Efficient Algorithms for Computing Betti Numbers of Semi-algebraic Sets :4

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Outline

Basic Semi-algebraic Sets Defined By Quadratic Inequalities

- Let R be a real closed field and let $S \subset \mathrm{R}^k$ be a basic, closed semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$, with $\deg(P_i) \leq 2$.
- Such sets are in fact quite general, since every semi-algebraic set can be defined by (quantified) formulas involving only quadratic polynomials (at the cost of increasing the number of variables and the size of the formula). Moreover, as in the case of general semi-algebraic sets, the Betti numbers of such sets can be exponentially large. For example, the set $S \subset \mathrm{R}^k$ defined by $X_1(X_1 - 1) \geq 0, \ldots, X_k(X_k - 1) \geq 0$, has $b_0(S) = 2^k$.
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Bounds on Betti Numbers of Sets Defined by Quadratic Inequalities

Theorem (B. 2003)

Let ℓ be any fixed number and R a real closed field. Let $S \subset R^k$ be defined by $P_1 \geq 0, \ldots, P_s \geq 0$, with $\deg(P_i) \leq 2$. Then,

$$b_{k-\ell}(S) \leq \binom{s}{\ell} k^{O(\ell)}.$$

Features of the bound

- For fixed $\ell \geq 0$ this gives a polynomial bound on the highest ℓ Betti numbers of S (which could possibly be non-zero).
- Similar bounds do not hold for sets defined by polynomials of degree greater than two. For instance, the set defined by the single quartic equation, $\sum_{i=1}^{k} X_i^2(X_i - 1)^2 - \varepsilon = 0$, will have $b_{k-1} = 2^k$, for small enough $\varepsilon > 0$.

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Bounds on the Projection

Theorem (with T.Zell, 2005)

Let R *be a real closed field and let* $S \subset \mathrm{R}^{k+m}$ *be a bounded basic semi-algebraic set defined by* $P_1 \geq 0, \ldots, P_\ell \geq 0$, *with* $P_i \in \mathrm{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_m], \deg(P_i) \leq 2, \ 1 \leq i \leq \ell$. *Let* $\pi : \mathrm{R}^{k+m} \rightarrow \mathrm{R}^m$ *be the projection onto the last m coordinates. For any* $q > 0, \ 0 \leq q \leq k$,

$$\sum_{i=0}^{q} b_i(\pi(S)) \leq (k+m)^{O(q\ell)}.$$

Computing the Euler-Poincaré Characteristic

Theorem (B. 2005)

For any fixed $\ell > 0$, we have an algorithm which given a set of ℓ polynomials, $\mathcal{P} = \{P_1, \ldots, P_\ell\} \subset \mathrm{R}[X_1, \ldots, X_k]$, with $\deg(P_i) \leq 2, 1 \leq i \leq \ell$, computes the Euler-Poincaré characteristic, $\chi(S)$, where S is the set defined by $P_1 \leq 0, \ldots, P_\ell \leq 0$. The complexity of the algorithm is

$$k^{O(\ell)}.$$

Betti Numbers

- For any fixed $\ell > 0$, we have an algorithm which given a set of s polynomials, $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{R}[X_1, \ldots, X_k]$, with $\deg(P_i) \leq 2, 1 \leq i \leq s$, computes $b_{k-1}(S), \ldots, b_{k-\ell}(S)$, where S is the set defined by $P_1 \geq 0, \ldots, P_s \geq 0$. The complexity of the algorithm is $s^{\ell+2} k^{2^{O(\ell)}}$.
- (with T. Zell) For fixed ℓ and q, there exists an algorithm for computing the first q Betti numbers of $\pi(S)$ in the case where $S \subset \mathrm{R}^{k+m}$ is a bounded basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_\ell \geq 0$, with $P_i \in \mathrm{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_m], \deg(P_i) \leq 2,\ 1 \leq i \leq \ell$. The complexity of the algorithm is $(k+m)^{2^{O(q\ell)}}$.

