Throughout, $A$ denotes a commutative ring with an identity element, and unless specified otherwise all modules referred to below are $A$-modules.

1. Let $G$ be a module and $E, F$ two submodules such that $E \subset F$. Prove the following.
   (a) If $F$ is a direct factor of $G$, $F/E$ is a direct factor of $G/E$. If $E$ is a direct factor of $F$, $E$ is a direct factor of $G$.
   (b) If $E$ is a direct factor of $G$, then $E$ is a direct factor of $F$. If $F/E$ is a direct factor of $G/E$, then $F$ is a direct factor of $G$.

2. (a) Give examples of two submodules $M, N$ of the $\mathbb{Z}$-module $E = \mathbb{Z} \oplus \mathbb{Z}$ such that $M$ and $N$ are direct factors of $E$, but $M + N$ is not a direct factor of $E$.
   (b) Show that the following sequence of $\mathbb{Z}$-module homomorphisms is a short exact sequence, which is not split,
      \[ 0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \]
      where the second map is multiplication by 2.

3. Let $(M_i)_{i \in I}$ be a (possibly infinite) family of $A$-modules and $(N_i \subset M_i)_{i \in I}$ a family of submodules. Prove that the modules
   \[ (\bigoplus_i M_i)/(\bigoplus_i N_i) \]
   and
   \[ \bigoplus_i (M_i/N_i) \]
   are isomorphic.

4. Recall that an $A$-module $E$ is called cyclic (or monogeneous) if it is generated by a single element. Prove the following.
   (a) Every simple module $M$ (i.e. a module not having any submodules other than 0 and itself) is cyclic.
   (b) Every sub-module of a cyclic module is isomorphic to a quotient module $b/a$ where $a, b$ are two left ideals of $A$ with $a \subset b$.
   (c) Every quotient module of a cyclic module is cyclic.
   (d) Sub-modules of cyclic modules are not necessarily cyclic.

5. Let $E, F$ be two $A$-modules and $u : E \to F$ a linear mapping.
   (a) Show that the mapping $(x, y) \mapsto (x, y - u(x))$ of the product module $E \times F$ to itself is an automorphism of $E \times F$.
   (b) Deduce that if there exists a linear mapping $v : F \to E$, and an $a \in E$, such that $v(u(a)) = a$, there exists an automorphism of $w$ of $E \times F$ such that $w(a, 0) = (0, u(a))$. 

ASSIGNMENT 1, DUE SEP 8, 2017
6. (The five lemma) Consider the following diagram of $A$-modules and linear maps in which the two rows are exact.

$$
\begin{array}{cccccc}
M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\
\bigg\downarrow f_1 & & \bigg\downarrow f_2 & & \bigg\downarrow f_3 & & \bigg\downarrow f_4 & & \bigg\downarrow f_5 \\
N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5
\end{array}
$$

Prove (by “diagram chasing”) the following.

(a) If $f_2$ and $f_4$ are injective, and $f_1$ is surjective, then $f_3$ is injective.
(b) If $f_2$ and $f_4$ are surjective, and $f_5$ is injective, then $f_3$ is surjective.
(In particular, if $f_1, f_2, f_4, f_5$ are isomorphisms, then so is $f_3$.)

7. In a similar vein to Exercise 6 look up the statement and prove by yourself the “snake lemma” (for $A$-modules). (The following link to a Youtube clip contains actually the entire proof if you are into 1970’s films. “It’s my turn”.) You do not need to submit this problem.