1. Arrange the following list of functions in ascending order of growth rate, i.e. if function $g(n)$ immediately follows $f(n)$ in your list then, it should be the case that $f(n) = O(g(n))$.

$$g_1(n) = 2^{\sqrt{\log n}}$$
$$g_2(n) = 2^n$$
$$g_3(n) = n^{4/3}$$
$$g_4(n) = n(\log n)^3$$
$$g_5(n) = n^{\log n}$$
$$g_6(n) = 2^{2^n}$$
$$g_7(n) = 2^{n^2}$$

The correct order is $g_1, g_4, g_3, g_5, g_2, g_7, g_6$. The easiest ones to tell are the polynomially-bounded functions and the purely exponential functions. $n(\log n)^3 = o(n^{4/3})$, $2^n = o(2^{n^2})$, $2^{n^2} = o(2^{2^n})$. When exponents and logarithms are combined, you have to be careful. Assuming all the logarithms are base 2, $2^{\sqrt{\log n}} = o(2^{\log n})$ but $2^{\log n} = n$, so $2^{\sqrt{\log n}}$ is actually sublinear. It is clear that $n^{4/3} = o(n^{\log n})$ since $4/3 = o(\log n)$. To compare it to the exponential functions, note that $n^{\log n} = 2^{\log(n^{\log n})} = 2^{\log n \log n} = 2^{(\log n)^2} = o(2^n)$.

2. Using Stirling's formula prove that

$$n! = o(n^n).$$

Proof: First note that $n! = \sqrt{2\pi n}(\frac{n}{e})^n(1+\Theta(\frac{1}{n})) = O(\sqrt{2\pi n}(\frac{n}{e})^n)$. Now $\lim_{n\to\infty} \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{n^n} = \lim \frac{\sqrt{2\pi n}}{e^n}$, and we use l'Hospital's rule to get $\lim \frac{\sqrt{2\pi}}{(2\sqrt{n})e^n} = 0$. Thus $n! = o(n^n)$.

3. Problems 4-1 and 4-4.
4-1

   (a) Master method: $a = 2$, $b = 2$, $f(n) = n^3$, $af(\frac{n}{b}) = \frac{1}{4}n^3$. Case 3$\Rightarrow T(n) = \Theta(n^3)$.
   (b) Master method: $a = 1$, $b = \frac{10}{9}$, $f(n) = n$, $af(\frac{n}{b}) = \frac{9}{10}n$. Case 3$\Rightarrow T(n) = \Theta(n)$.
   (c) Master method: $a = 16$, $b = 4$, $f(n) = n^2$. Case 2$\Rightarrow T(n) = \Theta(n^2 \lg n)$.
   (d) Master method: $a = 7$, $b = 3$, $f(n) = n^2$, $af(\frac{n}{b}) = \frac{7}{9}n^2$. Case 3$\Rightarrow T(n) = \Theta(n^2)$.
   (e) Master method: $a = 7$, $b = 2$, $f(n) = n^2$. Case 1$\Rightarrow T(n) = \Theta(n^{\log_2 7})$.
   (f) Master method: $a = 2$, $b = 4$, $f(n) = n^{\frac{1}{2}}$. Case 2$\Rightarrow T(n) = \Theta(\sqrt{n} \lg n)$.
   (g) $T(n) = T(n-1)+n = T(n-2)+(n-1)+n = T(n-3)+(n-2)+(n-1)+n = \ldots = T(0)+1+2+3+\ldots+(n-2)+(n-1)+n = T(0)+\sum_{i=1}^{n} i = T(0)+\frac{n(n+1)}{2} = \Theta(n^2)$.

(h) By expanding as in part (g) above, we get $T(n) = 1 + 1 + 1 + \ldots + 1 + T(2)$. The question is how many ones there are in this equation. If we say $n = 2^{2^k}$ for some k, $\sqrt{n} = 2^{2^{k-1}}$. Therefore there are k ones by the time we get down to $2 = 2^{2^0}$, so $T(n) = 1(k) + T(2) = \Theta(\lg\lg n)$. The substitution method can be used to verify this.

4-4

(a) Master method: $a = 3$, $b = 2$, $f(n) = n \lg n$. Case 1$\Rightarrow$ $T(n) = \Theta(n^{\log_2 3})$.

