Combinatorial Complexity in O-minimal Geometry

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Outline

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Semi-algebraic Sets and their Betti numbers

- Let $S \subset \mathbb{R}^k$ be defined by a Boolean formula whose atoms consists of $P > 0, P = 0, P < 0,\ P \in \mathcal{P}$, where $\mathcal{P}$ is a set of polynomials of degrees bounded by a parameter and $\#\mathcal{P} = n$.
- $$\sum_{i \geq 0} b_i(S) \leq n^{2k} O(d)^k.$$
- Bound for sign conditions:
$$\sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}}} b_i(\mathcal{R}(\sigma)) \leq \sum_{j=0}^{k-i} \binom{n}{j} 4^j d(2d-1)^{k-1} = n^{k-i} O(d)^k.$$

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Combinatorial Complexity

- Notice that the bounds in the previous page are products of two quantities – one that depends only on n (and k), and another part which is independent of n. We refer to the first part as the combinatorial part of the complexity, and the latter as the algebraic part.
- While understanding the algebraic part of the complexity is a very important problem, in several applications, most notably in discrete and computational geometry, it is the combinatorial part of the complexity that is of interest (the algebraic part is assumed to be bounded by a constant).

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- While understanding the algebraic part of the complexity is a very important problem, in several applications, most notably in discrete and computational geometry, it is the combinatorial part of the complexity that is of interest (the algebraic part is assumed to be bounded by a constant).

Definition of Arrangements

- Let $\mathcal{A} = \{S_1, \ldots, S_n\}$, with each S_i belonging to some "simple" class of sets.
- For $I \subset \{1, \ldots, n\}$, let $\mathcal{A}(I)$denote the set

$$\bigcap_{i \in I \subset [1 \ldots n]} S_i \cap \bigcap_{j \in [1 \ldots n] \setminus I} \mathbb{R}^k \setminus S_j,$$

  and it is customary to call a connected component of $\mathcal{A}_I$ a cell of the arrangement $\mathcal{A}$ and we denote by $\mathcal{C}(\mathcal{A})$ the set of all non-empty cells of the arrangement $\mathcal{A}$.
- The cardinality of $\mathcal{C}(\mathcal{A})$ is called the combinatorial complexity of the arrangement $\mathcal{A}$.

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Objects of Bounded Description Complexity

- The class of sets usually considered in the study of arrangements are sets with "bounded description complexity". This means that each set in the arrangement is defined by a first order formula in the language of ordered fields involving at most a constant number polynomials whose degrees are also bounded by a constant.
- Additionally, there is often a requirement that the sets be in "general position". The precise definition of "general position" varies with context, but often involves restrictions such as: the sets in the arrangements are smooth manifolds, intersecting transversally.

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Definition of O-minimal Structures

- The theory of o-minimal structures was developed by Van den Dries and others in part to show that the tame topological properties of semi-algebraic sets are consequences of few simple axioms.
- An o-minimal structure on a real closed field R is a sequence $\mathcal{S}(\mathrm{R}) = (\mathcal{S}_n)_{n \in \mathbb{N}}$.
  1. All algebraic subsets of R^n are in $\mathcal{S}_n$.
  2. The class $\mathcal{S}_n$ is closed under complementation and finite unions and intersections.
  3. If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$ then $A \times B \in \mathcal{S}_{m+n}$.
  4. If $\pi : \mathrm{R}^{n+1} \to \mathrm{R}^n$ is the projection map on the first n co-ordinates and $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$.
  5. The elements of $\mathcal{S}_1$ are precisely finite unions of points and intervals.

