Combinatorial Complexity in O-minimal Geometry

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Outline

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The Language of Arrangements

- Let $\mathcal{A} = \{S_1, \ldots, S_n\}$, with each $S_i \subset \mathrm{R}^k$ belonging to some "simple" class of sets (eg. hyperplanes, algebraic hypersurfaces of degree at most d, spheres, simplices etc.).
- For $I \subset \{1, \ldots, n\}$, let $\mathcal{A}(I)$denote the set

$$\bigcap_{i \in I \subset [1 \ldots n]} S_i \cap \bigcap_{j \in [1 \ldots n] \setminus I} \mathrm{R}^k \setminus S_j,$$

 and it is customary to call a connected component of $\mathcal{A}(I)$ a cell of the arrangement $\mathcal{A}$ and we denote by $\mathcal{C}(\mathcal{A})$ the set of all non-empty cells of the arrangement $\mathcal{A}$.
- The cardinality of $\mathcal{C}(\mathcal{A})$ is called the combinatorial complexity of the arrangement $\mathcal{A}$.

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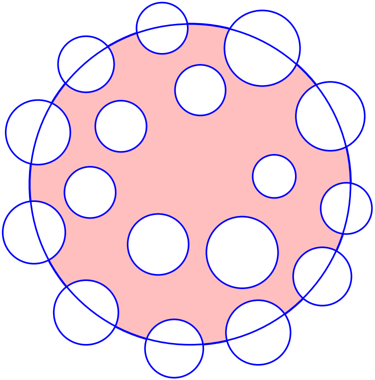
Arrangement of circles in $\mathbb{R}^2$



What does "simple" mean ?

- The class of sets usually considered in the study of arrangements are sets with "bounded description complexity". This means that each set in the arrangement is defined by a first order formula in the language of ordered fields involving at most a constant number polynomials whose degrees are also bounded by a constant.
- Additionally, there is often a requirement that the sets be in "general position". The precise definition of "general position" varies with context, but often involves restrictions such as: the sets in the arrangements are smooth manifolds, intersecting transversally.

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The Language of Semi-algebraic Geometry

- Let $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ be a set of polynomials with degrees bounded by d and $\#\mathcal{P} = n$.
- For $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$, we denote by
  - $\mathcal{R}(\sigma) = \{x \in \mathrm{R}^k \mid \mathrm{sign}(P(x)) = \sigma(P), \forall P \in \mathcal{P}\}$, and
  - $b_i(\sigma) = b_i(\mathcal{R}(\sigma))$.
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(B-Pollack-Roy, 2005)

$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_i(\sigma) \leq \sum_{j=0}^{k-i} \binom{n}{j} 4^j d(2d-1)^{k-1} = n^{k-i} O(d)^k.$$

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Complexity of Semi-algebraic Sets

- In the language of arrangements, the result in the previous slide implies (taking $i = 0$) that the combinatorial complexity of an arrangement of n algebraic hypersurfaces of fixed degree in R^k is bounded by $O(n^k)$ (d and k are to be considered fixed).
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Outline

Combinatorial Complexity

- Notice that the bound in the previous page are products of two quantities – one that depends only on n (and k), and another part which is independent of n. We refer to the first part as the combinatorial part of the complexity, and the latter as the algebraic part.
- While understanding the algebraic part of the complexity is a very important problem, in several applications, most notably in discrete and computational geometry, it is the combinatorial part of the complexity that is of interest (the algebraic part is assumed to be bounded by a constant).

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Admissible Sets

- Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R and let $T \subset \mathrm{R}^{k+\ell}$ be a fixed definable set.

$$\mathrm{R}^k \xleftarrow{\pi_1} T \subset \mathrm{R}^{k+\ell} \xrightarrow{\pi_2} \mathrm{R}^\ell$$

- We will call S of R^k to be a (T, π_1, π_2)-set if

$$S = T_{\mathbf{y}} = \pi_1(\pi_2^{-1}(\mathbf{y}) \cap T)$$

for some $\mathbf{y} \in \mathrm{R}^\ell$.

