Efficient Algorithms for Computing Betti Numbers of Semi-algebraic Sets

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Outline

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Semi-algebraic Sets and their Betti numbers

- The basic objects of real algebraic geometry are semi-algebraic sets. Subsets of R^k defined by a formula involving a finite number of polynomial equalities and inequalities. A basic semi-algebraic set is one defined by a conjunction of weak inequalities of the form $P \geq 0$.
- We will denote by $b_i(S)$ the i-th Betti number of S.
- The sum of the Betti numbers of S is bounded by $(sd)^{O(k)}$, where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.
- *Even though the Betti numbers are bounded singly exponentially in k, there is no known algorithm with single exponential complexity for computing them.*

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Some Motivations

- Studying topology of real algebraic varieties is an important mathematical problem.
- Semi-algebraic sets occur as configuration spaces in applications. Computing topological information of such spaces is important.
- Studying certain questions in quantitative real algebraic geometry. For instance, existence of single exponential sized triangulations.
- Recent work in complexity theory on the real version of counting complexity classes.
- Some ideas may be useful in designing algorithms for computing homology groups in other contexts.

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Previous Work

- Doubly exponential algorithms (with complexity $(sd)^{2^{O(k)}}$) for computing all the Betti numbers are known, since it is possible to obtain a triangulation of S in doubly exponential time using cylindrical algebraic decomposition (Collins,Schwartz-Sharir).
- Algorithms with single exponential complexity are known only for the problems of testing emptiness, computing the zero-th Betti number (i.e. the number of semi-algebraically connected components of S) the Euler-Poincaré characteristic of S, as well as the dimension of S.

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Summary

- (with Pollack, Roy) There exists an algorithm that takes as input the description of a $\mathcal{P}$-semi-algebraic set $S \subset \mathrm{R}^k$, and outputs $b_1(S)$. The complexity of the algorithm is

$$(sd)^{k^{O(1)}}$$

where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

- For any fixed $\ell > 0$, we can compute $b_0(S), \ldots, b_\ell(S)$ with complexity

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Basic Semi-algebraic Sets Defined By Quadratic Inequalities

- Let R be a real closed field and let $S \subset \mathrm{R}^k$ be a basic, closed semi-algebraic set defined by $P_1 \geq 0, \ldots, P_s \geq 0$, with $\deg(P_i) \leq 2$.
- Such sets are in fact quite general, since every semi-algebraic set can be defined by (quantified) formulas involving only quadratic polynomials (at the cost of increasing the number of variables and the size of the formula). Moreover, as in the case of general semi-algebraic sets, the Betti numbers of such sets can be exponentially large. For example, the set $S \subset \mathrm{R}^k$ defined by $X_1(X_1 - 1) \geq 0, \ldots, X_k(X_k - 1) \geq 0,$ has $b_0(S) = 2^k$.
- It is NP-hard to decide whether such a set is empty

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Bounds on Betti Numbers of Sets Defined by Quadratic Inequalities

Theorem (B. 2003)

Let ℓ be any fixed number and R a real closed field. Let $S \subset R^k$ be defined by $P_1 \geq 0, \ldots, P_s \geq 0$, with $\deg(P_i) \leq 2$. Then,

$$b_{k-\ell}(S) \leq \binom{s}{\ell} k^{O(\ell)}.$$

Features of the bound

- For fixed $\ell \geq 0$ this gives a polynomial bound on the highest ℓ Betti numbers of S (which could possibly be non-zero).
- Similar bounds do not hold for sets defined by polynomials of degree greater than two. For instance, the set defined by the single quartic equation, $\sum_{i=1}^{k} X_i^2(X_i - 1)^2 - \varepsilon = 0$, will have $b_{k-1} = 2^k$, for small enough $\varepsilon > 0$.