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Outline

Inclusion-Exclusion Property

Let $\mathcal{P} = \{P_1, \ldots, P_\ell\} \subset \mathrm{R}[X_0, X_1, \ldots, X_k]$ be a set of homogeneous quadratic polynomials, and let S be the basic closed semi-algebraic set defined on the unit sphere, $S^k \subset \mathrm{R}^{k+1}$, by the inequalities,

$$P_1 \leq 0, \ldots, P_\ell \leq 0.$$

We denote by S_i the subset of S^k defined by $P_i \leq 0$. Then, $S = \cap_{i=1}^{\ell} S_i$. For $J \subset \{1, \ldots, \ell\}$, we denote by $S^J = \cup_{j \in J} S_j$.

$$\chi(S) = \sum_{J \subset \{1,\ldots,\ell\}} (-1)^{\#(J)+1} \chi(S^J).$$

Thus, in order to compute $\chi(S)$ it suffices to compute $\chi(S^J)$ for each $J \subset \{1, \ldots, \ell\}$. Note that each S^J is a union of the sets S_j for $j \in J$.

Topology of unions

- For quadratic forms $P_1, \ldots, P_s$, we denote by $P = (P_1, \ldots, P_s) : \mathrm{R}^{k+1} \rightarrow \mathrm{R}^s$, the map defined by the polynomials $P_1, \ldots, P_s$.
- Let $A = \cup_{P \in \mathcal{P}} \{x \in S^k \mid P(x) \leq 0\}$. and $\Omega = \{\omega \in R^s \mid |\omega| = 1, \omega_i \leq 0, 1 \leq i \leq s\}$.
- For $\omega \in \Omega$ let $\omega P = \sum_{i=1}^{s} \omega_i P_i$. and let $B = \{(\omega, x) \mid \omega \in \Omega, x \in S^k \text{ and } \omega P(x) \geq 0\}$.
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$$B \xrightarrow{\phi_1} \Omega, \qquad B \xrightarrow{\phi_2} S^k$$

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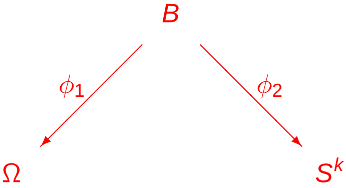
- 

Property of ϕ_2

Proposition (Agrachev)

The map ϕ_2 gives a homotopy equivalence between B and $\phi_2(B) = A$.

Proof

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We first prove that $\phi_2(B) = A$. If $x \in A$, then there exists some $i, 1 \leq i \leq s$, such that $P_i(x) \leq 0$. Then for $\omega = (-\delta_{1,i}, \ldots, -\delta_{s,i})$ (where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise), we see that $(\omega, x) \in B$. Conversely, if $x \in \phi_2(B)$, then there exists $\omega = (\omega_1, \ldots, \omega_s) \in \Omega$ such that, $\sum_{i=1}^{s} \omega_i P_i(x) \geq 0$. Since, $\omega_i \leq 0, 1 \leq i \leq s$, and not all $\omega_i = 0$, this implies that $P_i(x) \leq 0$ for some $i, 1 \leq i \leq s$. This shows that $x \in A$.

For $x \in \phi_2(B)$, the fibre $\phi_2^{-1}(x) = \{(\omega, x) \mid \omega \in \Omega \text{ such that } \omega P(x) \geq 0\}$, is a non-empty subset of Ω defined by a single linear inequality. From convexity considerations, all such fibres can clearly be retracted to their center of mass continuously. □

Property of ϕ_1

- We denote by $\Omega_j = \{\omega \in \Omega \mid \lambda_j(\omega P) \geq 0\}$, where $\lambda_j(\omega P)$ is the j-th eigenvalue of ωP.
- for $\omega \in \Omega_j \setminus \Omega_{j-1}$, the fiber $\phi_1^{-1}(\omega)$ is homotopy equivalent to a $(k-j)$-dimensional sphere.
- It follows that the Leray spectral sequence of the map ϕ_1 (converging to the cohomology $H^*(B) \cong H^*(A)$), has as its E_2 terms,

$$E_2^{pq} = H^p(\Omega_{k-q}, \Omega_{k-q-1}). \tag{1}$$

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Euler-Poincaré Characteristic of unions

