

Algebra 557: Week 2

1 Lengths, minimal and maximal conditions

Definition 1. A module M is *simple* if it has no proper sub-modules.

Remark 2. If M is a simple module and $0 \neq m \in M$, then $Am = M$. But, $Am = A/\text{ann}(m)$ which is simple if and only if $\text{ann}(m)$ is a maximal ideal. Hence, all simple modules are of the form A/\mathfrak{m} for some maximal ideal \mathfrak{m} of A .

Definition 3. If a module M admits a composition series then the length of the series is called the *length* of M (denoted $\ell(M)$). If M admits no composition series then we say that $\ell(M) = \infty$.

Theorem 4. If $N \subset M$ is a submodule then

$$\ell(M) = \ell(M/N) + \ell(N).$$

More generally, if there is an exact sequence

$$0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0$$

then we have that

$$\sum_i (-1)^i \ell(M_i) = 0.$$

Example 5. Let A be a ring with a finitely generated maximal ideal \mathfrak{m} . Then for each $\nu > 0$ we have that $\ell(A/\mathfrak{m}^\nu) < \infty$. This is because we have that

$$\ell(A/\mathfrak{m}^\nu) = \ell(A/\mathfrak{m}) + \ell(\mathfrak{m}/\mathfrak{m}^2) + \cdots + \ell(\mathfrak{m}^{\nu-1}/\mathfrak{m}^\nu).$$

The modules $\mathfrak{m}^{p-1}/\mathfrak{m}^p$ are A/\mathfrak{m} -vector spaces which are finite dimensional since \mathfrak{m} is f.g. and thus have finite lengths.

This example also shows that $\ell(A)$ is finite if $\mathfrak{m}^n = 0$ for some n and also that the ring A/\mathfrak{m}^ν is an Artinian ring.

Definition 6. A module M is said to be *Noetherian* (resp. *Artinian*) if it satisfies the a.c.c (resp. d.c.c) on the set of its sub-modules.

Remark 7. The a.c.c (resp. d.c.c) can be equivalently stated as the property that every set of sub-modules has a maximal (resp. minimal) element.

Remark 8. A module M is both Noetherian and Artinian if and only if it has finite length.

Proof. If N, N' are submodules of M and $N \supsetneq N'$, then clearly $\ell(N) > \ell(N')$. This implies that if $\ell(M) < \infty$, then every ascending or descending sequence of sub-modules of M must stabilize after a finite number of terms.

Conversely, if M is both Noetherian and Artinian, then using the d.c.c we can obtain a minimal non-zero submodule M_0 of M . Now let M_1 be a minimal submodule amongst all submodules properly containing M_0 . Continuing this way we obtain an ascending sequence of submodules, $0 \subset M_0 \subset M_1 \subset M_2 \subset \dots$ which must stabilize at M by the a.c.c. Moreover, by construction the quotients M_{i+1}/M_i are all simple, and thus we have a composition series for M proving that $\ell(M) < \infty$. \square

Remark 9. Quotients, homomorphic images, sub-modules of Noetherian (resp. Artinian) modules are Noetherian (resp. Artinian). But a subring of a Noetherian ring need not be Noetherian. (Take your favorite non-Noetherian which is an integral domain and consider it as a sub-ring of its field of fractions).

Theorem 10. *If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is a short exact sequence of A -modules and both M', M'' are Noetherian (resp. Artinian) then M is Noetherian (resp. Artinian).*

Proof. It is sufficient to prove the following claim. Abusing notation a little we will consider M' as a sub-module of M . Now suppose we have two sub-modules N_1, N_2 of M , with $N_1 \subset N_2$, and such that $N_1 \cap M' = N_2 \cap M'$, and $\phi(N_1) = \phi(N_2)$, where ϕ is the last surjection in the short exact sequence. We claim that in this case $N_1 = N_2$.

Suppose not and let $x \in N_2 \setminus N_1$. Then since $\phi(N_1) = \phi(N_2)$, there exists $y \in N_1$, such that $\phi(x) = \phi(y)$ implying that $x - y \in \ker(\phi) = M'$. Hence, $x - y \in M' \cap N_2 = M' \cap N_1$. Since $y \in N_1$, this implies that $x \in N_1$, which is a contradiction. \square

Corollary 11. *If A is Noetherian (resp. Artinian) then any f.g. A -module is also Noetherian (resp. Artinian).*

Proof. Reduce to the case of a free module A^n (since M is a quotient of such a module) and then apply the theorem in the case of A^n and use induction on n . \square

Theorem 12. (Hilbert) *If A is a Noetherian module, then so is $A[X]$.*

Before proving Theorem 12 we first prove a lemma.