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Bounds on the Projection

Theorem (with T.Zell, 2005)

Let R *be a real closed field and let* $S \subset \mathrm{R}^{k+m}$ *be a bounded basic semi-algebraic set defined by* $P_1 \geq 0, \ldots, P_\ell \geq 0$, *with* $P_i \in \mathrm{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_m], \deg(P_i) \leq 2,\ 1 \leq i \leq \ell$. *Let* $\pi : \mathrm{R}^{k+m} \to \mathrm{R}^m$ *be the projection onto the last* m *coordinates. For any* $q > 0,\ 0 \leq q \leq k$,

$$\sum_{i=0}^{q} b_i(\pi(S)) \leq (k+m)^{O(q\ell)}.$$

Main Results in the Quadratic Case

- For any fixed $\ell > 0$, we have an algorithm which given a set of s polynomials, $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{R}[X_1, \ldots, X_k]$, with $\deg(P_i) \leq 2, 1 \leq i \leq s$, computes $b_{k-1}(S), \ldots, b_{k-\ell}(S)$, where S is the set defined by $P_1 \geq 0, \ldots, P_s \geq 0$. The complexity of the algorithm is $s^{\ell+2} k^{2^{O(\ell)}}$.
- (with Zell) For fixed ℓ and q, there exists an algorithm for computing the first q Betti numbers of $\pi(S)$ in the case where $S \subset \mathrm{R}^{k+m}$ is a bounded basic semi-algebraic set defined by $P_1 \geq 0, \ldots, P_\ell \geq 0$, with $P_i \in \mathrm{R}[X_1, \ldots, X_k, Y_1, \ldots, Y_m], \deg(P_i) \leq 2,\ 1 \leq i \leq \ell$. The complexity of the algorithm is $(k+m)^{2^{O(q\ell)}}$.

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Main Steps in the General Case

- First reduce to the closed and bounded case using a recent construction of Gabrielov and Vorobjov.
- Compute using a parametrized version of the connecting algorithm a covering of the given closed and bounded semi-algebraic set by closed and bounded sets which are moreover contractible.
- Using the Roadmap Algorithm compute the connected components of the pairwise and triple-wise intersections of the elements of the covering and their incidences.

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Result from Algebraic Topology

Proposition

Let $A_1, \ldots, A_n$ be sub-complexes of a finite simplicial complex A such that $A = A_1 \cup \cdots \cup A_n$ and each A_i is acyclic, that is $H^0(A_i) = \mathbb{Q}$ and $H^q(A_i) = 0$ for all $q > 0$. Then,
$b_1(A) = \dim(\mathrm{Ker}(\delta_2)) - \dim(\mathrm{Im}(\delta_1))$, *with*

$$\prod_i H^0(A_i) \xrightarrow{\delta_1} \prod_{i<j} H^0(A_{i,j}) \xrightarrow{\delta_2} \prod_{i<j<\ell} H^0(A_{i,j,\ell})$$

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Connecting paths

- Given a semi-algebraic set $S \subset \mathrm{R}^k$, and two points $x, y \in S$, there exists an algorithm (Roadmap) with single exponential complexity which can decide whether x and y are in the same connected component of S and if so output a semi-algebraic path connecting in S connecting x to y.
- Fix a finite set of distinguised points in every connected component of S and for $x \in S$, let $\gamma(x)$ denote the connecting path computed by the algorithm connecting x to a distinguished point.

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Important Property of Connecting Path

- The connecting path $\gamma(x)$ consists of two consecutive parts, $\gamma_0(x)$ and $\Gamma_1(x)$. The path $\gamma_0(x)$ is contained in $\mathrm{RM}(S)$ and the path $\Gamma_1(x)$ is contained in S_{x_1}.
- Moreover, $\Gamma_1(x)$ can again be decomposed into two parts, $\gamma_1(x)$ and $\Gamma_2(x)$ with $\Gamma_2(x)$ contained in $S_{\bar{x}_2}$ and so on.
- If $y = (y_1, \ldots, y_k) \in S$ is another point such that $x_1 \neq y_1$, then the images of $\Gamma_1(x)$ and $\Gamma_1(y)$ are disjoint. If the image of $\gamma_0(y)$ (which is contained in S) follows the same sequence of curve segments as $\gamma_0(x)$ starting at p, then it is clear that the images of the paths $\gamma(x)$ and $\gamma(y)$ has the property that they are identical upto a point and they are disjoint after it.

