# Algebra 557: Week 2

## 1 Lengths, minimal and maximal conditions

**Definition 1.** *A module $M$ is simple if it has no proper sub-modules.*

**Remark 2.** If $M$ is a simple module and $0 \neq m \in M$, then $Am = M$. But, $Am = A/\operatorname{ann}(m)$ which is simple if and only if $\operatorname{ann}(m)$ is a maximal ideal. *Hence, all simple modules are of the form $A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of $A$.*

**Definition 3.** *If a module $M$ admits a composition series then the length of the series is called the length of $M$ (denoted $\ell(M)$). If $M$ admits no composition series then we say that $\ell(M) = \infty$.*

**Theorem 4.** *If $N \subset M$ is a submodule then*

$$\ell(M) = \ell(M/N) + \ell(N).$$

More generally, if there is an exact sequence

$$0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_n \longrightarrow 0$$

then we have that

$$\sum_i (-1)^i \ell(M_i) = 0.$$

**Example 5.** Let $A$ be a ring with a finitely generated maximal ideal $\mathfrak{m}$. Then for each $\nu > 0$ we have that $\ell(A/\mathfrak{m}^\nu) < \infty$. This is because we have that

$$\ell(A/\mathfrak{m}^\nu) = \ell(A/\mathfrak{m}) + \ell(\mathfrak{m}/\mathfrak{m}^2) + \cdots + \ell(\mathfrak{m}^\nu/\mathfrak{m}^{\nu-1}).$$

The modules $\mathfrak{m}^{p-1}/\mathfrak{m}^p$ are $A/\mathfrak{m}$-vector spaces which are finite dimensional since $\mathfrak{m}$ is f.g. and thus have finite lengths.

This example also shows that $\ell(A)$ is finite if $\mathfrak{m}^n = 0$ for some $n$ and also that the ring $A/\mathfrak{m}^\nu$ is an Artinian ring.

**Definition 6.** *A module $M$ is said to be Noetherian (resp. Artinian) if it satisfies the a.c.c (resp. d.c.c) on the set of its sub-modules.*

**Remark 7.** The a.c.c (resp. d.c.c) can be equivalently stated as the property that every set of sub-modules has a maximal (resp. minimal) element.

**Remark 8.** *A module $M$ is both Noetherian and Artinian if and only if it has finite length.*

**Proof.** If $N$, $N'$ are submodules of $M$ and $N \supsetneq N'$, then clearly $\ell(N) > \ell(N')$. This implies that if $\ell(M) < \infty$, then every ascending or descending sequence of sub-modules of $M$ must stabilize after a finite number of terms.

Conversely, if $M$ is both Noetherian and Artinian, then using the d.c.c we can obtain a minimal non-zero submodule $M_0$ of $M$. Now let $M_1$ be a minimal submodule amongst all submodules properly containing $M_0$. Continuing this way we obtain an ascending sequence of submodules, $0 \subset M_0 \subset M_1 \subset M_2 \subset \cdots$ which must stabilize at $M$ by the a.c.c. Moreover, by construction the quotients $M_{i+1}/M_i$ are all simple, and thus we have a composition series for $M$ proving that $\ell(M) < \infty$. $\square$

**Remark 9.** Quotients, homomorphic images, sub-modules of Noetherian (resp. Artinian) modules are Noetherian (resp. Artinian). But a subring of a Noetherian ring need not be Noetherian. (Take your favorite non-Noetherian which is an integral domain and consider it as a sub-ring of its field of fractions).

**Theorem 10.** *If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is a short exact sequence of $A$-modules and both $M'$, $M''$ are Noetherian (resp. Artinian) then $M$ is Noetherian (resp. Artinian).*

**Proof.** It is sufficient to prove the following claim. Abusing notation a little we will consider $M'$ as a sub-module of $M$. Now suppose we have two sub-modules $N_1, N_2$ of $M$, with $N_1 \subset N_2$, and such that $N_1 \cap M' = N_2 \cap M'$, and $\phi(N_1) = \phi(N_2)$, where $\phi$ is the last surjection in the short exact sequence. We claim that in this case $N_1 = N_2$.

Suppose not and let $x \in N_2 \setminus N_1$. Then since $\phi(N_1) = \phi(N_2)$, there exists $y \in N_1$, such that $\phi(x) = \phi(y)$ implying that $x - y \in \ker(\phi) = M'$. Hence, $x - y \in M' \cap N_2 = M' \cap N_1$. Since $y \in N_1$, this implies that $x \in N_1$, which is a contradiction. $\square$

**Corollary 11.** *If $A$ is Noetherian (resp. Artinian) then any f.g. $A$-module is also Noetherian (resp. Artinian).*

**Proof.** Reduce to the case of a free module $A^n$ (since $M$ is a quotient of such a module) and then apply the theorem in the case of $A^n$ and use induction on $n$. $\square$

**Theorem 12.** *(Hilbert)If $A$ is a Noetherian module, then so is $A[X]$.*

Before proving Theorem 12 we first prove a lemma.

