Efficient Algorithms for Computing Betti Numbers of Semi-algebraic Sets :3

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Outline

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Singly Exponential Covering

Theorem

There exists an algorithm that takes as input the description of a $\mathcal{P}$-closed semi-algebraic set $S \subset \mathrm{R}^k$, and outputs a covering of S by a family of subsets of S which are closed and contractible. The complexity of the algorithm, as well as the complexity of the covering, is

$$(sd)^{k^{O(1)}},$$

where $s = \#(\mathcal{P})$ *and* $d = \max_{P \in \mathcal{P}} \deg(P)$.

Outline

Main Theorem

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There exists an algorithm that takes as input the description of a $\mathcal{P}$-semi-algebraic set $S \subset \mathrm{R}^k$, and outputs $b_0(S), \ldots, b_\ell(S)$. The complexity of the algorithm is

$$(sd)^{k^{O(\ell)}},$$

where $s = \#(\mathcal{P})$ *and* $d = \max_{P \in \mathcal{P}} \deg(P)$.

Main Ingredients

- The first ingredient is a result discussed in the previous lecture, which enables us to compute a singly exponential sized covering of the given closed and bounded semi-algebraic set, consisting of closed, contractible semi-algebraic sets, in single exponential time. The number and the degrees of the polynomials used to define the sets in this covering are also bounded singly exponentially.
- The second ingredient is an algorithm which uses the covering algorithm recursively and computes in singly exponential time a complex whose homology groups are isomorphic to the first ℓ homology groups of the input set. This complex is of singly exponential size for fixed ℓ.

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Outline

Mayer-Vietoris exact sequence

- Let $A_1, \ldots, A_n$ be subcomplexes of a finite simplicial complex A such that $A = A_1 \cup \cdots \cup A_n$. Let $C^i(A)$ denote the R-vector space of i co-chains of A, and $C^*(A) = \oplus_i C^i(A)$.
- We will denote by $A_{\alpha_0,\ldots,\alpha_p}$ the subcomplex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.
- The following sequence of homomorphisms is exact.

$$0 \longrightarrow C^*(A) \xrightarrow{r} \prod_{\alpha_0} C^*(A_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0 < \alpha_1} C^*(A_{\alpha_0,\alpha_1})$$

$$\cdots \xrightarrow{\delta} \prod_{\alpha_0 < \cdots < \alpha_p} C^*(A_{\alpha_0,\ldots,\alpha_p}) \cdots \xrightarrow{\delta} \prod_{\alpha_0 < \cdots < \alpha_{p+1}} C^*(A_{\alpha_0,\ldots,\alpha_{p+1}}) \cdots \xrightarrow{\delta} \cdots$$

Mayer-Vietoris exact sequence

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Mayer-Vietoris Double Complex

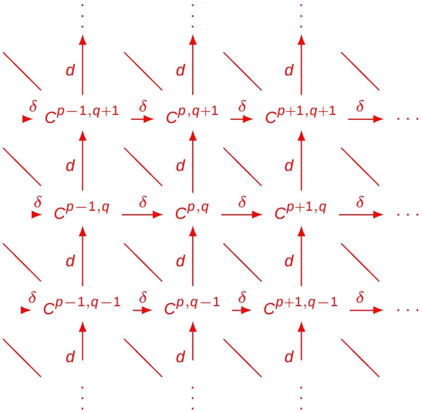
$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
0 & \longrightarrow & \prod_{\alpha_0} C^3(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^3(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^3(A_{\alpha_0} \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
0 & \longrightarrow & \prod_{\alpha_0} C^2(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^2(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^2(A_{\alpha_0} \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
0 & \longrightarrow & \prod_{\alpha_0} C^1(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^1(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^1(A_{\alpha_0} \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
0 & \longrightarrow & \prod_{\alpha_0} C^0(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^0(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^0(A_{\alpha_0} \\
 & & \uparrow d & & \uparrow d & & \uparrow d \\
 & & 0 & & 0 & & 0
\end{array}
$$