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Examples of O-minimal Structures I

- Our first example of an o-minimal structure $\mathcal{S}(\mathrm{R})$, is the o-minimal structure over a real closed field R where each $\mathcal{S}_n$ is exactly the class of semi-algebraic subsets of R^n.
- Let $\mathcal{S}_n$ be the images in $\mathbb{R}^n$ under the projection maps $\mathbb{R}^{n+k} \to \mathbb{R}^n$ of sets of the form $\{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+k} \mid P(\mathbf{x}, \mathbf{y}, e^{\mathbf{x}}, e^{\mathbf{y}}) = 0\}$, where P is a real polynomial in $2(n+k)$ variables, and $e^{\mathbf{x}} = (e^{x_1}, \ldots, e^{x_n})$ and $e^{\mathbf{y}} = (e^{y_1}, \ldots, e^{y_k})$. We will denote this o-minimal structure over $\mathbb{R}$ by $\mathcal{S}_{\exp}(\mathbb{R})$.

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Examples of O-minimal Structures II

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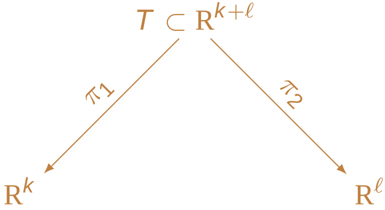
  (A restricted analytic function in N variables is an analytic function defined on an open neighborhood of $[0,1]^N$ restricted to $[0,1]^N$ (and extended by 0 outside)).
  We will denote this o-minimal structure over $\mathbb{R}$ by $\mathcal{S}_{\mathrm{ana}}(\mathbb{R})$.

- Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure on a real closed field R and let $T \subset \mathrm{R}^{k+\ell}$ be a fixed definable set.



- We will call S of R^k to be a (T, π_1, π_2)-set if

$$S = T_{\mathbf{y}} = \pi_1(\pi_2^{-1}(\mathbf{y}) \cap T)$$

for some $\mathbf{y} \in \mathrm{R}^\ell$.

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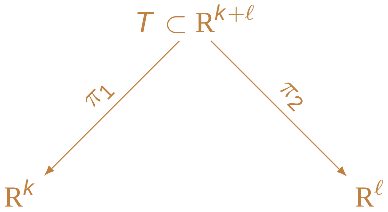

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Example I

Let $\mathcal{S}(\mathrm{R}) = \mathcal{S}_{\mathrm{sa}}(\mathrm{R})$ and Let $T \subset \mathrm{R}^{2k+1}$ be the semi-algebraic set defined by

$$T = \{(x_1, \ldots, x_k, a_1, \ldots, a_k, b) \mid \langle \mathbf{a}, \mathbf{x} \rangle - b = 0\}$$

(where we denote $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{x} = (x_1, \ldots, x_k)$), and π_1 and π_2 are the projections onto the first k and last $k+1$ co-ordinates respectively. A (T, π_1, π_2)-set is clearly a hyperplane in R^k and vice versa.

Example II

Let $\mathcal{S}(\mathbb{R}) = \mathcal{S}_{\exp}(\mathbb{R})$ and

$$T = \{(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_m, a_1, \ldots, a_m) \mid \mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_m \in \mathbb{R}^k, \\ a_1, \ldots, a_m \in \mathbb{R}, x_1, \ldots, x_k > 0, \sum_{i=0}^{m} a_i \mathbf{x}^{\mathbf{y}_i} = 0\},$$

with $\pi_1 : \mathbb{R}^{k+m(k+1)} \to \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^{k+m(k+1)} \to \mathbb{R}^{m(k+1)}$ be the projections onto the first k and the last $m(k+1)$ co-ordinates respectively. The (T, π_1, π_2)-sets in this example include (amongst others) all semi-algebraic sets consisting of intersections with the positive orthant of all real algebraic sets defined by a polynomial having at most m monomials (different sets of monomials are allowed to occur in different polynomials).

Outline

$\mathcal{A}$-sets I

Let $\mathcal{A} = \{S_1, \ldots, S_n\}$, such that each $S_i \subset \mathrm{R}^k$ is a (T, π_1, π_2)-set. For $I \subset \{1, \ldots, n\}$, we let $\mathcal{A}(I)$ denote the set

$$\bigcap_{i \in I \subset [1 \ldots n]} S_i \quad \cap \bigcap_{j \in [1 \ldots n] \setminus I} \mathrm{R}^k \setminus S_j, \tag{1}$$

and we will call such a set to be a basic $\mathcal{A}$-set. We will denote by, $\mathcal{C}(\mathcal{A})$, the set of non-empty connected components of all basic $\mathcal{A}$-sets.