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Parametrized Paths

Definition

A parametrized path γ is a continuous semi-algebraic mapping from $V \subset \mathrm{R}^{k+1} \to \mathrm{R}^k$, a semi-algebraic continuous function $\ell : U \to [0, +\infty)$, with $U = \pi_{1\ldots k}(V) \subset \mathrm{R}^k$, and a in R^k, such that

1. $V = \{(x, t) \mid x \in U, 0 \le t \le \ell(x)\}$,
2. $\forall x \in U, \ \gamma(x, 0) = a$,
3. $\forall x \in U, \ \gamma(x, \ell(x)) = x$,
4. $\forall x \in U, \forall y \in U, \forall s \in [0, \ell(x)], \forall t \in [0, \ell(y)]$ $(\gamma(x, s) = \gamma(y, t) \Rightarrow s = t)$,
5. $\forall x \in U, \forall y \in U, \forall s \in [0, \min(\ell(x), \ell(y))]$ $(\gamma(x, s) = \gamma(y, s) \Rightarrow \forall t \le s, \ \gamma(x, t) = \gamma(y, t))$.

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Useful property of Parametrized Paths

Proposition

Let $\gamma : V \to \mathrm{R}^k$ be a parametrized path such that $U = \pi_{1\ldots k}(V)$ is closed and bounded. Then, the image of γ is semi-algebraically contractible.

Computing Parametrized Paths

- Given a closed and bounded semi-algebraic set $S \subset \mathrm{R}^k$, there exists an algorithm which outputs,
- a finite set of t polynomials $\mathcal{A} \subset \mathrm{R}[X_1, \ldots, X_k]$,
- for every $\sigma \in \mathrm{Sign}(\mathcal{A}, S)$, a parametrized path $\gamma_\sigma : V_\sigma \to \mathrm{R}^k$, with base $U_\sigma = \mathcal{R}(\sigma)$, such that for each $y \in \mathcal{R}(\sigma)$, $\mathrm{Im}\ \gamma_\sigma(y, \cdot)$ is a semi-algebraic path which connects the point y to a distinguished point a_σ.
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Ensuring Contractibility

For each $\sigma \in \mathrm{Sign}(\mathcal{A}, S)$, since $\mathcal{R}(\sigma, S)$ is not necessarily closed and bounded, $\mathrm{Im}\ \gamma_\sigma$ might not be contractible. In order to ensure contractibility, we restrict the base of γ_σ to a slightly smaller set which is closed, using infinitesimals.

Ensuring the Covering Property

- The images of the parametrized paths obtained after shrinking their bases do not necessarily cover S.
- We enlarge them, preserving contractibility, to recover a covering of S.
- It is necessary to use $2t$ infinitesimals in the shrinking and enlargement process to work correctly.

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Outline

Main Ideas

- Consider S as the intersection of the individual sets, S_i defined by $P_i \geq 0$.
- The top dimensional homology groups of S are isomorphic to those of the total complex associated to a suitable truncation of the Mayer-Vietoris double complex.
- The terms appearing in the truncated complex depend on the unions of the S_i's taken at most $\ell + 2$ at a time. There are at most $\sum_{j=1}^{\ell+2} \binom{s}{j} = O(s^{\ell+2})$ such sets.
- Moreover, for such semi-algebraic sets we are able to compute in polynomial (in k) time a complex, whose homology groups are isomorphic to those of the given sets.

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Outline

Generalized Mayer-Vietoris Exact Sequence

Proposition

Let $A = A_1 \cap \cdots \cap A_n$ and $A^{\alpha_0,\ldots,\alpha_p}$ denote the union, $A_{\alpha_0} \cup \cdots \cup A_{\alpha_p}$. The following sequence is exact.

