NOT TO BE SUBMITTED

Throughout this assignment, unless otherwise stated, k is an algebraically closed field of char 0.

1. Let E be a finite dimensional k-vector space, u, v two diagonalizable endomorphisms of E, and suppose that u, v commute. Prove that $u \circ v$ is also diagonalizable.
2. Let E be a finite dimensional k-vector space and let F be a set of endomorphisms of E such that every pair of elements of F commute.
   (a) For each $u \in F$, and each eigenvalue λ of u, prove that the eigenspace of u belonging to λ is closed under each endomorphism of F.
   (b) Prove that the endomorphisms of F have a common eigenvector (with possibly different eigenvalues for different $u \in F$).
   (c) Use Part (2a) and an induction on the dimension of E to prove that there exists a basis of E consisting of common eigenvectors of F (in other words the endomorphisms of F are *simultaneously diagonalizable*).
3. Let E be a finite dimensional k-vector space and $u \in \mathrm{End}(E)$. Consider $\mathrm{End}(E)$ as a k-vector space, and denote by $\mathrm{ad}(u)$ the element of $\mathrm{End}(\mathrm{End}(E))$ defined by $\mathrm{ad}(u)(v) = u \circ v - v \circ u$.
   (a) Prove that (using the notation introduced in class for the additive Jordan decomposition)
   $$\mathrm{ad}(u)_s = \mathrm{ad}(u_s), \mathrm{ad}(u)_n = \mathrm{ad}(u_n).$$
   (b) Prove that $\mathrm{ad}(u)$ is diagonalizable if and only if u is diagonalizable.
4. For every $n > 0$, let $M_n(k)$ denote the k-vector space of $n \times n$ matrices with entries in k. Prove that there exists a basis B of $M_n(k)$, having the property that for each $X \in B$, and every diagonal matrix H,
   $$[H, X] = HX - XH = \alpha(H)X,$$
   where α is a linear functional on the subspace of diagonal matrices. (Hint. Use the results of Problems (2) and (3)).
5. Let E be a finite dimensional complex vector space with an inner product.
   (a) Prove that for any subspace M of E, the orthogonal projection, p_M to M is self-adjoint as an endomorphism of E.
   (b) Conversely, show that if p is an endomorphism of E such that
   $$p = p^* = p^2,$$
   then there exists a subspace of M of E such that $p = p_M$.
   (c) Let M, N be two subspaces of E, and let M' (resp. N') be the space of all $x \in M$ (resp. $x \in N$) orthogonal to $M \cap N$. Prove that p_M, p_N commute if and only if M' and N' are orthogonal. If this condition is satisfied then, then show that

(i)
$$p_{M\cap N} = p_M \circ p_N,$$
(ii)
$$p_{M+N} = p_M + p_N - p_M \circ p_N.$$

6. (optional) Let V be a finite dimensional real vector space, and J and endomorphism of V such that $J^2 = -1_V$. (Such a J is called a *complex structure* on V)
   (a) Prove that the dimension of V is even.
   (b) Let $V_{\mathbb{C}} = V \otimes \mathbb{C}$ denote the complexification of V. Here we consider $\mathbb{C}$ as a two dimensional real vector space and define a $\mathbb{C}$-vector space structure on $V \otimes \mathbb{C}$ by $z \cdot (v \otimes w) = v \otimes (wz)$. The endomorphism J extends in an obvious way to an endomorphism of $V_{\mathbb{C}}$. Prove that $V_{\mathbb{C}}$ is a direct sum of the eigenspaces of J.
   (c) Prove that V as a basis of the form $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$.