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Example I

Let $\mathcal{S}(\mathrm{R}) = \mathcal{S}_{\mathrm{sa}}(\mathrm{R})$ and Let $T \subset \mathrm{R}^{2k+1}$ be the semi-algebraic set defined by

$$T = \{(x_1, \ldots, x_k, a_1, \ldots, a_k, b) \mid \langle \mathbf{a}, \mathbf{x} \rangle - b = 0\}$$

(where we denote $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{x} = (x_1, \ldots, x_k)$), and π_1 and π_2 are the projections onto the first k and last $k+1$ co-ordinates respectively. A (T, π_1, π_2)-set is clearly a hyperplane in R^k and vice versa.

Example II

Let $\mathcal{S}(\mathbb{R}) = \mathcal{S}_{\exp}(\mathbb{R})$ and

$$T = \{(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_m, a_1, \ldots, a_m) \mid \mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_m \in \mathbb{R}^k,$$

$$a_1, \ldots, a_m \in \mathbb{R}, x_1, \ldots, x_k > 0, \sum_{i=0}^{m} a_i \mathbf{x}^{\mathbf{y}_i} = 0\},$$

with $\pi_1 : \mathbb{R}^{k+m(k+1)} \to \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^{k+m(k+1)} \to \mathbb{R}^{m(k+1)}$ be the projections onto the first k and the last $m(k+1)$ co-ordinates respectively. The (T, π_1, π_2)-sets in this example include (amongst others) all semi-algebraic sets consisting of intersections with the positive orthant of all real algebraic sets defined by a polynomial having at most m monomials (different sets of monomials are allowed to occur in different polynomials).

Outline

$\mathcal{A}$-sets I

Let $\mathcal{A} = \{S_1, \ldots, S_n\}$, such that each $S_i \subset \mathrm{R}^k$ is a (T, π_1, π_2)-set. For $I \subset \{1, \ldots, n\}$, we let $\mathcal{A}(I)$ denote the set

$$\bigcap_{i \in I \subset [1 \ldots n]} S_i \cap \bigcap_{j \in [1 \ldots n] \setminus I} \mathrm{R}^k \setminus S_j, \tag{1}$$

and we will call such a set to be a basic $\mathcal{A}$-set. We will denote by, $\mathcal{C}(\mathcal{A})$, the set of non-empty connected components of all basic $\mathcal{A}$-sets.

$\mathcal{A}$-sets II

- We will call definable subsets $S \subset \mathrm{R}^k$ defined by a first order formula with atoms

$$x \in S_i, 1 \leq i \leq n,$$

y an $\mathcal{A}$-set. An $\mathcal{A}$-set is thus a union of basic $\mathcal{A}$-sets.

- In case T is closed and the Boolean formula contains no negation we will call S an $\mathcal{A}$-closed set.

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Bounds on Betti Numbers I

Theorem

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exists a constant $C = C(T) > 0$ depending only on T, such that for any (T, π_1, π_2)-family $\mathcal{A} = \{S_1, \ldots, S_n\}$ of subsets of R^k the following holds. For every $i, 0 \leq i \leq k$,

$$\sum_{D \in \mathcal{C}(\mathcal{A})} b_i(D) \leq C \cdot n^{k-i}.$$

In particular, the combinatorial complexity of $\mathcal{A}$, is at most $C \cdot n^k$. The topological complexity of any m cells in the arrangement $\mathcal{A}$ is bounded by $m + C \cdot n^{k-1}$.

Lower dimensional

Theorem

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R and let $T \subset \mathrm{R}^{k+\ell}$, $V \subset \mathrm{R}^k$ be closed definable sets with $\dim(V) = k'$. Then, there exists a constant $C = C(T, V) > 0$ depending only on T and V, such that for any (T, π_1, π_2)-family, $\mathcal{A} = \{S_1, \ldots, S_n\}$, of subsets of R^k, and for every $i, 0 \leq i \leq k'$,

$$\sum_{D \in \mathcal{C}(\mathcal{A}, V)} b_i(D) \leq C \cdot n^{k'-i}.$$

In particular, the combinatorial complexity of $\mathcal{A}$ restricted to V, is bounded by $C \cdot n^{k'}$.