**Lemma 13.** *Let $I \subset A[X]$ be an ideal and for $i \geq 0$ let $L_i(I) \subset A$ denote the set of elements of $A$ occuring as the leading coefficient of some element of $I$ of degree $i$. Then $L_0(I) \subset L_1(I) \subset \cdots$ is an ascending sequence of ideals of $A$. Morever, if $J$ is another ideal of $A$ with $I \subset J$, and $L_i(I) = L_i(J)$ for all $i \geq 0$, then $I = J$.*

**Proof.** The first part is obvious. For the second part use induction on degree. $\square$

We now prove Theorem 12.

**Proof.** (of Theorem 12) Let $I_0 \subset I_1 \subset I_2 \subset \cdots$ be any ascending sequence of ideasl. Then for each fixed $i \geq 0$, $L_i(I_0) \subset L_i(I_1) \subset \cdots$ is an ascending sequence of ideals of $A$. Also, for each fixed $j \geq 0$, $L_0(I_j) \subset L_1(I_j) \subset \cdots$ is also an ascending sequence of ideals of $A$. Using the maximal property there exists a maximal ideal $L_p(I_q)$ in the double family $L_i(I_j)$. Now using the acc on A, for each $i$ there exists $n(i)$ such that $L_i(I_j) = L_i(I_{n(i)})$ for all $j \geq n(i)$. Clearly for $i \geq p$, we can take $n(i) = q$. Thus, we can choose $n(i) = n_0$ (independent of $i$). Now for any $j \geq n_0$, we have that $L_i(I_{n_0}) = L_i(I_j)$, for all $i \geq 0$. By the previous lemma we have that $I_j = I_{n_0}$. $\square$

**Theorem 14.** *A ring $A$ is Noetherian if all its prime ideals are f.g.*

**Proof.** Let $\Gamma$ be the set of ideals of $A$ which are not f.g. and suppose that $\Gamma \neq \emptyset$. Then, $\Gamma$ is inductively ordered (verify !) and hence admits a maximal element (say $I$). Then, $I$ cannot be prime. Hence, there exists $x, y \in A$, s.t. $x, y \notin I, xy \in I$. By maximality of $I$, $I + Ax$ is f.g. (say with generators $u_1, \ldots, u_m, x$, $u_1, \ldots, u_m \in I$. Since $xy \in I$, $y \in (I\colon x)$ and again by maximality of $I$ we have that $(I\colon x)$ is f.g. say with generators $v_{1,\ldots,} v_n$. Then, $u_1, \ldots, u_m, v_1\, x, \ldots, v_n\, x$ together generate $I$, which is a contradiction. □

**Example 15.** $\mathbb{Z}$ is Noetherian but not Artinian as a $\mathbb{Z}$-module.

**Theorem 16.** *(Akizuki) An Artinian ring $A$ is Noetherian.*

**Proof.** By previous remark it suffices to prove that $A$ has finite length. Since $A$ is Artinian it must be semi-local (with maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$). Otherwise, we will have a infinite descending chain of ideals $\mathfrak{p}_1 \supset \mathfrak{p}_1\mathfrak{p}_2 \supset \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 \supset \cdots$ (this is a strictly descending sequence since each $\mathfrak{p}_i$ is co-prime to $\mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_{i-1} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{i-1}$ and thus $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_{i-1} \not\subset \mathfrak{p}_i$).

Now let $I = \mathfrak{p}_1\cdots\mathfrak{p}_n\ (= \mathrm{rad}(A))$. Consider the descending sequence of ideals $I \supset I^2 \supset I^3 \supset \cdots$. By the dcc on $A$, this sequence has to stabilize. Let $I^s = I^{s+1} = \cdots$. Let $J = (0\colon I^s)_A$ (i.e $J$ is the annhilator of $I^s$). We now prove that $J = A$ which will imply that $I^s = 0$.

Suppose $J \neq A$. Then by the dcc there exists a minimal ideal $J'$ amongst all ideals strictly containing $J$. Let $x \in J' \setminus J$. Then by the minimality of $J'$, $J' = J + Ax$. Consider now the ideal $J + Ix \subset J'$. If $J' = J + Ix$, then by Nakayama Lemma we would have (since $J'/J$ is f.g. as a module) $J' = J$. Thus, $J \subset J + Ix \subsetneq J'$, and again by the minimality of $J'$ this implies that $J = J + Ix$, implying that $Ix \subset J$, or in other words $x \in (J\colon I)_A$. But $(J\colon I) = ((0\colon I^s)\colon I) = (0\colon I^{s+1}) = (0\colon I^s) = J$, and so we have a contradiction. Hence, $J = A$.

