $\{\mu\}_k = (k - k_0, \mu_1, \mu_2, \ldots, \mu_\ell) \vdash k$.

- It is a consequence of the hook-length formula that

  \[
  \dim_{\mathbb{F}}(S^{\{\mu\}_k}) = \frac{\dim_{\mathbb{F}}(S^{\mu})}{|\mu|!} P_\mu(k),
  \]

  where $P_\mu(T)$ is a monic polynomial having distinct integer roots, and $\deg(P_\mu) = |\mu|$. 
For any fixed number $p \geq 0$ we pose the following question.

**Question**

Does there exist a polynomial $P_{F,p,\mu}(k)$ such that for all sufficiently large $k$, $m_{p,\{\mu\}_k}(V_k, F) = P_{F,p,\mu}(k)$? Note that a positive answer would imply that

$$\dim_F(H^p(V_k, F))_{\{\mu\}_k} = \frac{\dim_F(S_{\mu})}{|\mu|!} P_{F,p,\mu}(k)P_{\mu}(k)$$

is also given by a polynomial for all large enough $k$. A stronger question is to ask for a bound on the degree of $P_{F,p,\mu}(k)$ as a function of $d, \mu$ and $p$.

The conjecture holds in the “key example”.
For any fixed number $p \geq 0$ we pose the following question.

**Question**

*Does there exist a polynomial $P_{F,p,\mu}(k)$ such that for all sufficiently large $k$, $m_{p,\{\mu\}_k}(V_k, F) = P_{F,p,\mu}(k)$?* Note that a positive answer would imply that

$$\dim_F(H^p(V_k, F))_{\{\mu\}_k} = \frac{\dim_F(S_{\mu})}{|\mu|!} P_{F,p,\mu}(k) P_{\mu}(k)$$

is also given by a polynomial for all large enough $k$.

A stronger question is to ask for a bound on the degree of $P_{F,p,\mu}(k)$ as a function of $d, \mu$ and $p$.

The conjecture holds in the “key example”.
For any fixed number $p \geq 0$ we pose the following question.

**Question**

Does there exist a polynomial $P_{F,p,\mu}(k)$ such that for all sufficiently large $k$, $m_{p,\{\mu\}_k}(V_k, F) = P_{F,p,\mu}(k)$? Note that a positive answer would imply that

$$\dim_F(H^p(V_k, F))_{\{\mu\}_k} = \frac{\dim_F(S_{\mu})}{|\mu|!} P_{F,p,\mu}(k) P_{\mu}(k)$$

is also given by a polynomial for all large enough $k$.

A stronger question is to ask for a bound on the degree of $P_{F,p,\mu}(k)$ as a function of $d, \mu$ and $p$.

The conjecture holds in the “key example”.