Topological Complexity of $\mathcal{A}$-sets

Theorem

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R, and let $T \subset \mathrm{R}^{k+\ell}$ be a definable set. Then, there exists a constant $C = C(T, V) > 0$ such that for any (T, π_1, π_2)-family, $\mathcal{A}$ with $\#\mathcal{A} = n$, and an $\mathcal{A}$-set $S \subset \mathrm{R}^k$,

$$\sum_{i \geq 0} b_i(S) \leq C \cdot n^k$$

Topological Complexity of Projections

Theorem (Topological Complexity of Projections)

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure, and let $T \subset \mathrm{R}^{k+\ell}$ be a definable, closed and bounded set. Let $k = k_1 + k_2$ and let $\pi_3 : \mathrm{R}^k \to \mathrm{R}^{k_2}$ denote the projection map on the last k_2 co-ordinates.
Then, there exists a constant $C = C(T) > 0$ such that for any (T, π_1, π_2)-family, $\mathcal{A}$, with $|\mathcal{A}| = n$, and an $\mathcal{A}$-closed set $S \subset \mathrm{R}^k$,

$$\sum_{i=0}^{k_2} b_i(\pi_3(S)) \leq C \cdot n^{(k_1+1)k_2}.$$

Outline

Definition of cdcd

A cdcd of R^k is a finite partition of R^k into definable sets $(C_i)_{i \in I}$ (called the cells of the cdcd) satisfying the following properties.
If $k = 1$ then a cdcd of R is given by a finite set of points $a_1 < \cdots < a_N$ and the cells of the cdcd are the singletons $\{a_i\}$ as well as the open intervals, $(\infty, a_1), (a_1, a_2), \ldots, (a_N, \infty)$.
If $k > 1$, then a cdcd of R^k is given by a cdcd, $(C_i')_{i \in I'}$, of R^{k-1} and for each $i \in I'$, a collection of cells, $\mathcal{C}_i$ defined by,

$$\mathcal{C}_i = \{\phi_i(C_i' \times D_j) \mid j \in J_i\},$$

Definition II

where

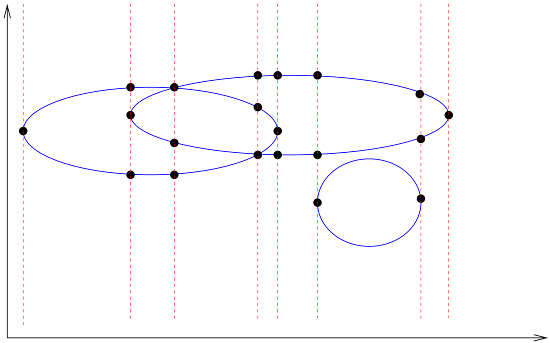
$$\phi_i : C'_i \times \mathrm{R} \to \mathrm{R}^k$$

is a definable homemorphism satisfying $\pi \circ \phi = \pi$, $(D_j)_{j \in J_i}$ is a cdcd of R, and $\pi : \mathrm{R}^k \to \mathrm{R}^{k-1}$ is the projection map onto the first $k-1$ coordinates. The cdcd of R^k is then given by

$$\bigcup_{i \in I'} \mathcal{C}_i.$$

Given a family of definable subsets $\mathcal{A} = \{S_1, \ldots, S_n\}$ of R^k, we say that a cdcd is adapted to $\mathcal{A}$, if each S_i is a union of cells of the given cdcd.

Easier to understand with a picture ....