(b) Master method does not apply since $\frac{n}{\lg n} \neq O(n^{1-\epsilon})$ and $\frac{n}{\lg n} \neq \Omega(n^{1+\epsilon})$. We can find some bounds, however, as follows. Let $T_1(n) = 5T_1(\frac{n}{5}) + n$ and $T_2(n) = 5T_2(\frac{n}{5}) + n^{1-\epsilon}$ for any $0 < \epsilon < 1$. By the master method, $T_1 = \Theta(n \lg n)$ and $T_2 = \Theta(n)$. But $T(n) \leq T_1(n)$ and $T(n) = \Omega(T_2(n))$, so we may conclude $T(n) = O(n \lg n)$ and $T(n) = \Omega(n)$.

(c) Master method: $a = 4$, $b = 2$, $f(n) = n^{5/2}$, $af(\frac{n}{b}) = \frac{1}{\sqrt{2}}n^{5/2}$. Case 3$\Rightarrow$ $T(n) = \Theta(n^2\sqrt{n})$.

(d) Master method does not apply because of the +5 in the recursion. We guess that the +5 doesn't add much, however, and we may use the substitution method to show that $T(n) \leq n \lg n$ for sufficiently large n. First note that $\lg(\frac{n}{3} + 5) \leq \lg(\frac{n}{2})$ for $n \geq 30$. So $T(n) \leq 3(\frac{n}{3} + 5)\lg(\frac{n}{3} + 5) + \frac{n}{2} \leq (n + 15)\lg\frac{n}{2} + \frac{n}{2} = n \lg n - n + 15\lg n - 15 + \frac{n}{2} \leq n \lg n$ if $n \geq 15 \lg n + \frac{n}{2}$, which occurs when $n \geq 256$. Note also $T(n) \geq 3T(\frac{n}{3}) + \frac{n}{2} = \Omega(n \lg n)$. Thus $T(n) = \Theta(n \lg n)$.

(e) Same as part (b) above with 2 instead of 5.

(f) If we assume $T(n) = 8n$, we can use the substitution method to show $T(n) = 8(\frac{n}{2}) + 8(\frac{n}{4}) + 8(\frac{n}{8}) + n = 4n + 2n + n + n = 8n$. Thus $T(n) = \Theta(n)$.

(g) $T(n) = T(n-1) + \frac{1}{n} = \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots + 1 + T(0) = H_n + T(0) = \Theta(\lg n)$. This is the harmonic series.

(h) $T(n) = T(n-1) + \lg n = \lg n + \lg(n-1) + \lg(n-2) + \ldots + \lg 2 + T(1) = \sum_{i=2}^{n} \lg i + T(1) \leq \sum_{i=2}^{n} \lg n + T(1) \leq n \lg n + T(1) = O(n \lg n)$. Note also $T(n) = \sum_{i=2}^{n/2} \lg i + \sum_{i=n/2}^{n} \lg i + T(1) \geq \sum_{i=n/2}^{n} \lg i \geq \sum_{i=n/2}^{n} \lg\frac{n}{2} = \frac{n}{2}\lg\frac{n}{2} = \Omega(n \lg n)$. Thus $T(n) = \Theta(n \lg n)$.

(i) $T(n) = T(n-2) + 2\lg n = 2\lg n + 2\lg(n-2) + 2\lg(n-4) + \ldots + 2\lg 2 \geq \lg n + \lg(n-1) + \lg(n-2) + \lg(n-3) + \lg(n-4) + \lg(n-5) + \ldots + \lg 2 + \lg 1 = \sum_{i=1}^{n} \lg i = \Omega(n \lg n)$. But also $T(n) = 2\lg n + 2\lg(n-2) + 2\lg(n-4) + \ldots + 2\lg 2 \leq 2\lg n + 2\lg(n-1) + 2\lg(n-2) + 2\lg(n-3) + 2\lg(n-4) + 2\lg(n-5) + \ldots + 2\lg 2 + 2\lg 1 = 2\sum_{i=1}^{n} \lg i = O(n \lg n)$. Thus $T(n) = \Theta(n \lg n)$.