Lemma 13. *Let $I \subset A[X]$ be an ideal and for $i \geq 0$ let $L_i(I) \subset A$ denote the set of elements of A occurring as the leading coefficient of some element of I of degree i . Then $L_0(I) \subset L_1(I) \subset \dots$ is an ascending sequence of ideals of A . Moreover, if J is another ideal of A with $I \subset J$, and $L_i(I) = L_i(J)$ for all $i \geq 0$, then $I = J$.*

Proof. The first part is obvious. For the second part use induction on degree. \square

We now prove Theorem 12.

Proof. (of Theorem 12) Let $I_0 \subset I_1 \subset I_2 \subset \dots$ be any ascending sequence of ideals. Then for each fixed $i \geq 0$, $L_i(I_0) \subset L_i(I_1) \subset \dots$ is an ascending sequence of ideals of A . Also, for each fixed $j \geq 0$, $L_0(I_j) \subset L_1(I_j) \subset \dots$ is also an ascending sequence of ideals of A . Using the maximal property there exists a maximal ideal $L_p(I_q)$ in the double family $L_i(I_j)$. Now using the acc on A , for each i there exists $n(i)$ such that $L_i(I_j) = L_i(I_{n(i)})$ for all $j \geq n(i)$. Clearly for $i \geq p$, we can take $n(i) = q$. Thus, we can choose $n(i) = n_0$ (independent of i). Now for any $j \geq n_0$, we have that $L_i(I_{n_0}) = L_i(I_j)$, for all $i \geq 0$. By the previous lemma we have that $I_j = I_{n_0}$. \square

Theorem 14. *A ring A is Noetherian if all its prime ideals are f.g.*

Proof. Let Γ be the set of ideals of A which are not f.g. and suppose that $\Gamma \neq \emptyset$. Then, Γ is inductively ordered (verify !) and hence admits a maximal element (say I). Then, I cannot be prime. Hence, there exists $x, y \in A$, s.t. $x, y \notin I, xy \in I$. By maximality of I , $I + Ax$ is f.g. (say with generators $u_1, \dots, u_m, x, u_1, \dots, u_m \in I$. Since $xy \in I, y \in (I : x)$ and again by maximality of I we have that $(I : x)$ is f.g. say with generators v_1, \dots, v_n . Then, $u_1, \dots, u_m, v_1 x, \dots, v_n x$ together generate I , which is a contradiction. \square

Example 15. \mathbb{Z} is Noetherian but not Artinian as a \mathbb{Z} -module.

Theorem 16. *(Akizuki) An Artinian ring A is Noetherian.*

Proof. By previous remark it suffices to prove that A has finite length. Since A is Artinian it must be semi-local (with maximal ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$). Otherwise, we will have a infinite descending chain of ideals $\mathfrak{p}_1 \supset \mathfrak{p}_1 \mathfrak{p}_2 \supset \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3 \supset \dots$ (this is a strictly descending sequence since each \mathfrak{p}_i is co-prime to $\mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_{i-1} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_{i-1}$ and thus $\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_{i-1} \not\subset \mathfrak{p}_i$).

Now let $I = \mathfrak{p}_1 \dots \mathfrak{p}_n (= \text{rad}(A))$. Consider the descending sequence of ideals $I \supset I^2 \supset I^3 \supset \dots$. By the dcc on A , this sequence has to stabilize. Let $I^s = I^{s+1} = \dots$. Let $J = (0 : I^s)_A$ (i.e J is the annihilator of I^s). We now prove that $J = A$ which will imply that $I^s = 0$.

Suppose $J \neq A$. Then by the dcc there exists a minimal ideal J' amongst all ideals strictly containing J . Let $x \in J' \setminus J$. Then by the minimality of J' , $J' = J + Ax$. Consider now the ideal $J + Ix \subset J'$. If $J' = J + Ix$, then by Nakayama Lemma we would have (since J'/J is f.g. as a module) $J' = J$. Thus, $J \subset J + Ix \subsetneq J'$, and again by the minimality of J' this implies that $J = J + Ix$, implying that $Ix \subset J$, or in other words $x \in (J : I)_A$. But $(J : I) = ((0 : I^s) : I) = (0 : I^{s+1}) = (0 : I^s) = J$, and so we have a contradiction. Hence, $J = A$.