Quantitative cylindrical definable cell decomposition I

Theorem (Quantitative cylindrical definable cell decomposition)

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R, and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exist constants $C_1, C_2 > 0$ depending only on T, and definable sets,

$$\{T_i\}_{i \in I}, \quad T_i \subset \mathrm{R}^k \times \mathrm{R}^{2(2^k-1)\cdot\ell},$$

depending only on T, with $|I| \le C_1$, such that for any (T, π_1, π_2)-family, $\mathcal{A} = \{S_1, \ldots, S_n\}$ with $S_i = T_{\mathbf{y}_i}, \mathbf{y}_i \in \mathrm{R}^\ell, 1 \le i \le n$, some sub-collection of the sets

Quantitative cylindrical definable cell decomposition II

Theorem (Quantitative cylindrical definable cell decomposition)

$$\pi^{\le k}_{k+2(2^k-1)\cdot\ell}\left({\pi^{>k}_{k+2(2^k-1)\cdot\ell}}^{-1}(\mathbf{y}_{i_1},\ldots,\mathbf{y}_{i_{2(2^k-1)}}) \cap T_i\right),$$
$$i \in I,\ 1 \le i_1,\ldots,i_{2(2^k-1)} \le n,$$

form a cdcd of R^k *compatible with* $\mathcal{A}$. *Moreover, the cdcd has at most* $C_2 \cdot n^{2(2^k-1)}$ *cells.*

An important point (in combinatorial applications) is that each cell in the cdcd defined above depends on at most $2(2^k-1)$ *of the elements of* $\mathcal{A}$.

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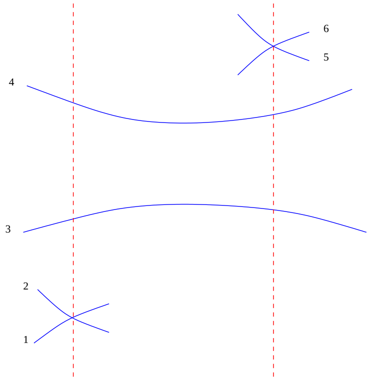
If $k = 2$ then $2(2^k - 1) = 6$



Outline

Ramsey-type Theorem

Theorem

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R, and let $T \subset \mathrm{R}^{k+\ell}$ be a definable set. Then, there exists a constant $1 > \varepsilon = \varepsilon(T) > 0$ depending only on T, such that for any (T, π_1, π_2)-family, $\mathcal{A} = \{S_1, \ldots, S_n\}$, there exists two subfamilies $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$, with $|\mathcal{A}_1|, |\mathcal{A}_2| \geq \varepsilon n$, and either,

- *for all $S_i \in \mathcal{A}_1$ and $S_j \in \mathcal{A}_2$, $S_i \cap S_j \neq \emptyset$, OR*
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Unions of definable families

Suppose that $T_1, \ldots, T_m \subset \mathrm{R}^{k+\ell}$ are closed, definable sets, $\pi_1 : \mathrm{R}^{k+\ell} \to \mathrm{R}^k$ and $\pi_2 : \mathrm{R}^{k+\ell} \to \mathrm{R}^\ell$ the two projections.

Lemma

For any collection of (T_i, π_1, π_2) families $\mathcal{A}_i$, $1 \leq i \leq m$, the family $\cup_{1 \leq i \leq m} \mathcal{A}_i$ is a (T', π'_1, π'_2) family where,

$$T' = \bigcup_{i=1}^{m} T_i \times \{e_i\} \subset \mathrm{R}^{k+\ell+m},$$

with e_i the i-th standard basis vector in R^m, and $\pi'_1 : \mathrm{R}^{k+\ell+m} \to \mathrm{R}^k$ and $\pi'_2 : \mathrm{R}^{k+\ell+m} \to \mathrm{R}^{\ell+m}$, the projections onto the first k and the last $\ell + m$ coordinates respectively.

Hardt's Triviality Theorem

Theorem (Hardt, 1980)

Given any definable set $S \subset \mathbb{R}^{k_1+k_2}$, there exists a finite partition of $\mathbb{R}^{k_2}$ into definable sets $\{T_i\}_{i \in I}$ such that S is definably trivial over each T_i.

This means that for each $i \in I$ and any point $\mathbf{z} \in T_i$, the pre-image $\pi_S^{-1}(T_i)$ is definably homeomorphic to $\pi_S^{-1}(\mathbf{z}) \times T_i$ by a fiber preserving homeomorphism. In particular, for each $i \in I$, all fibers $\pi_S^{-1}(\mathbf{z}), \mathbf{z} \in T_i$ are definably homeomorphic.

