Throughout this assignment, unless otherwise stated, $k$ is an algebraically closed field of char 0.

1. Let $E$ be a finite dimensional $k$-vector space, $u, v$ two diagonalizable endomorphisms of $E$, and suppose that $u, v$ commute. Prove that $u \circ v$ is also diagonalizable.

2. Let $E$ be a finite dimensional $k$-vector space and let $F$ be a set of endomorphisms of $E$ such that every pair of elements of $F$ commute.
   (a) For each $u \in F$, and each eigenvalue $\lambda$ of $u$, prove that the eigenspace of $u$ belonging to $\lambda$ is closed under each endomorphism of $F$.
   (b) Prove that the endomorphisms of $F$ have a common eigenvector (with possibly different eigenvalues for different $u \in F$).
   (c) Use Part (2a) and an induction on the dimension of $E$ to prove that there exists a basis of $E$ consisting of common eigenvectors of $F$ (in other words the endomorphisms of $F$ are simultaneously diagonalizable).

3. Let $E$ be a finite dimensional $k$-vector space and $u \in \text{End}(E)$. Consider $\text{End}(E)$ as a $k$-vector space, and denote by $\text{ad}(u)$ the element of $\text{End}(\text{End}(E))$ defined by $\text{ad}(u)(v) = u \circ v - v \circ u$.
   (a) Prove that (using the notation introduced in class for the additive Jordan decomposition)
   \[ \text{ad}(u)_s = \text{ad}(u_s), \text{ad}(u)_n = \text{ad}(u_n). \]
   (b) Prove that $\text{ad}(u)$ is diagonalizable if and only if $u$ is diagonalizable.

4. For every $n > 0$, let $M_n(k)$ denote the $k$-vector space of $n \times n$ matrices with entries in $k$. Prove that there exists a basis $B$ of $M_n(k)$, having the property that for each $X \in B$, and every diagonal matrix $H$,
   \[ [H, X] = HX - XH = \alpha(H)X, \]
   where $\alpha$ is a linear functional on the subspace of diagonal matrices. (Hint. Use the results of Problems (2) and (3)).

5. Let $E$ be a finite dimensional complex vector space with an inner product.
   (a) Prove that for any subspace $M$ of $E$, the orthogonal projection, $p_M$ to $M$ is self-adjoint as an endomorphism of $E$.
   (b) Conversely, show that if $p$ is an endomorphism of $E$ such that
   \[ p = p^* = p^2, \]
   then there exists a subspace of $M$ of $E$ such that $p = p_M$.
   (c) Let $M, N$ be two subspaces of $E$, and let $M'$ (resp. $N'$) be the space of all $x \in M$ (resp. $x \in N$) orthogonal to $M \cap N$. Prove that $p_M, p_N$ commute if and only if $M'$ and $N'$ are orthogonal. If this condition is satisfied then, then show that
(i) \[ p_{M \cap N} = p_M \circ p_N, \]

(ii) \[ p_{M+N} = p_M + p_N - p_M \circ p_N. \]

6. (optional) Let \( V \) be a finite dimensional real vector space, and \( J \) and endomorphism of \( V \) such that \( J^2 = -1_V \). (Such a \( J \) is called a complex structure on \( V \))

(a) Prove that the dimension of \( V \) is even.

(b) Let \( V_C = V \otimes \mathbb{C} \) denote the complexification of \( V \). Here we consider \( \mathbb{C} \) as a two dimensional real vector space and define a \( \mathbb{C} \)-vector space structure on \( V \otimes \mathbb{C} \) by \( z \cdot (v \otimes w) = v \otimes (wz) \). The endomorphism \( J \) extends in an obvious way to an endomorphism of \( V_C \). Prove that \( V_C \) is a direct sum of the eigenspaces of \( J \).

(c) Prove that \( V \) as a basis of the form \( \{e_1, \ldots, e_n, Je_1, \ldots, Je_n\} \).