$$0 \longrightarrow C_\bullet(A) \xrightarrow{i} \bigoplus_{\alpha_0} C_\bullet(A^{\alpha_0}) \xrightarrow{\delta} \bigoplus_{\alpha_0<\alpha_1} C_\bullet(A^{\alpha_0,\alpha_1}) \xrightarrow{\delta} \cdots$$

$$\xrightarrow{\delta} \bigoplus_{\alpha_0<\cdots<\alpha_p} C_\bullet(A^{\alpha_0,\ldots,\alpha_p}) \xrightarrow{\delta} \bigoplus_{\alpha_0<\cdots<\alpha_{p+1}} C_\bullet(A^{\alpha_0,\ldots,\alpha_{p+1}}) \xrightarrow{\delta} \cdots,$$

where i is induced by inclusion and the connecting homomorphisms δ are defined as follows: for $c \in \oplus_{\alpha_0<\cdots<\alpha_p} C_\bullet(A^{\alpha_0,\ldots,\alpha_p})$, $(\delta c)_{\alpha_0,\ldots,\alpha_{p+1}} = \sum_{0\le i\le p+1} (-1)^i c_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_{p+1}}$.

Mayer-Vietoris Double Complex

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longrightarrow & \bigoplus_{\alpha_0} C_k(A^{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1} C_k(A^{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} C_k(A^{\alpha_0,\alpha_1,\alpha_2}) \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longrightarrow & \bigoplus_{\alpha_0} C_{k-1}(A^{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1} C_{k-1}(A^{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} C_{k-1}(A^{\alpha_0,\alpha_1,\alpha_2}) \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longrightarrow & \bigoplus_{\alpha_0} C_{k-2}(A^{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1} C_{k-2}(A^{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} C_{k-2}(A^{\alpha_0,\alpha_1,\alpha_2}) \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
0 & \longrightarrow & \bigoplus_{\alpha_0} C_{k-3}(A^{\alpha_0}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1} C_{k-3}(A^{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \bigoplus_{\alpha_0<\alpha_1<\alpha_2} C_{k-3}(A^{\alpha_0,\alpha_1,\alpha_2}) \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
 & & \vdots & & \vdots & & \vdots
\end{array}
$$

The Associated Total Complex

- The i-th homology group of A, $H_i(A)$ is isomorphic to the i-th homology group of the associated total complex of the double complex described above.
- For $0 \leq i \leq k$,

$$H_i(A) \cong H^i(\mathrm{Tot}^\bullet(\mathcal{N}^{\bullet,\bullet})).$$

Moreover, if we denote by $\mathcal{N}_\ell^{\bullet,\bullet}$ the truncated complex defined by,

$$\begin{aligned} \mathcal{N}_\ell^{p,q} &= \mathcal{N}^{p,q}, \quad 0 \leq p+k-q \leq \ell+1, \\ &= 0, \quad \text{otherwise}, \end{aligned}$$

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Computing a quasi-isomorphic complex

- We cannot hope to compute even the truncated complex $\mathcal{N}_\ell^{\bullet,\bullet}$ since we do not know how to compute triangulations efficiently.
- We overcome this problem by computing another double complex $\mathcal{D}_\ell^{\bullet,\bullet}$, such that there exists a homomorphism of double complexes,

$$\psi : \mathcal{D}_\ell^{\bullet,\bullet} \to \mathcal{N}_\ell^{\bullet,\bullet},$$

  which induces an isomorphism between the E_1 terms of the spectral sequences associated to the double complexes $\mathcal{D}_\ell^{\bullet,\bullet}$ and $\mathcal{N}_\ell^{\bullet,\bullet}$.
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Topology of unions

- For quadratic forms $P_1, \ldots, P_s$, we denote by $P = (P_1, \ldots, P_s) : \mathrm{R}^{k+1} \to \mathrm{R}^s$, the map defined by the polynomials $P_1, \ldots, P_s$.
- Let $A = \cup_{P \in \mathcal{P}} \{x \in S^k \mid P(x) \leq 0\}$. and $\Omega = \{\omega \in R^s \mid |\omega| = 1, \omega_i \leq 0, 1 \leq i \leq s\}$.
- For $\omega \in \Omega$ let $\omega P = \sum_{i=1}^{s} \omega_i P_i$. and let $B = \{(\omega, x) \mid \omega \in \Omega, x \in S^k$ and $\omega P(x) \geq 0\}$.
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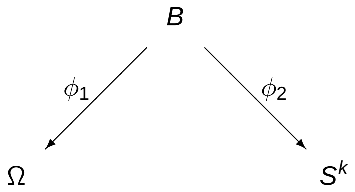
- 

Property of ϕ_2

Proposition (Agrachev)

The map ϕ_2 gives a homotopy equivalence between B and $\phi_2(B) = A$.