Outline

Double Complex

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
C^{0,2} & \xrightarrow{\delta} & C^{1,2} & \xrightarrow{\delta} & C^{2,2} & \xrightarrow{\delta} & \cdots \\
d \uparrow & & d \uparrow & & d \uparrow & & \\
C^{0,1} & \xrightarrow{\delta} & C^{1,1} & \xrightarrow{\delta} & C^{2,1} & \xrightarrow{\delta} & \cdots \\
d \uparrow & & d \uparrow & & d \uparrow & & \\
C^{0,0} & \xrightarrow{\delta} & C^{1,0} & \xrightarrow{\delta} & C^{2,0} & \xrightarrow{\delta} & \cdots
\end{array}
$$

The Associated Total Complex



Spectral Sequences of a Double Complex

- A sequence of vector spaces progressively approximating the homology of the total complex. More precisely,
- a sequence of bi-graded vector spaces and differentials $(E_r, d_r : E_r^{p,q} \to E_r^{p+r,q-r+1})$,
- $E_{r+1} = H(E_r, d_r)$,
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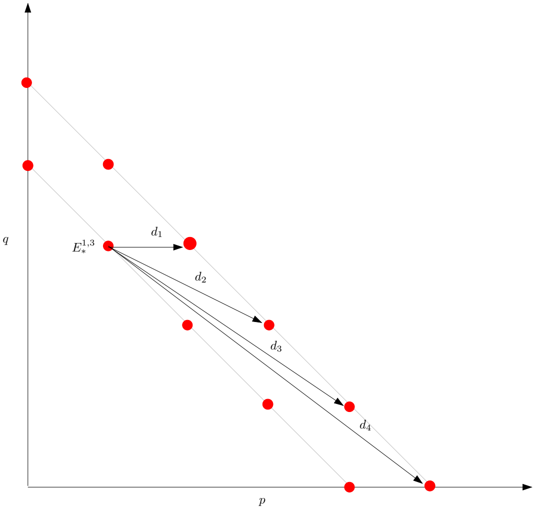
Spectral Sequence



Two Spectral Sequences

- There are two spectral sequences associated with $\mathcal{M}^{p,q}$ both converging to $H_D^*(\mathcal{M})$. The first terms of these are:
- $$'E_1 = H_\delta(\mathcal{M}), 'E_2 = H_d H_\delta(\mathcal{M})$$
- $$''E_1 = H_d(\mathcal{M}), ''E_2 = H_\delta H_d(\mathcal{M})$$

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Homomorphisms of Double Complexes

Given two (first quadrant) double complexes, $C^{\bullet,\bullet}$ and $\bar{C}^{\bullet,\bullet}$, a homomorphism of double complexes,

$$\phi : C^{\bullet,\bullet} \longrightarrow \bar{C}^{\bullet,\bullet},$$

is a collection of homomorphisms, $\phi^{p,q} : C^{p,q} \longrightarrow \bar{C}^{p,q}$, such that the following diagrams commute.

$$\begin{array}{ccc} C^{p,q} & \xrightarrow{\delta} & C^{p+1,q} \\ \downarrow \phi^{p,q} & & \downarrow \phi^{p+1,q} \\ \bar{C}^{p,q} & \xrightarrow{\delta} & \bar{C}^{p+1,q} \end{array}$$

$$\begin{array}{ccc} C^{p,q} & \xrightarrow{d} & C^{p,q+1} \\ \downarrow \phi^{p,q} & & \downarrow \phi^{p,q+1} \\ \bar{C}^{p,q} & \xrightarrow{d} & \bar{C}^{p,q+1} \end{array}$$

Comparison Theorem

Proposition

A homomorphism of double complexes,

$$\phi : C^{\bullet,\bullet} \longrightarrow \bar{C}^{\bullet,\bullet},$$

induces homomorphisms, $'\phi_s : {}'E_s \longrightarrow {}'\bar{E}_s$ *(respectively,* $''\phi_s : {}''E_s \longrightarrow {}''\bar{E}_s$*) between the associated spectral sequences. If* $'\phi_s$ *(respectively,* $''\phi_s$*) is an isomorphism for some* $s \geq 1$*, then* $'E_r^{p,q}$ *and* $'\bar{E}_r^{p,q}$ *(repectively,* $''E_r^{p,q}$ *and* $''\bar{E}_r^{p,q}$ *) are isomorphic for all* $r \geq s$*. In particular, the induced homomorphism,*

$$\phi : \mathrm{Tot}^{\bullet}(C^{\bullet,\bullet}) \rightarrow \mathrm{Tot}^{\bullet}(\bar{C}^{\bullet,\bullet})$$

is a quasi-isomorphism.

