CS 3510
Practice Final
Time: 2hrs.

Write your name in the top left corner. Attempt all questions. You must show all work in order to obtain credit

1. (a) Find a tight upper bound for the function $T(n)$ satisfying the recurrence,
$$T(n) = T(\sqrt{n}) + 1.$$

2. Compare the asymptotic rates of growth of the functions $n!$ and $F(n)$ where $F(n)$ is the n-th Fibonacci number defined by, $F(n) = F(n-1) + F(n-2), F(1) = F(2) = 1.$

(a) Describe an $O(n \log n)$ algorithm of or sorting. Prove that your algorithm runs in $O(n \log n)$ time.

(b) Show that any algorithm for sorting must take $\Omega(n \log n)$ time in the worst case in the comparison model. State clearly your assumptions on the algorithm.

3. Let $G = (V, E)$ be a weighted undirected graph with non-negative edge weights. Let $s \in V$. Prove or disprove that a shortest path tree with source s is always a minimum spanning tree.

4. Explain Dijkstra's algorithm on the following weighted graph.

5. (a) What are the 4-th roots of unity ?

(b) Compute $\mathrm{DFT}_{\omega_4}(P)$ where P is the polynomial $1 + X + X^2 + X^3$.

(c) Compute the inverse $\mathrm{DFT}_{\omega_4^{-1}}$ of the vector you computed in the previous question. How does your result relate to the original polynomial P ?

6. Prove that a polynomial of degree d is uniquely determined by its values at $d+1$ distinct points. Underlying the sentence in your proof where you make use of the fact that the points are distinct.

7. Does the set of elements $\{1, 2, 3, 4, 5, 6, 7\}$ form a group under multiplication mod 8 ? Explain your answer.

8. (a) What is a primitive element in the group Z_m^* ?

(b) List the primitive elements in the group Z_9^*.

(c) What is the number of primitive elements in the group Z_{18}^* ?

9. Describe an efficient algorithm for the following problem. Given integers a, and n, the problem is to determine the congruence class of the multiplicative inverse of a mod n if it exists. State the complexity of your algorithm in terms of the bit sizes of a and n.

10. Find *all* solutions (if any) to the modular equation,

$$75x \equiv 20 (\text{mod } 55).$$

11. Find the congruence class a mod 60 such that,

$$\begin{aligned} a &\equiv 1 \bmod 3, \\ a &\equiv 2 \bmod 4, \\ a &\equiv 3 \bmod 5. \end{aligned}$$

12. Assuming that $3SAT$ is NP-complete, prove that the CLIQUE problem is NP-complete.