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This means that for each $i \in I$ and any point $\mathbf{z} \in T_i$, the pre-image $\pi_S^{-1}(T_i)$ is definably homeomorphic to $\pi_S^{-1}(\mathbf{z}) \times T_i$ by a fiber preserving homeomorphism. In particular, for each $i \in I$, all fibers $\pi_S^{-1}(\mathbf{z}), \mathbf{z} \in T_i$ are definably homeomorphic.

Notation

Given closed definable sets $X \subset V \subset \mathrm{R}^k$, and $\varepsilon > 0$, we denote

$$\mathrm{OT}(X, V, \varepsilon) = \{\mathbf{x} \in V \mid d_X(\mathbf{x}) < \varepsilon\},$$

$$\mathrm{CT}(X, V, \varepsilon) = \{\mathbf{x} \in V \mid d_X(\mathbf{x}) \leq \varepsilon\},$$

$$\mathrm{BT}(X, V, \varepsilon) = \{\mathbf{x} \in V \mid d_X(\mathbf{x}) = \varepsilon\},$$

and finally for $\varepsilon_1 > \varepsilon_2 > 0$ we define

$$\mathrm{Ann}(X, V, \varepsilon_1, \varepsilon_2) = \{\mathbf{x} \in V \mid \varepsilon_2 < d_X(\mathbf{x}) < \varepsilon_1\},$$

$$\overline{\mathrm{Ann}}(X, V, \varepsilon_1, \varepsilon_2) = \{\mathbf{x} \in V \mid \varepsilon_2 \leq d_X(\mathbf{x}) \leq \varepsilon_1\}.$$

Key Proposition

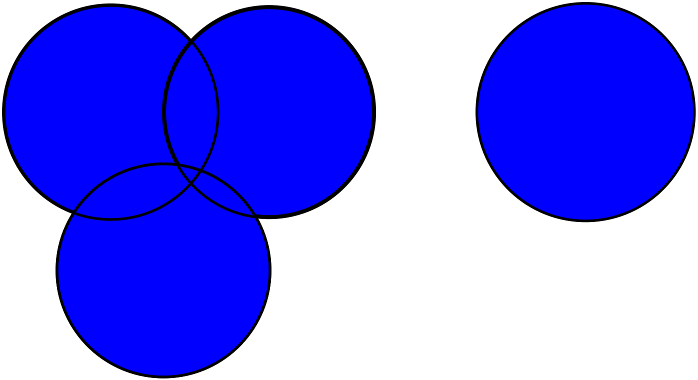
Proposition

Let $\mathcal{A} = \{S_1, \ldots, S_n\}$ be a collection of closed definable subsets of R^k and let $V \subset \mathrm{R}^k$ be a closed, and bounded definable set. Then, for all sufficiently small $1 \gg \varepsilon_1 \gg \varepsilon_2 > 0$ the following holds. For any connected component, C, of $\mathcal{A}(I) \cap V$, $I \subset [1 \ldots n]$, there exists a connected component, D, of the definable set,

$$\bigcap_{1 \le i \le n} \mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c \cap V$$

such that D is definably homotopy equivalent to C.

Picture



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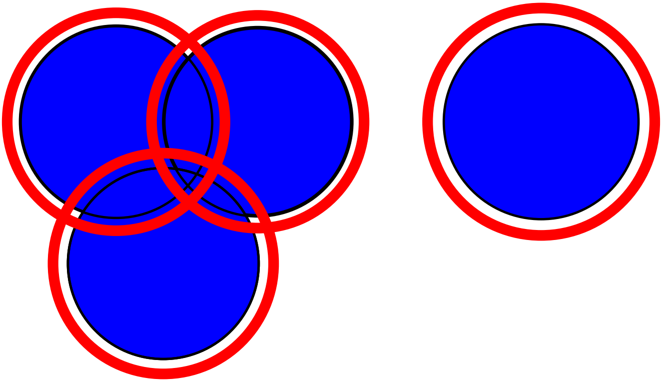

Proof of Theorem on Topological Complexity

- For $1 \leq i \leq n$, let $\mathbf{y}_i \in \mathrm{R}^\ell$ such that

$$S_i = T_{\mathbf{y}_i},$$

and let

$$A_i(\varepsilon_1, \varepsilon_2) = \mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c \cap V.$$