(j) We can use the substitution method to prove that $T(n) \leq cn \lg\lg n$. $T(n) = \sqrt{n}T(\sqrt{n}) + n \leq \sqrt{n}c\sqrt{n}\lg\lg\sqrt{n} + n = cn \lg\lg n^{\frac{1}{2}} + n = cn \lg(\frac{1}{2}\lg n) + n =$

$cn \lg \frac{1}{2} + cn \lg \lg n + n = cn \lg \lg n - cn + n \leq cn \lg \lg n$ if $c \geq 1$. We can likewise show $T(n) \geq cn \lg \lg n$ for some c. Thus $T(n) = \Theta(n \lg \lg n)$.

4. Problem 5.2-5.
Let X_{ij} indicate that the pair (i, j) is an inversion. Then by linearity of expectation, $E[\sum_{i,j} X_{ij}] = \sum_{i,j} E[X_{ij}] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} p_{ij}$, where p_{ij} is the probability that the pair (i, j) is an inversion. If you consider a permutation at random, it is plausible that the probability of any given pair being flipped or not is 50-50. To make sure of this, we calculate p_{ij} by conditioning on the value of $A[i]$, which will be exactly one of $1, \ldots, n$. So $p_{ij} = \Pr[A[j] < A[i]] = \sum_{a=1}^{n} \Pr[A[j] < a | A[i] = a] Pr[A[i] = a] = \sum_{a=1}^{n} \frac{a-1}{n-1} \cdot \frac{1}{n} = \frac{1}{n(n-1)} \sum_{a=1}^{n} (a - 1) = \frac{1}{n(n-1)} \cdot \frac{n(n-1)}{2} = \frac{1}{2}$. So $\sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{1}{2} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i+1}^{n} 1 = \frac{1}{2} \sum_{i=1}^{n} (n - i) = \frac{1}{2} \frac{n(n-1)}{2} = \frac{1}{4} n(n - 1)$.

5. Problem 6-3.

   (a) Remember ∞ is used to represent empty cells so that the less than or equal as you move right or down relationship may be maintained.

$$\begin{bmatrix} 2 & 4 & 8 & 9 \\ 3 & 14 & \infty & \infty \\ 5 & 16 & \infty & \infty \\ 12 & \infty & \infty & \infty \end{bmatrix}$$

   (b) If a cell is ∞, then everything to the right of it must also be ∞. Thus if $Y[1, 1]$ is ∞, the whole top row is ∞. If a cell is ∞, then everything below it must also be ∞. Thus if the whole top row is ∞, the whole matrix is ∞. Conversely, if a cell is finite, then everything above it must also be finite. Thus if $Y[m, n]$ is finite, the whole last column is finite. If a cell finite, then everything to the left of it must be finite. Thus if the whole last column is finite, the entire matrix is finite.

   (c) The Young tableau is actually very similar in concept to a heap. The minimum element is always $Y[1, 1]$, so we return that for Extract-Min. To "re-Youngify" the matrix, insert an ∞ at $Y[1, 1]$ and bubble it downward and rightward until you hit the edge of the array or another ∞ (as far as possible in each direction). At worst it will end up in the lower-right corner, and this would take $m + n$ steps.

   (d) Again this is similar to a heap. In this case, insert your new element into $Y[m, n]$. This is necessarily ∞ because the Young tableau is nonfull, and if there's an ∞ anywhere it will be here. Then bubble the element up and to the left until you hit the edge of the array or an element smaller than the new entry (as far as possible in each direction).

   (e) Note that Extract-Min and Insert are $O(n)$ for this problem since $n = m$ in a square tableau. Take the list of n^2 numbers and insert them one at a time into the tableau; this is $n^2 O(n) = O(n^3)$. Then repeatedly use Extract-Min to get the next element in sorted order; this is also $n^2 O(n) = O(n^3)$. The total time is $O(n^3)$.

(f) We want to find a given number k in a Young tableau. In order to achieve $O(n + m)$ time, we need to eliminate either an entire row or an entire column at every step. We start in the upper right corner, $Y[i, j]$ with $i = 1$ and $j = n$, and proceed left and down. If $k = Y[i, j]$, we're done. If $k < Y[i, j]$, the rest of column i is also greater than k, so this column is eliminated and we decrement j. If $k > Y[i, j]$, then nothing to the left can be greater than or equal to k, so this row is eliminated and increment i. The loop invariant that shows this works is that at any point i, j, we've already eliminated everything above and to the right. If i becomes $n + 1$ or j becomes 0, everything was eliminated and k is not in Y.