Outline

${}'E_1$

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
C^3(A) & 0 & 0 \\
C^2(A) & 0 & 0 \\
C^1(A) & 0 & 0 \\
C^0(A) & 0 & 0
\end{array}
$$

$'E_2$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ H^3(A) & 0 & 0 \\ H^2(A) & 0 & 0 \\ H^1(A) & 0 & 0 \\ H^0(A) & 0 & 0 \end{array}$$

$''E_1$

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\prod_{\alpha_0} H^3(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^3(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^3(A_{\alpha_0,\alpha_1,\alpha_2}) \\
\prod_{\alpha_0} H^2(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^2(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^2(A_{\alpha_0,\alpha_1,\alpha_2}) \\
\prod_{\alpha_0} H^1(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^1(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^1(A_{\alpha_0,\alpha_1,\alpha_2}) \\
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\end{array}
$$

$''E_1$ in this case

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
0 & \prod_{\alpha_0<\alpha_1} H^3(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^3(A_{\alpha_0,\alpha_1,\alpha_2}) \\
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\end{array}
$$

Convergence of the Mayer-Vietoris Spectral Sequence

The following proposition is classical and follows from the exactness of the generalized Mayer-Vietoris sequence.

Proposition

The spectral sequences, $'E_r, ''E_r$, associated to $\mathcal{N}^{\bullet,\bullet}(A)$ converge to $H^*(A, \mathbb{Q})$ and thus,

$$H^*(\mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet}(A))) \cong H^*(A, \mathbb{Q}).$$

Moreover, the homomorphism $\psi : C^\bullet(A) \to \mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet}(A))$ induced by the homomorphism r (in the generalized Mayer-Vietoris sequence) is a quasi-isomorphism.

Admssible Subsets

- Consider a fixed family of polynomials, $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, as well as a fixed $\mathcal{P}$-closed and bounded semi-algebraic set, $S \subset \mathrm{R}^k$. We also fix a number, $\ell, 0 \leq \ell \leq k$.
- We identify certain closed and bounded semi-algebraic subsets of S (which we call the admissible subsets of S). We associate to each admissible subset $X \subset S$, its level denoted $\mathrm{level}(X)$, with $\mathrm{level}(S) = 0$.
- For each such admissible subset, $X \subset S$, we define a double complex, $\mathcal{M}^{\bullet,\bullet}(X)$, such that

$$H^i(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(X))) \cong H^i(X, \mathbb{Q}),\ 0 \leq i \leq \ell - \mathrm{level}(X).$$

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Main Idea

- If the sets occuring in the covering of X are all acyclic, then the first column of the Mayer-Vietoris double complex is zero except at the first row.
- In order to compute $b_0(X), \ldots, b_{\ell-\mathrm{level}(X)}(X)$, it suffices to compute a suitable truncation of the Mayer-Vietoris double complex.
- However, we do not know how to efficiently compute (even the truncated) Mayer-Vietoris double complex.
- However, making use of the covering construction recursively, we are able to compute another double complex, $\mathcal{M}^{\bullet,\bullet}(X)$, which has much smaller size but whose associated total complex is quasi-isomorphic to the truncated Mayer-Vietoris double complex.

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Admissible Sets

- Given any closed and bounded semi-algebraic set $X \subset \mathrm{R}^k$, we will denote by $\mathcal{C}'(X)$, a fixed covering of X by a finite family of closed, bounded and acyclic semi-algebraic sets.
- We have that, $V \subset X$ for each $V \in \mathcal{C}'(X)$ and $X = \cup_{V \in \mathcal{C}'(X)} V$. We will index the sets in $\mathcal{C}'(X)$ as $V_1, \ldots, V_{n_X}$ where $n_X = \#\mathcal{C}'(X)$, and for $1 \leq \alpha_0 < \cdots < \alpha_p \leq n_X$, we will denote $V_{\alpha_0,\ldots,\alpha_p} = \bigcap_{0 \leq i \leq p} V_{\alpha_i}$. For $I \subset J \subset \{1, \ldots, n_X\}$ we will call V_I an *ancestor* of V_J and X an ancestor of all the V_I's. We will transitively close the ancestor relation.