- $\chi(A) = \chi^{BM}(A) = \sum_{j=0}^{k+1} \chi^{BM}(\Omega_j \setminus \Omega_{j-1})(1 + (-1)^{(k-j)})$.
- Let $Z = (Z_1, \ldots, Z_s)$ be variables and let $M(Z)$ be the symmetric matrix corresponding to the quadratic form $Z \cdot P = Z_1 P_1 + \cdots + Z_s P_s$. The entries of $M(Z)$ depend linearly on Z. Let,

  $$F(Z, T) = \det(M(Z) + T \cdot I_{k+1}) = T^{k+1} + C_k T^k + \cdots + C_0,$$

  where each $C_i \in \mathrm{R}[Z_1, \ldots, Z_s]$ is a polynomial of degree at most $k+1$. It follows from Descarte's rule of signs that for any $z \in \Omega$, $\mathrm{index}(zP)$ is equal to the number of sign variations in the sequence $C_0(z), \ldots, C_k(z), +1$.
- Additivity of the (Borel-Moore) Euler-Poincaré characteristic gives, $\chi(A) = \chi^{BM}(A) = \sum_{\sigma \in \mathrm{SIGN}(\mathcal{C}, \Omega)} \chi^{BM}(\mathcal{R}(\sigma, \Omega)) \cdot (1 + (-1)^{(k-n(\sigma))})$.

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Main Ideas

- Consider S as the intersection of the individual sets, S_i defined by $P_i \geq 0$.
- The top dimensional homology groups of S are isomorphic to those of the total complex associated to a suitable truncation of the Mayer-Vietoris double complex.
- The terms appearing in the truncated complex depend on the unions of the S_i's taken at most $\ell + 2$ at a time. There are at most $\sum_{j=1}^{\ell+2} \binom{s}{j} = O(s^{\ell+2})$ such sets.
- Moreover, for such semi-algebraic sets we are able to compute in polynomial (in k) time a complex, whose homology groups are isomorphic to those of the given sets.

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Generalized Mayer-Vietoris Exact Sequence

Proposition

Let $A = A_1 \cap \cdots \cap A_n$ and $A^{\alpha_0,\ldots,\alpha_p}$ denote the union, $A_{\alpha_0} \cup \cdots \cup A_{\alpha_p}$. The following sequence is exact.

$$0 \longrightarrow C_\bullet(A) \xrightarrow{i} \bigoplus_{\alpha_0} C_\bullet(A^{\alpha_0}) \xrightarrow{\delta} \bigoplus_{\alpha_0<\alpha_1} C_\bullet(A^{\alpha_0,\alpha_1}) \xrightarrow{\delta} \cdots$$

$$\xrightarrow{\delta} \bigoplus_{\alpha_0<\cdots<\alpha_p} C_\bullet(A^{\alpha_0,\ldots,\alpha_p}) \xrightarrow{\delta} \bigoplus_{\alpha_0<\cdots<\alpha_{p+1}} C_\bullet(A^{\alpha_0,\ldots,\alpha_{p+1}}) \xrightarrow{\delta} \cdots,$$

where i is induced by inclusion and the connecting homomorphisms δ are defined as follows:
for $c \in \oplus_{\alpha_0<\cdots<\alpha_p} C_\bullet(A^{\alpha_0,\ldots,\alpha_p})$,
$(\delta c)_{\alpha_0,\ldots,\alpha_{p+1}} = \sum_{0\leq i\leq p+1}(-1)^i c_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_{p+1}}$.