- Applying Mayer-Vietoris inequalities we have for $0 \leq i \leq k'$,

$$b_i(\bigcap_{j=1}^{n} A_j(\varepsilon_1, \varepsilon_2)) \leq b_{k'}(V) + \sum_{j=1}^{k'-i} \sum_{J \subset \{1,\ldots,n\}, \#(J)=j} \Big(b_{i+j-1}(A^J(\varepsilon_1, \varepsilon_2))$$

where $A^J(\varepsilon_1, \varepsilon_2) = \cup_{j \in J} A_j(\varepsilon_1, \varepsilon_2)$.

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Proof of Theorem on Topological Complexity (cont).

- Notice that each $\mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c, 1 \le i \le n$, is a $(\mathrm{Ann}(T, \varepsilon_1, \varepsilon_2)^c, \pi_1, \pi_2)$-set and moreover,

$$\mathrm{Ann}(S_i, \varepsilon_1, \varepsilon_2)^c = T_{\mathbf{y}_i} \cap \mathrm{Ann}(T, \varepsilon_1, \varepsilon_2)^c; \; 1 \le i \le n.$$

- For $J \subset [1 \ldots n]$, we denote

$$S^J(\varepsilon_1, \varepsilon_2) = \bigcup_{j \in J} \mathrm{Ann}(S_j, \varepsilon_1, \varepsilon_2)^c.$$

There are only a finite number (depending on T) of topological types amongst $S^J(\varepsilon_1, \varepsilon_2)$. Restricting all the sets to V in the above argument, we obtain that there are only finitely many (depending on T and V) of topological types amongst the sets $A^J(\varepsilon_1, \varepsilon_2) = S^J(\varepsilon_1, \varepsilon_2) \cap V$.

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Proof of Theorem on topological complexity(cont).

- Thus, there exists a constant $C(T,V)$ such that

$$C(T,V) \geq \max_{J \subset \{1,\ldots,n\}} \left(b_{i+j-1}(A^J(\varepsilon_1,\varepsilon_2)) + b_{k'}(V) \right) + b_{k'}(V).$$

- It follows from the previous Proposition that

$$\sum_{D \in \mathcal{C}(\mathcal{A},V)} b_i(D) \leq C \cdot n^{k'-i}.$$

Proof of Theorem on topological complexity(cont).

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Proof of Theorem for $\mathcal{A}$-sets

Key proposition:

Proposition

Let $\mathcal{A} = \{S_1, \ldots, S_n\}$ be a collection of closed definable subsets of R^k and let $V \subset \mathrm{R}^k$ be a closed, and bounded definable set and let S be an $(\mathcal{A}, V)$-closed set. Then, for all sufficiently small $1 \gg \varepsilon_1 \gg \varepsilon_2 \cdots \gg \varepsilon_n > 0$,

$$b(S) \leq \sum_{D \in \mathcal{C}(\mathcal{B}, V)} b(D),$$

where

$$\mathcal{B} = \bigcup_{i=1}^{n} \{S_i, \mathrm{BT}(S_i, \varepsilon_i), \mathrm{OT}(S_i, 2\varepsilon_i)^c\}.$$

Proof of Theorem on projections

Notice that for each $p, 0 \leq p \leq k_2$, and any $\mathcal{A}$-closed set $S \subset \mathrm{R}^{k_1+k_2}$, $W^p_{\pi_3}(S) \subset \mathrm{R}^{(p+1)k_1+k_2}$ is an $\mathcal{A}^p$-closed set where,

$$\mathcal{A}^p = \bigcup_{j=0}^{p} \mathcal{A}^{p,j},$$

$$\mathcal{A}^{p,j} = \bigcup_{i=1}^{n} \{S_i^{p,j}\},$$

where $S_i^{p,j} \subset \mathrm{R}^{(p+1)k_1+k_2}$ is defined by,

$$S_i^{p,j} = \{(\mathbf{x}_0, \ldots, \mathbf{x}_p, \mathbf{y}) \mid \mathbf{x}_j \in \mathrm{R}^{k_1}, \mathbf{y} \in \mathrm{R}^{k_2}, (\mathbf{x}_j, \mathbf{y}) \in S_i\}.$$

and $W^i_f(X) = \{(\mathbf{x}_0, \ldots, \mathbf{x}_i) \in X^{i+1} \mid f(\mathbf{x}_0) = \cdots = f(\mathbf{x}_i)\}$.