Now consider the descending sequence of ideals terminating at $I^s = 0$,

$$A \supset \mathfrak{p}_1 \supset \mathfrak{p}_1 \mathfrak{p}_2 \supset \dots \supset \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_n = I \supset I \mathfrak{p}_1 \supset I \mathfrak{p}_1 \mathfrak{p}_2 \supset \dots \supset I^2 \supset \dots \supset I^s = 0.$$

Notice that the successive quotients are all of the form $M/\mathfrak{p}_i M$ which is a A/\mathfrak{p}_i -vector space, finite dimensional by the dcc condition. This implies that $\ell(A) < \infty$. \square

Theorem 17. *In an Artinian ring every prime ideal is a maximal ideal.*

Proof. An Artinian ring A which is a domain must be a field, since if there is a non-invertible non-zero $a \in A$, we would have a strictly descending sequence of ideals $A \supset aA \supset a^2A \supset \dots$.

Moreover, quotients of Artinian rings are Artinian. \square

Theorem 18. *In a Noetherian ring every ideal contains a product of prime ideals. In particular, (0) is a product of prime ideals.*

Proof. Let Γ be the set of ideals not satisfying the condition of the theorem. Suppose $\Gamma \neq \emptyset$. Then by the Noetherian property of the ring A , Γ contains a maximal element (say \mathfrak{a}). Now \mathfrak{a} cannot be prime. Hence, there exists $x, y \notin \mathfrak{a}$, s.t. $xy \in \mathfrak{a}$. Then the ideals both strictly contain \mathfrak{a} , and by the maximality of \mathfrak{a} , both these ideals contain prodcuts of prime. Moreover, it is clear that $(\mathfrak{a} + Ax)(\mathfrak{a} + Ay) = \mathfrak{a}$, which contradicts the fact that $\mathfrak{a} \in \Gamma$. \square

Theorem 19. *The property that (0) is a product of maximal ideals along with one of the chain conditions imply the other chain condition.*

Proof. Let $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Consider the sequence of sub-modules

$$A \supset \mathfrak{m}_1 \supset \mathfrak{m}_1 \mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n = 0.$$

Each quotient $\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1} / \mathfrak{m}_1 \cdots \mathfrak{m}_i$ is a A/\mathfrak{m}_i -vector space which is finite dimensional by any one of the chain conditions. This shows that A is of finite length and hence satisfies both acc and dcc. \square

Theorem 20. *A ring A is Artinian if and only if A is Noetherian and every prime ideal is maximal.*

Proof. One direction is already shown. Now suppose that A is Noetherian and every prime ideal is maximal. We prove that (0) is an intersection of maximal ideals. But since A is Noetherian, we have by above that (0) is a product of prime ideals, but by assumption every prime ideal is maximal. \square

Theorem 21. *If a f.g. faithful A -module M is Noetherian, then so is A .*

Proof. Let x_1, \dots, x_n be a set of generators of M , and consider the injective (since M is faithful) homomorphism $\phi: A \rightarrow M^n$, sending $a \mapsto (ax_1, \dots, ax_n)$. Since M is Noetherian, so is M^n , and since A is a sub-module of a Noetherian module it is Noetherian too. \square

Theorem 22. *If B is a f.g. faithful module over A such that the set of sub-modules of B of the form IB where I is an ideal of A satisfy the a.c.c, then A is Noetherian.*

Proof. By the previous theorem it suffices to prove that B is Noetherian. Suppose that B is not Noetherian. Consider the set of ideals I of A satisfying B/IB is not Noetherian. This set is non-empty since (0) is in it, and inductively ordered. By hypothesis there exists a maximal element I_0 , which has the property that B/I_0B is not Noetherian, but for all ideals $I \supset I_0$, B/I_0 is Noetherian. We now replace B by the module B/IB , and replace A by $A/\text{ann}(B/IB)$. Then, B is a f.g., faithful, non-Noetherian module. Now consider the set of sub-modules N of B such that B/N is faithful. This set is non-empty and inductively ordered (verify!). Choose N_0 to be a maximal element. Replace B by B/N_0 to obtain a module having the following properties:

- i. B is non-Noetherian, but for every proper ideal $I \subset A$, $I \neq 0$, B/I is Noetherian.

ii. B is faithful, but for every sub-module $N \subset B, N \neq 0, B/N$ is not faithful.

Consider any sub-module $N \subset B$. Then by (ii) above B/N is not faithful, and hence $\text{ann}(B/N) \neq 0$. Let $a \in \text{ann}(B/N), a \neq 0$. Then, $aB \subset N$. Then by (i) B/aB is Noetherian, and hence N/aB is f.g. (since it is a sub-module of B/aB). Further, since B is f.g. so is aB . Hence, N is f.g. Hence, B is Noetherian and we have reached a contradiction. \square