PRACTICE MIDTERM 2, 2017

Please write your name in the top left corner. Attempt all questions. Time 50 min.

1. (20 pts)
   (a) Let G be a group, H be a subgroup of G and $g \in G$. Suppose that $gHg^{-1} \subset H$. Is it always true that in this case $gHg^{-1} = H$?
   (b) Is the group $GL(3, \mathbb{R})$ abelian ?
   (c) Is the group $GL(3, \mathbb{R})/SL(3, \mathbb{R})$ abelian ?
   (d) How many distinct subgroups does the group Z_{24} have ?
   (e) How many elements of $Z_{10000000}$ have order 20 ?
   (f) Let p be a prime number. Can a group of order p^3 be non-abelian ?
   (g) Let p be a prime number. Can a group of order p^2 be non-abelian ?
   (h) Suppose that G is a group and N a normal subgroup of G, and $H = G/N$. Is it always true that G is isomorphic to the product group $N \times H$?
   (i) Is it always true that the center, $Z(G)$, of a group G is a normal subgroup of G ?
   (j) Is it true that every non-abelian group has a nontrivial center ?
   (k) Let p be a prime. Is it true that every group of order p^3 has a non-trivial center ?
   (l) Let G be the cyclic group order 24. How many distinct subgroups of G are isomorphic to $Z_2 \times Z_2$?
2. (10 pts)
   (a) State the class equation.
   (b) Using the class equation prove that if p is a prime, then every group whose order is a positive power of p has a nontrivial center.
3. (10 pts) Let H be a group and $G = H \times H$. For each one of the following subgroups of G prove or disprove (with an example) that the subgroup is a normal subgroup of G.
   (a) The subgroup $D = \{(h, h) | h \in H\}$ (the diagonal subgroup).
   (b) The subgroup $K = \{(h, e) | h \in H\}$ (where e is the identity element of G).
4. (10 pts) Let p be a prime. Prove that every finite abelian group whose order is divisible by p must have a subgroup isomorphic to Z_p.