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Admissible Sets (cont.)

- We now associate to certain closed semi-algebraic subsets X of S (which we call the admissible subsets of S), a covering, $\mathcal{C}(X)$, of X by closed, bounded, acyclic semi-algebraic sets, obtained by enlarging the covering $\mathcal{C}'(X)$.
- The set S itself is admissible of level 0 and $\mathcal{C}(S) = \mathcal{C}'(S)$. All intersections of the sets in $\mathcal{C}(S)$ taken upto $\ell + 2$ at a time are admissible and have level 1.

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Admissible Sets (cont.)

- The admissible subsets of S are the smallest family of subsets of S containing the above sets and satisfying the following. For any admissible subset $X \subset S$ at level i, we define $\mathcal{C}(X)$ as follows. Let $\{Y_1, \ldots, Y_N\}$ be the set of admissible sets which are ancestors of X. Then,

$$\mathcal{C}(X) = \bigcup_{U_i \in \mathcal{C}(Y_i), 1 \leq i \leq N} \mathcal{C}'(U_1 \cap \cdots \cap U_N \cap X).$$

  All intersections of the sets in $\mathcal{C}(X)$ taken at most $\ell - i + 2$ at a time are admissible, have level $i + 1$, and have X as an ancestor. For $I \subset J \subset \{1, \ldots, n_X\}$, V_I is an ancestor of V_J and X is an ancestor of all the V_I's. Moreover, for $V \in \mathcal{C}'(U_1 \cap \cdots \cap U_N \cap X)$, each U_i is an ancestor of V. This clearly implies that each $V \in \mathcal{C}(X)$ has a unique ancestor ineach $\mathcal{C}(Y_i)$ (namely, U_i).

Complexity of computing $\mathcal{C}'(X)$

We have a procedure (recall last lecture) for computing $\mathcal{C}'(X)$, for any given $\mathcal{P}'$-closed and bounded semi-algebraic set, X, such that the number and the degrees of the polynomials appearing in the output of this procedure is bounded by $(mD)^{k^{O(1)}}$. where $\#\mathcal{P}' = m$ and $\deg(P) \leq D$, for $P \in \mathcal{P}'$.

Complexity of computing $\mathcal{C}'(X)$

Proposition

Let $S \subset \mathrm{R}^k$ be a $\mathcal{P}$-closed semi-algebraic set, where $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ is a family of s polynomials of degree at most d. Then the number of admissible sets, the number of polynomials used to define them, the degrees of these polynomials, are all bounded by $(sd)^{k^{O(\ell)}}$.

Proof.

By induction on $\mathrm{level}(X)$. □

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Double complex Associated to an Admissible Set

Given the different coverings described above, we now associate to each admissible set $X \subset S$ a double complex, $\mathcal{M}^{\bullet,\bullet}(X)$, satisfying the following:

1.
$$H^i(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(X)), \mathbb{Q}) \cong H^i(X, \mathbb{Q}), \text{ for } 0 \le i \le \ell - \mathrm{level}(X). \tag{1}$$

2. For every admissible set Y, such that X is an ancestor of Y, and $\mathrm{level}(X) = \mathrm{level}(Y)$, a restriction homomorphism: $r_{X,Y} : \mathcal{M}^{\bullet,\bullet}(X) \to \mathcal{M}^{\bullet,\bullet}(Y)$, which induces the restriction homomorphisms between the cohomology groups:
$$r : H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$$
for $0 \le i \le \ell - \mathrm{level}(X)$ via the isomorphisms in (1).

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$$r : H^i(X,\mathbb{Q}) \to H^i(Y,\mathbb{Q})$$

for $0 \le i \le \ell - \mathrm{level}(X)$ via the isomorphisms in (1).