Mayer-Vietoris Double Complex

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longrightarrow & \bigoplus_{\alpha_0} C_k(A^{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1} C_k(A^{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} C_k(A^{\alpha_0,\alpha_1,\alpha_2}) \\
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 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longrightarrow & \bigoplus_{\alpha_0} C_{k-2}(A^{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1} C_{k-2}(A^{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} C_{k-2}(A^{\alpha_0,\alpha_1,\alpha_2}) \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longrightarrow & \bigoplus_{\alpha_0} C_{k-3}(A^{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1} C_{k-3}(A^{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} C_{k-3}(A^{\alpha_0,\alpha_1,\alpha_2}) \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
 & & \vdots & & \vdots & & \vdots
\end{array}
$$

The Associated Total Complex

- The i-th homology group of A, $H_i(A)$ is isomorphic to the i-th homology group of the associated total complex of the double complex described above.
- For $0 \leq i \leq k$,

$$H_i(A) \cong H^i(\mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet})).$$

Moreover, if we denote by $\mathcal{N}_\ell^{\bullet,\bullet}$ the truncated complex defined by,

$$\begin{aligned} \mathcal{N}_\ell^{p,q} &= \mathcal{N}^{p,q}, \quad 0 \leq p+k-q \leq \ell+1, \\ &= 0, \quad \text{otherwise}, \end{aligned}$$

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Computing a quasi-isomorphic complex

- We cannot hope to compute even the truncated complex $\mathcal{N}_\ell^{\bullet,\bullet}$ since we do not know how to compute triangulations efficiently.
- We overcome this problem by computing another double complex $\mathcal{D}_\ell^{\bullet,\bullet}$, such that there exists a homomorphism of double complexes,

$$\psi : \mathcal{D}_\ell^{\bullet,\bullet} \rightarrow \mathcal{N}_\ell^{\bullet,\bullet},$$

  which induces an isomorphism between the E_1 terms of the spectral sequences associated to the double complexes $\mathcal{D}_\ell^{\bullet,\bullet}$ and $\mathcal{N}_\ell^{\bullet,\bullet}$.
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Outline

Cohomological Descent

- Let, $X \subset \mathrm{R}^m$ and $Y \subset \mathrm{R}^n$ be open semi-algebraic sets, and let $f : X \to Y$ be a semi-algebraic, continuous surjection, which is also an open mapping (it takes open sets to open sets).
- We denote by $W_f^i(X)$ the $(i+1)$-fold fibered product of X over f, that is,
$W_f^i(X) = \{(x_0, \ldots, x_i) \in X^{i+1} \mid f(x_0) = \cdots = f(x_i)\}$.

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Descent Spectral Sequence

- We have an exact sequence analogous to the Mayer-Vietoris exact sequence.

$$0 \longrightarrow \bar{C}^{\bullet}(Y) \xrightarrow{f^*} \bar{C}^{\bullet}(W_f^0(X)) \xrightarrow{\delta^0} \bar{C}^{\bullet}(W_f^1(X)) \xrightarrow{\delta^1} \cdots \bar{C}^{\bullet}(W_f^p(X))$$

Idea behind the algorithm

- Notice that the fibered product of q sets each defined by ℓ quadatic inequalities in $k + m$ variables is defined by $q\ell$ quadratic inequalities in $qk + m$ variables.
- Using the polynomial time algorithm described previously for computing a complex whose cohomology groups are isomorphic to those of a given semi-algebraic set defined by a constant number of quadratic inequalities,we are able to construct a certain double complex, whose associated total complex is quasi-isomorphic to (implying having isomorphic homology groups) a suitable truncation of the one obtained from the cohomological descent spectral sequence mentioned above. This complex is of much smaller size and can be computed in polynomial time.

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Open Problems

- Single exponential algorithm for computing triangulation (or even stratification) of semi-algebraic sets ?
- Single exponential time algorithm for computing all the Betti numbers of semi-algebraic sets ?
- Algorithm for deciding connectivity of a given algebraic set whose complexity is $d^{O(k)}$ (instead of $d^{O(k^2)}$) ?
- Algorithm for deciding connectivity of semi-algebraic sets defined by ℓ quadratic inequalities in time $k^{O(\ell)}$ (instead of $k^{2^{O(\ell)}}$)?

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