$\mathcal{A}$-sets II

We will call definable subsets $S \subset \mathrm{R}^k$ defined by a Boolean formula whose atoms are of the form, $x \in S_i, 1 \leq i \leq n$, a $\mathcal{A}$-set. A $\mathcal{A}$-set is thus a union of basic $\mathcal{A}$-sets. If T is closed, and the Boolean formula defining S has no negations, then S is closed by definition (since each S_i is closed) and we call such a set an $\mathcal{A}$-closed set.
Moreover, if V is any closed definable subset of R^k, and S is an $\mathcal{A}$-set (resp. $\mathcal{A}$-closed set), then we will call $S \cap V$ to be an $(\mathcal{A}, V)$-set (resp. $(\mathcal{A}, V)$-closed set).

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   - Combinatorial and Algebraic Complexity
2. Arrangements
3. O-minimal Structures
   - Examples of O-minimal Structures
4. Admissible Sets
   - Examples of Admissible Sets
   - $\mathcal{A}$-sets
5. Results
   - Bounds on Betti Numbers
   - Cylindrical Definable Cell Decomposition
   - Application: Generalization of a Theorem due to Alon et al.

Bounds on Betti Numbers I

Theorem

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exists a constant $C = C(T) > 0$ depending only on T, such that for any (T, π_1, π_2)-family $\mathcal{A} = \{S_1, \ldots, S_n\}$ of subsets of R^k the following holds. For every $i, 0 \leq i \leq k$,

$$\sum_{D \in \mathcal{C}(\mathcal{A})} b_i(D) \leq C \cdot n^{k-i}.$$

In particular, the combinatorial complexity of $\mathcal{A}$, is at most $C \cdot n^k$. The topological complexity of any m cells in the arrangement $\mathcal{A}$ is bounded by $m + C \cdot n^{k-1}$.

Lower dimensional

Theorem

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R and let $T \subset \mathrm{R}^{k+\ell}$, $V \subset \mathrm{R}^k$ be closed definable sets with $\dim(V) = k'$. Then, there exists a constant $C = C(T, V) > 0$ depending only on T and V, such that for any (T, π_1, π_2)-family, $\mathcal{A} = \{S_1, \ldots, S_n\}$, of subsets of R^k, and for every $i, 0 \leq i \leq k'$,

$$\sum_{D \in \mathcal{C}(\mathcal{A}, V)} b_i(D) \leq C \cdot n^{k'-i}.$$

In particular, the combinatorial complexity of $\mathcal{A}$ restricted to V, is bounded by $C \cdot n^{k'}$.

Topological Complexity of Projections

Theorem (Topological Complexity of Projections)

Let $\mathcal{S}(\mathbb{R})$ be an o-minimal structure, and let $T \subset \mathbb{R}^{k+\ell}$ be a definable, closed and bounded set. Let $k = k_1 + k_2$ and let $\pi_3 : \mathbb{R}^k \to \mathbb{R}^{k_2}$ denote the projection map on the last k_2 co-ordinates.
Then, there exists a constant $C = C(T) > 0$ such that for any (T, π_1, π_2)-family, $\mathcal{A}$, with $|\mathcal{A}| = n$, and an $\mathcal{A}$-closed set $S \subset \mathbb{R}^k$,

$$\sum_{i=0}^{k_2} b_i(\pi_3(S)) \leq C \cdot n^{(k_1+1)k_2}.$$

Outline

Definition I

A cdcd of R^k is a finite partition of R^k into definable sets $(C_i)_{i\in I}$ (called the cells of the cdcd) satisfying the following properties.
If $k = 1$ then a cdcd of R is given by a finite set of points $a_1 < \cdots < a_N$ and the cells of the cdcd are the singletons $\{a_i\}$ as well as the open intervals, $(\infty, a_1), (a_1, a_2), \ldots, (a_N, \infty)$.
If $k > 1$, then a cdcd of R^k is given by a cdcd, $(C_i')_{i\in I'}$, of R^{k-1} and for each $i \in I'$, a collection of cells, $\mathcal{C}_i$ defined by,