Construction of $\mathcal{M}^{\bullet,\bullet}(X)$

We now describe the construction of the double complex $\mathcal{M}^{\bullet,\bullet}(X)$ and prove that it has the properties stated above. The double complex $\mathcal{M}^{\bullet,\bullet}(X)$ is constructed inductively using induction on $\mathrm{level}(X)$:
The base case is when $\mathrm{level}(X) = \ell$. In this case the double complex, $\mathcal{M}^{\bullet,\bullet}(X)$ is defined by:

$$\begin{aligned}
\mathcal{M}^{0,0}(X) &= \bigoplus_{U_{\alpha_0} \in \mathcal{C}(X)} C^0(U_{\alpha_0}),\\
\mathcal{M}^{1,0}(X) &= \bigoplus_{U_{\alpha_0}, U_{\alpha_1} \in \mathcal{C}(X), \alpha_0 < \alpha_1} C^0(U_{\alpha_0,\alpha_1}),\\
\mathcal{M}^{p,q}(X) &= 0, \text{ if } q > 0 \text{ or } p > 1.
\end{aligned}$$

Here $C^0(Y)$ is the $\mathbb{Q}$-vector space of $\mathbb{Q}$ valued locally constant functions on Y.

Diagramatically

$$
\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow \\
\bigoplus_{U_{\alpha_0} \in \mathcal{C}(X)} C^0(U_{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{U_{\alpha_0}, U_{\alpha_1} \in \mathcal{C}(X), \alpha_0 < \alpha_1} C^0(U_{\alpha_0,\alpha_1}) & \longrightarrow & 0
\end{array}
$$

Definition of the restriction homomorphism

For every admissible set Y, such that X is an ancestor of Y, and $\mathrm{level}(X) = \mathrm{level}(Y) = \ell$, we define $r_{X,Y} : \mathcal{M}^{0,0}(X) \to \mathcal{M}^{0,0}(Y)$, as follows. Recall that, $\mathcal{M}^{0,0}(X) = \bigoplus_{U \in \mathcal{C}(X)} C^0(U)$, and $\mathcal{M}^{0,0}(Y) = \bigoplus_{V \in \mathcal{C}(Y)} C^0(V)$.

Also, by definition of $\mathcal{C}(Y)$, we have that for each $V \in \mathcal{C}(Y)$ there is a unique $U \in \mathcal{C}(X)$ (which we will denote by $a(V)$) such that U is an ancestor of V.

For $x \in \mathcal{M}^{0,0}(X)$ and $V \in \mathcal{C}(Y)$ we define,

$$r_{X,Y}(x)_V = x_{a(V)}|_V.$$

We define $r_{X,Y} : \mathcal{M}^{1,0}(X) \to \mathcal{M}^{1,0}(Y)$, in a similar manner. More precisely, for $x \in \mathcal{M}^{0,0}(X)$ and $V, V' \in \mathcal{C}(Y)$, we define

$$r_{X,Y}(x)_{V,V'} = x_{a(V),a(V')}|_{V \cap V'}.$$

The inductive step

In general the $\mathcal{M}^{p,q}(X)$ are defined as follows using induction on $\mathrm{level}(X)$ and with $n = \ell - \mathrm{level}(X) + 1$.

$$
\begin{array}{ll}
\mathcal{M}^{0,0}(X) = \bigoplus_{U_{\alpha_0} \in \mathcal{C}(X)} C^0(U_{\alpha_0}), & \\
\mathcal{M}^{0,q}(X) = 0, & 0 < q, \\
\mathcal{M}^{p,q}(X) = \bigoplus_{\alpha_0 < \cdots < \alpha_p,\ U_{\alpha_i} \in \mathcal{C}(X)} \mathrm{Tot}^q(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p})), & 0 < p,\ 0 < p+q \leq \\
\mathcal{M}^{p,q}(X) = 0, & \text{else.}
\end{array}
$$

