ASSIGNMENT 7. DUE IN CLASS OCT 20, 2017.

1. The aim of this exercise is to clarify some subtle aspects of the definition of normal subgroups. Let G be a group and H a subgroup of G.
   (a) Prove that if $gHg^{-1} \subset H$ for every $g \in G$, then $gHg^{-1} = H$ for every $g \in G$. (Hence, in order to check if a subgroup H of a group G is a normal subgroup of G, it suffices to check that $ghg^{-1} \in H$ for every $g \in G$ and $h \in H$).
   (b) Prove that for each $g \in G$, gHg^{-1} is a subgroup of G which is isomorphic to H.
   (c) Give an example of a group G and a subgroup H of G such that $gHg^{-1} \neq H$.
   (d) Prove that if H is a finite subgroup of G, and $g \in G$ such that $gHg^{-1} \subset H$, then $gHg^{-1} = H$.
   (e) Why is the statement in (1d) *not* an obvious corollary of (1a) ?
   (f) But beware that if H is not finite, then it can happen that for some $g \in G$, $gHg^{-1} \subset H$, but $gHg^{-1} \neq H$. The goal of the following exercise is to construct such an example. Let $G = S_{\mathbb{Z}}$ be the group of all bjiections from $\mathbb{Z} \to \mathbb{Z}$ (the group operation being composition). Let H be the subset of G consisting of all bijections $f \in G$ such that $f(x) = x$ for all $x \leq 0$.
      (i) Prove that H is a subgroup of G.
      (ii) Let $g : \mathbb{Z} \to \mathbb{Z}$ be the map defined by $g(x) = x + 1$. Prove that g is a bijection and hence an element of G.
      (iii) Prove that $gHg^{-1} \subset H$.
      (iv) Prove that $gHg^{-1} \neq H$.
2. Let G be a group H, N subgroups of G and let N be a normal subgroup G.
   (a) Prove that $H \cap N$ is a normal subgroup of H.
   (b) Let $NH = \{nh \mid n \in N, h \in H\}$ and $HN = \{hn \mid h \in H, n \in N\}$. Prove that $NH = HN$ and that HN is a subgroup of G.
   (c) Observe that N is a subgroup of HN. Prove that N is a normal subgroup of HN. (Notice that if G' is a subgroup of G containing N, then N is also a subgroup of G', but not necessarily a normal subgroup of G'. Why ?)
   (d) (Optional) Prove that HN/N is isomorphic to $H/H \cap N$. (This often goes by the name "Second Isomorphism Theorem").