$$\mathcal{C}_i = \{\phi_i(C_i' \times D_j) \mid j \in J_i\},$$

Definition II

where

$$\phi_i : C'_i \times \mathrm{R} \to \mathrm{R}^k$$

is a definable homemorphism satisfying $\pi \circ \phi = \pi$, $(D_j)_{j \in J_i}$ is a cdcd of R, and $\pi : \mathrm{R}^k \to \mathrm{R}^{k-1}$ is the projection map onto the first $k - 1$ coordinates. The cdcd of R^k is then given by

$$\bigcup_{i \in I'} \mathcal{C}_i.$$

Given a family of definable subsets $\mathcal{A} = \{S_1, \ldots, S_n\}$ of R^k, we say that a cdcd is adapted to $\mathcal{A}$, if each S_i is a union of cells of the given cdcd.

Quantitative cylindrical definable cell decomposition I

Theorem (Quantitative cylindrical definable cell decomposition)

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R, and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exist constants $C_1, C_2 > 0$ depending only on T, and definable sets,

$$\{T_i\}_{i\in I}, \quad T_i \subset \mathrm{R}^k \times \mathrm{R}^{2(2^k-1)\cdot\ell},$$

depending only on T, with $|I| \leq C_1$, such that for any (T, π_1, π_2)-family, $\mathcal{A} = \{S_1, \ldots, S_n\}$ with $S_i = T_{\mathbf{y}_i}, \mathbf{y}_i \in \mathrm{R}^\ell, 1 \leq i \leq n$, some sub-collection of the sets

Quantitative cylindrical definable cell decomposition II

Theorem (Quantitative cylindrical definable cell decomposition)

$$\pi^{\leq k}_{k+2(2^k-1)\cdot\ell}\left(\pi^{>k}_{k+2(2^k-1)\cdot\ell}{}^{-1}(\mathbf{y}_{i_1},\ldots,\mathbf{y}_{i_{2(2^k-1)}})\cap T_i\right),$$
$$i \in I,\ 1 \leq i_1,\ldots,i_{2(2^k-1)} \leq n,$$

form a cdcd of R^k *compatible with* $\mathcal{A}$. *Moreover, the cdcd has at most* $C_2 \cdot n^{2(2^k-1)}$ *cells.*

Outline

Ramsey type theorem

Theorem

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R, and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exists a constant $1 > \varepsilon = \varepsilon(T) > 0$ depending only on T, such that for any (T, π_1, π_2)-family, $\mathcal{A} = \{S_1, \ldots, S_n\}$, there exists two subfamilies $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$, with $|\mathcal{A}_1|, |\mathcal{A}_2| \geq \varepsilon n$, and either,

- *for all $S_i \in \mathcal{A}_1$ and $S_j \in \mathcal{A}_2$, $S_i \cap S_j \neq \emptyset$ or*
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Theorem

Let $\mathcal{S}(\mathrm{R})$ be an o-minimal structure over a real closed field R, and let $T \subset \mathrm{R}^{k+\ell}$ be a closed definable set. Then, there exists a constant $1 > \varepsilon = \varepsilon(T) > 0$ depending only on T, such that for any (T, π_1, π_2)-family, $\mathcal{A} = \{S_1, \ldots, S_n\}$, there exists two subfamilies $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$, with $|\mathcal{A}_1|, |\mathcal{A}_2| \geq \varepsilon n$, and either,

- *for all $S_i \in \mathcal{A}_1$ and $S_j \in \mathcal{A}_2$, $S_i \cap S_j \neq \emptyset$ or*
- *for all $S_i \in \mathcal{A}_1$ and $S_j \in \mathcal{A}_2$, $S_i \cap S_j = \emptyset$.*