1. (a) If \( p_1, \ldots, p_t \) are prime numbers, prove that \( p_1 \cdots p_t + 1 \) is not divisible by any one of the primes \( p_1, \ldots, p_t \).

(b) Prove using contradiction and part (a) that the number of primes is infinite.

2. Recall from class the equivalence relation on integers defined by \( m \sim n \) if and only if 4 divides \( m - n \). Denote by \([n]\) the equivalence class of an integer \( n \) under this equivalence relation.

(a) Prove that all primes greater than 2 belong either to \([1]\) or \([3]\).

(b) Prove that the equivalence class \([1]\) is closed under multiplication (i.e. the product of two elements of \([1]\) belongs to \([1]\)).

(c) Prove that the number of primes in \([3]\) (i.e. the number of primes that can be written in the form \( 4k + 3 \)) is infinite.

3. Using Euclidean division compute the gcd of 1024 and 560 and express the gcd as \( 1024x + 560y \) where \( x, y \) are integers (Bézout identity).

4. Prove using the principle of mathematical induction that
\[
\sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2.
\]

5. Using the binomial formula and the principle of induction prove that \( n^p - n \) is divisible by \( p \) for any natural number \( n \) and prime \( p \).