Proof

Proof.

We first prove that $\phi_2(B) = A$. If $x \in A$, then there exists some $i, 1 \leq i \leq s$, such that $P_i(x) \leq 0$. Then for $\omega = (-\delta_{1,i}, \ldots, -\delta_{s,i})$ (where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise), we see that $(\omega, x) \in B$. Conversely, if $x \in \phi_2(B)$, then there exists $\omega = (\omega_1, \ldots, \omega_s) \in \Omega$ such that, $\sum_{i=1}^{s} \omega_i P_i(x) \geq 0$. Since, $\omega_i \leq 0, 1 \leq i \leq s$, and not all $\omega_i = 0$, this implies that $P_i(x) \leq 0$ for some $i, 1 \leq i \leq s$. This shows that $x \in A$.

For $x \in \phi_2(B)$, the fibre $\phi_2^{-1}(x) = \{(\omega, x) \mid \omega \in \Omega \text{ such that } \omega P(x) \geq 0\}$, is a non-empty subset of Ω defined by a single linear inequality. From convexity considerations, all such fibres can clearly be retracted to their center of mass continuously. □

Property of ϕ_1

- We denote by $\Omega_j = \{\omega \in \Omega \mid \lambda_j(\omega P) \geq 0\}$, where $\lambda_j(\omega P)$ is the j-th eigenvalue of ωP.
- for $\omega \in \Omega_j \setminus \Omega_{j-1}$, the fiber $\phi_1^{-1}(\omega)$ is homotopy equivalent to a $(k - j)$-dimensional sphere.
- It follows that the Leray spectral sequence of the map ϕ_1 (converging to the cohomology $H^*(B) \cong H^*(A)$), has as its E_2 terms,

$$E_2^{pq} = H^p(\Omega_{k-q}, \Omega_{k-q-1}). \quad (2)$$

- Algorithm follows from an index invariant triangulation of Ω. Complexity is doubly exponential in ℓ.

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Cohomological Descent

- Let, $X \subset \mathrm{R}^m$ and $Y \subset \mathrm{R}^n$ be semi-algebraic sets, and let $f : X \to Y$ be a semi-algebraic, continuous surjection, which is also an open mapping (it takes open sets to open sets).
- We denote by $W^i_f(X)$ the $(i+1)$-fold fibered product of X over f, that is, $W^i_f(X) = \{(x_0, \ldots, x_i) \in X^{i+1} \mid f(x_0) = \cdots = f(x_i)\}$.
- For any semi-algebraic set S, we will denote by $\bar{C}^\bullet(S)$ be the Alexander-Spanier co-chain complex of S.

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Descent Spectral Sequence

- We have an exact sequence analogous to the Mayer-Vietoris exact sequence.

$$0 \longrightarrow \bar{C}^\bullet(Y) \xrightarrow{f^*} \bar{C}^\bullet(W_f^0(X)) \xrightarrow{\delta^0} \bar{C}^\bullet(W_f^1(X)) \xrightarrow{\delta^1} \cdots \bar{C}^\bullet(W_f^p(X))$$

Idea behind the algorithm

- Notice that the fibered product of q sets each defined by ℓ quadatic inequalities is defined by $q + \ell$ quadratic inequalities.
- Using the polynomial time algorithm described previously for computing a complex whose cohomology groups are isomorphic to those of a given semi-algebraic set defined by a constant number of quadratic inequalities,we are able to construct a certain double complex, whose associated total complex is quasi-isomorphic to (implying having isomorphic homology groups) a suitable truncation of the one obtained from the cohomological descent spectral sequence mentioned above. This complex is of much smaller size and can be computed in polynomial time.

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