Proof of Theorem on Projections (cont).

- Also, note that $\mathcal{A}^{p,j}$ is a $(T^{p,j}, \pi_1^p, \pi_2^p)$ family, where

$$T^{p,j} = \{(\mathbf{x}_0, \ldots, \mathbf{x}_p, \mathbf{y}, \mathbf{z}) \mid \mathbf{x}_j \in \mathrm{R}^{k_1}, \mathbf{y} \in \mathrm{R}^{k_2}, \mathbf{z} \in \mathrm{R}^{\ell}, (\mathbf{x}_j, \mathbf{y}, \mathbf{z}) \in T, \text{ for some } j, 0 \le j \le p\}.$$

  and $\pi_1^p : \mathrm{R}^{(p+1)k_1+k_2+\ell} \longrightarrow \mathrm{R}^{(p+1)k_1+k_2}$, and $\pi_2^p : \mathrm{R}^{(p+1)k_1+k_2+\ell} \longrightarrow \mathrm{R}^{\ell}$ are the appropriate projections.
- Since each $T^{p,j}$ is determined by T, we have using previous lemma that $\mathcal{A}^p$ is a (T', π_1', π_2')-family for some definable T' determined by T.

Proof of Theorem on Projections (cont).

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Proof of Theorem on projections (cont).

Now $W^p_{\pi_3}(S) \subset \mathrm{R}^{(p+1)k_1+k_2}$ is a $\mathcal{A}^p$-closed set and $\#\mathcal{A}^p = (p+1)n$. Applying previous theorem we get, for each p and j, $0 \leq p, j < k_2$,

$$b_j(W^p_{\pi_3}(S)) \leq C_1(T) \cdot n^{(p+1)k_1+k_2}$$

The theorem now follows, since for each $q, 0 \leq q < k_2$,

$$b_q(\pi_3(S)) \leq \sum_{i+j=q} b_j(W^i_{\pi_3}(S)) \leq C_2(T)\cdot n^{(q+1)k_1+k_2} \leq C(T)\cdot n^{(k_1+1)k_2}.$$

Fibers of a definable map

- Let $S \subset \mathrm{R}^{k_1+k_2}$ be a definable set, and let $\pi : \mathrm{R}^{k_1+k_2} \rightarrow \mathrm{R}^{k_2}$ be the projection map on the last k_2 co-ordinates. We denote by $\pi_S = \pi|_S$.
- For $\mathbf{z} \in \mathrm{R}^{k_2}$, let $S_{\mathbf{z}} = S \cap \pi^{-1}(\mathbf{z})$.
- Question: How many "topological types" occur amongst the $S_{\mathbf{z}}$'s as $\mathbf{z}$ varies over R^{k_2} ?
- As an application: how many topological types occur amongst real or complex hypersurfaces defined by a polynomial of degree d in k variables ?

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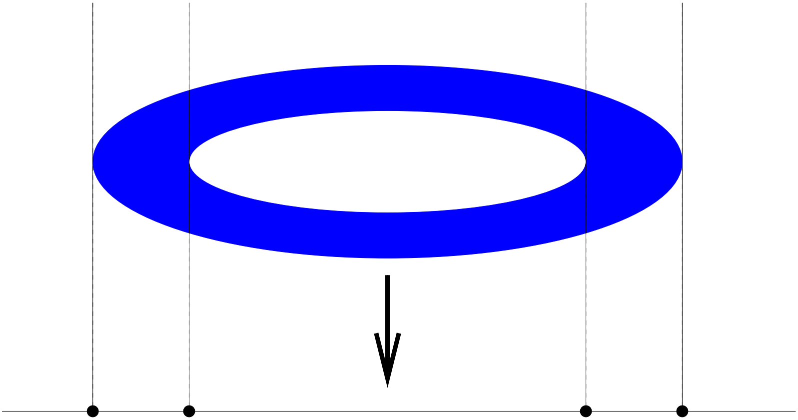
Definable map