Diagrammatically

$$
\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \bigoplus_{\alpha_0<\alpha_1} \mathrm{Tot}^{n-1}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1})) & \xrightarrow{\delta} & 0 & & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \bigoplus_{\alpha_0<\alpha_1} \mathrm{Tot}^{n-2}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1})) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} \mathrm{Tot}^{n-2}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1,\alpha_2})) & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & \longrightarrow & \bigoplus_{\alpha_0<\alpha_1} \mathrm{Tot}^{2}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1})) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} \mathrm{Tot}^{2}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1,\alpha_2})) & & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \bigoplus_{\alpha_0<\alpha_1} \mathrm{Tot}^{1}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1})) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} \mathrm{Tot}^{1}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1,\alpha_2})) & & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
\bigoplus_{\alpha_0\in\mathcal{C}_X} C^0(U_{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1} \mathrm{Tot}^{0}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1})) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} \mathrm{Tot}^{0}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\alpha_1,\alpha_2})) & \cdots & \bigoplus_{\alpha_0<\cdots<\alpha_n} \mathrm{Tot}^{0}(
\end{array}
$$

Key proposition

Proposition

For each admissible subset $X \subset S$ the double complex $\mathcal{M}^{\bullet,\bullet}(X)$ satisfies the following properties:

1. $H^i(\mathrm{Tot}^{\bullet}(\mathcal{M}^{\bullet,\bullet}(X)), \mathbb{Q}) \cong H^i(X, \mathbb{Q})$ *for* $0 \leq i \leq \ell - \mathrm{level}(X)$.
2. *For every admissible set Y, such that X is an ancestor of Y, and* $\mathrm{level}(X) = \mathrm{level}(Y)$, *the homomorphism,* $r_{X,Y} : \mathcal{M}^{\bullet,\bullet}(X) \to \mathcal{M}^{\bullet,\bullet}(Y)$ *induces the restriction homomorphisms between the cohomology groups:*

$$r : H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$$

for $0 \leq i \leq \ell - \mathrm{level}(X)$ *via the isomorphisms in (1).*

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2. *For every admissible set Y, such that X is an ancestor of Y, and $\mathrm{level}(X) = \mathrm{level}(Y)$, the homomorphism, $r_{X,Y} : \mathcal{M}^{\bullet,\bullet}(X) \to \mathcal{M}^{\bullet,\bullet}(Y)$ induces the restriction homomorphisms between the cohomology groups:*

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for $0 \leq i \leq \ell - \mathrm{level}(X)$ *via the isomorphisms in (1).*

Proof Idea

Proof.

The proof is by induction on $\mathrm{level}(X)$. If $\mathrm{level}(X) = \ell$ then the proposition is clear. Otherwise, by induction we can assume that the proposition is true for all admissilbe sets of the form, $U_{\alpha_0,\ldots,\alpha_p}$ with $U_{\alpha_i} \in \mathcal{C}(X)$. Thus, the p-th column of the complex, $\mathcal{M}^{\bullet,\bullet}(X)$, is the direct sum of the complexes,

$$\begin{array}{c}
\mathrm{Tot}^{n-p}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p})) \\
\uparrow \\
\vdots \\
\mathrm{Tot}^{1}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p})) \\
\uparrow \\
\mathrm{Tot}^{0}(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p}))
\end{array}$$

Proof (cont.)

Proof.

By induction hypothesis, $H^i(\mathrm{Tot}^\bullet(\mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p}))) \cong H^i(U_{\alpha_0,\ldots,\alpha_p})$. Moreover, the homomorphism, $r_{U_{\alpha_0,\ldots,\alpha_p},U_{\alpha_0,\ldots,\alpha_{p+1}}} : \mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_p}) \to \mathcal{M}^{\bullet,\bullet}(U_{\alpha_0,\ldots,\alpha_{p+1}})$, induces the restriction homomorphisms between the cohomology groups:

$$r : H^i(U_{\alpha_0,\ldots,\alpha_p}, \mathbb{Q}) \to H^i(U_{\alpha_0,\ldots,\alpha_{p+1}}, \mathbb{Q}).$$

The proposition follows from comparing the spectral sequence of $\mathcal{M}^{\bullet,\bullet}(X)$ with that of the truncated Mayer-Vietoris double complex associated to the covering, $\mathcal{C}(X)$, of X, which are isomorphic from the $'E_1$ term onwards. □