Now consider the descending sequence of ideals terminating at $I^s = 0$,

$$A \supset \mathfrak{p}_1 \supset \mathfrak{p}_1\mathfrak{p}_2 \supset \cdots \supset \mathfrak{p}_1\mathfrak{p}_2\cdots\mathfrak{p}_n = I \supset I\mathfrak{p}_1 \supset I\mathfrak{p}_1\mathfrak{p}_2 \supset \cdots \supset I^2 \supset \cdots \supset I^s = 0.$$

Notice that the successive quotients are all of the form $M/\mathfrak{p}_i M$ which is a $A/\mathfrak{p}_i$-vector space, finite dimensional by the dcc condition. This implies that $\ell(A) < \infty$. □

**Theorem 17.** *In an Artinian ring every prime ideal is a maximal ideal.*

**Proof.** An Artinian ring $A$ which is a domain must be a field, since if there is a non-invertible non-zero $a \in A$, we would have a strictly descending sequence of ideals $A \supset aA \supset a^2A \supset \cdots$.

Moreover, quotients of Artinian rings are Artinian. □

**Theorem 18.** *In a Noetherian ring every ideal contains a product of prime ideals. In particular,* $(0)$ *is a product of prime ideals.*

**Proof.** Let $\Gamma$ be the set of ideals not satisfying the condition of the theorem. Suppose $\Gamma \neq \emptyset$. Then by the Noetherian property of the ring $A$, $\Gamma$ contains a maximal element (say $\mathfrak{a}$). Now $\mathfrak{a}$ cannot be prime. Hence, there exists $x$, $y \notin \mathfrak{a}$, s.t. $xy \in \mathfrak{a}$. Then the ideals both strictly contain $\mathfrak{a}$, and by the maximaility of $\mathfrak{a}$, both these ideals contain proodcuts of prime. Moreover, it is clear that $(\mathfrak{a} + Ax)(\mathfrak{a} + Ay) = \mathfrak{a}$, which contradicts the fact that $\mathfrak{a} \in \Gamma$. ☐

**Theorem 19.** *The property that* $(0)$ *is a product of maximal ideals along with one of the chain conditions imply the other chain condition.*

**Proof.** Let $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Consider the sequence of sub-modules

$$A \supset \mathfrak{m}_1 \supset \mathfrak{m}_1\mathfrak{m}_2 \supset \cdots \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n = 0.$$

Each quotient $\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1}/\mathfrak{m}_1 \cdots \mathfrak{m}_i$ is a $A/\mathfrak{m}_i$-vector space which is finite dimensional by any one of the chain conditions. This shows that $A$ is of finite length and hence satisfies both acc and dcc. ☐

**Theorem 20.** *A ring* $A$ *is Artinian if and only if* $A$ *is Noetherian and every prime ideal is maximal.*

**Proof.** One direction is already shown. Now suppose that $A$ is Noetherian and every prime ideal is maximal. We prove that $(0)$ is an intersection of maximal ideals. But since $A$ is Noetherian, we have by above that $(0)$ is a product of prime ideals, but by assumption every prime ideal is maximal. ☐

**Theorem 21.** *If a f.g. faithful* $A$*-module* $M$ *is Noetherian, then so is* $A$*.*

**Proof.** Let $x_1, \ldots, x_n$ be a set of generators of $M$, and consider the injective (since $M$ is faithful) homomorphism $\phi: A \longrightarrow M^n$, sending $a \longmapsto (ax_1, \ldots, ax_n)$. Since $M$ is Noetherian, so is $M^n$, and since $A$ is a sub-module of a Noetherian module it is Noetherian too. ☐

**Theorem 22.** *If* $B$ *is a f.g. faithful module over* $A$ *such that the set of sub-modules of* $B$ *of the form* $IB$ *where* $I$ *is an ideal of* $A$ *satisfy the a.c.c, then* $A$ *is Noetherian.*

**Proof.** By the previous theorem it suffices to prove that $B$ is Noetherian. Suppose that $B$ is not Noetherian. Consider the set of ideals $I$ of $A$ satisfying $B/IB$ is not Noetherian. This set os non-empty since $(0)$ is in it, and inductively ordered. By hypothesis there exists a maximal element $I_0$, which has the property that $B/I_0B$ is not Noetherian, but for all ideals $I \supset I_0$, $B/I_0$ is Noetherian. We now replace $B$ by the module $B/IB$, and replace $A$ by $A/\mathrm{ann}(B/IB)$. Then, $B$ is a f.g., faithful, non-Noetherian module. Now consider the set of sub-modules $N$ of $B$ such that $B/N$ is faithful. This set is non-empty and inductively ordered (verify!). Choose $N_0$ to be a maximal element. Replace $B$ by $B/N_0$ to obtain a module having the following properties:

i. $B$ is non-Noetherian, but for every proper ideal $I \subset A$, $I \neq 0$, $B/I$ is Noetherian.

ii. $B$ is faithful, but for every sub-module $N \subset B, N \neq 0$, $B/N$ is not faithful.

Consider any sub-module $N \subset B$. Then by (ii) above $B/N$ is not faithful, and hence $\mathrm{ann}(B/N) \neq 0$. Let $a \in \mathrm{ann}(B/N), a \neq 0$. Then, $aB \subset N$. Then by (i) $B/aB$ is Noetherian, and hence $N/aB$ is f.g. (since it is a sub-module of $B/aB$). Further, since $B$ is f.g. so is $aB$. Hence, $N$ is f.g. Hence, $B$ is Noetherian and we have reached a contradiction. $\square$