Complexity of the Hardt partition

- Hardt's theorem is a corollary of the existence of *cylindrical cell decompositions* for definable sets.
- This implies a double exponential (in $k_1 k_2$) upper bound on the cardinality of I.
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The Semi-algebraic Case

Theorem (B., Vorobjov, 2007)

Let $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_{k_1}, Y_1, \ldots, Y_{k_2}]$, with $\deg(P) \leq d$ for each $P \in \mathcal{P}$ and $\#\mathcal{P} = n$, and let $\pi : \mathrm{R}^{k_1+k_2} \to \mathrm{R}^{k_2}$ be the projection map on the Y-cordinates. Then, for any fixed $\mathcal{P}$-semi-algebraic set S the number of different homotopy types of fibers $\pi^{-1}(\mathbf{y}) \cap S, \mathbf{y} \in \pi(S)$ is bounded by

$$(2^{k_1} n k_2 d)^{O(k_1 k_2)}.$$

Open Problem: Can one prove a single exponential bound like the one above on the number of homeomorphism types ?

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The o-minimal case

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over R, $T \subset \mathrm{R}^{k_1+k_2+\ell}$ a closed definable set, and

$$\begin{aligned}
\pi_1 &: \mathrm{R}^{k_1+k_2+\ell} \rightarrow \mathrm{R}^{k_1+k_2}, \\
\pi_2 &: \mathrm{R}^{k_1+k_2+\ell} \rightarrow \mathrm{R}^{\ell}, \\
\pi_3 &: \mathrm{R}^{k_1+k_2} \rightarrow \mathrm{R}^{k_2}
\end{aligned}$$

the projection maps as depicted below.

$$\begin{array}{ccc}
\mathrm{R}^{k_1+k_2+\ell} & \xrightarrow{\pi_1} & \mathrm{R}^{k_1+k_2} \\
\Big\downarrow{\scriptstyle \pi_2} & & \Big\downarrow{\scriptstyle \pi_3} \\
\mathrm{R}^{\ell} & & \mathrm{R}^{k_2}
\end{array}$$

Bounding the number of homotopy types

Theorem (B. 2007)

For any collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ *of subsets of* $\mathrm{R}^{k_1+k_2}$*, and* $\mathbf{z} \in \mathrm{R}^{k_2}$*, let* $\mathcal{A}_{\mathbf{z}}$ *denote the collection of subsets of* R^{k_1}*,*

$$\{A_{1,\mathbf{z}}, \ldots, A_{n,\mathbf{z}}\},$$

where $A_{i,\mathbf{z}} = A_i \cap \pi_3^{-1}(\mathbf{z}),\ 1 \leq i \leq n$*. Then, there exists a constant* $C = C(T) > 0$*, such that for any family* $\mathcal{A} = \{A_1, \ldots, A_n\}$ *of definable sets, where each* $A_i = \pi_1(T \cap \pi_2^{-1}(\mathbf{y}_i))$*, for some* $\mathbf{y}_i \in \mathrm{R}^{\ell}$*, and any fixed* $\mathcal{A}$*-set* S*, the number of homotopy types of the fibers* $S \cap \pi_3 - 1(\mathbf{z}),\ \mathbf{z} \in \mathrm{R}^{k_2}$*, is bounded by* $C \cdot n^{(k_1+3)k_2}$*.*

Some Open problems

1. Try to prove all the known results on combinatorial complexity of arrangements in the o-minimal setting. (Note that we are not allowed to use “general position” assumptions such as transversality etc., or other tricks such as “linearization” which strongly depend on the semi-algebraicity of the objects.)
2. Prove a singly exponential upper bound on the number of homeomorphism types (not just homotopy types) of the fibers of a definable map. This would be interesting in the special cases of semi-algebraic or semi-Pfaffian sets.

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