Exercises 10.3

1. Let $V = \mathbb{R}^3$ with the standard inner product and let

$$S = \{ u_1, u_2, u_3 \} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Use routine \texttt{gschmidt} in Matlab to obtain an orthonormal basis $T$ and then find the coordinates of $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ relative to $T$. Record the orthonormal basis and the coordinates of $x$ below.

2. Let $V = \mathbb{R}^4$ with the standard inner product and let

$$S = \{ u_1, u_2, u_3, u_4 \} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Use routine \texttt{gschmidt} in Matlab to obtain an orthonormal basis $T$ and then find the coordinates of $x = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ relative to $T$. Record the orthonormal basis and the coordinates of $x$ below.

3. Let $V = \mathbb{R}^4$ with the standard inner product and let
\[ S = \{u_1, u_2, u_3, u_4\} = \left\{ \begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \end{bmatrix}, \begin{bmatrix} .5 \\ -.5 \\ -.5 \\ .5 \end{bmatrix}, \begin{bmatrix} -.5 \\ -.5 \\ -.5 \\ .5 \end{bmatrix}, \begin{bmatrix} -.5 \\ -.5 \\ -.5 \\ .5 \end{bmatrix} \right\} \]

a) Is \( S \) an orthonormal basis?  

Circle one:  Yes  No  

Explain your answer.

b) In MATLAB form the matrix \( T \) whose columns are the vectors in \( S \). Generate a random vector in \( \mathbb{R}^4 \) using command \( x = \text{rand}(4,1) \) and then compute \( \| x \| \) and \( \| Tx \| \). How are the values of the norms related? Repeat the experiment for another arbitrary vector.

4. Let \( v_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \) and \( v_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \). In MATLAB form the matrix \( A = [v_1 \ v_2] \) and then use command \( \text{gschmidt}(A) \). Explain the meaning of the display generated.

5. Let \( A = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix} \).

a) In MATLAB use command \( A' \). Record the result.  

\[ A' = \]

b) In MATLAB use command \( C = A'A \). Record the result.  

\[ C = \]

c) What is the relation between \( C \) and \( C' \)?
d) Experiment with other complex matrices $A$ to confirm or reject your answer in part c).

Circle one: confirmed not confirmed.

6. A complex matrix $A$ is called Hermitian if it is equal to its conjugate transpose. The command $A'$ gives the conjugate transpose in MATLAB.

a) How can you use MATLAB to determine if the matrix $A$ below is Hermitian?

$$A = \begin{bmatrix} 2 & 3 - 3i \\ 3 + 3i & 5 \end{bmatrix}$$

b) Compute $r = x' \times A \times x$ for the complex vector below.

$$x = \begin{bmatrix} i \\ 1 - i \end{bmatrix} \quad r = \text{_________________________}$$

Is $r$ a real number? (Circle one:) YES NO

c) Experiment with other complex vectors $x$ to determine whether $x'Ax$ will always be a real number. (Circle one:)

Always a real number for this matrix $A$. Not always a real number.

d) Experiment with another Hermitian matrix $A$ and arbitrary vector $x$ to see if $r = x' \times A \times x$ is always a real number.

(Circle one:) Always a real number. Not always a real number.

7. Let $V = R^4$ with the standard inner product and let

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ -1 \end{bmatrix}.$$

a) Find an orthonormal basis for $R^4$ containing scalar multiples of the vectors $v_1$ and $v_2$. Record your result below.
b) Find an orthonormal basis for \( \mathbb{R}^4 \) containing scalar multiples of the vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \). Record your result below.