ASSIGNMENT 2, DUE SEP 15, 2017

Throughout, A denotes a commutative ring with an identity element, and unless specified otherwise all modules referred to below are A-modules.

1. Let E be an A-module, prove that the covariant functor $\mathrm{Hom}_A(E, \cdot)$ is *left* exact. (Make sure you state what "left exactness" means for covariant functors taking A-modules to A-modules.)
2. Let $(M_i)_{i\in I}, (N_j)_{j\in J}$ be two families of A-modules Prove that:
   (a) There is a canonical isomorphism
   $$\mathrm{Hom}_A(\bigoplus_{i\in I} M_i, \prod_{j\in J} N_j) \cong \prod_{(i,j)\in I\times J} \mathrm{Hom}_A(M_i, N_j).$$
   (b) If $(M_i')_{i\in I}, (N_j')_{j\in J}$ are another two families of A-modules, and $(u_i : M_i \to M_i')_{i\in I}, (v_j : N_j \to N_j')_{j\in J}$ families of linear maps, prove that the following diagram (where the horizontal arrrows are the canonical isomorphisms from part (a)) is commutative:
   $$\begin{array}{ccc} \mathrm{Hom}_A(\bigoplus_{i\in I} M_i', \prod_{j\in J} N_j') & \longrightarrow & \prod_{(i,j)\in I\times J} \mathrm{Hom}_A(M_i', N_j') \\ {\scriptstyle \mathrm{Hom}(\oplus_{i\in I} u_i, \prod_{j\in J} v_j)}\uparrow & & \uparrow{\scriptstyle \prod_{i,j} \mathrm{Hom}(u_i, v_j)} \\ \mathrm{Hom}_A(\bigoplus_{i\in I} M_i, \prod_{j\in J} N_j) & \longrightarrow & \prod_{(i,j)\in I\times J} \mathrm{Hom}_A(M_i, N_j) \end{array}$$
3. Prove that for a direct sum E of a family of submodules $(M_i)_{i\in I}$ to be projective, it is necessary and sufficient that each M_i is projective.
4. (a) If A has no zero-divisors, show that every element $\neq 0$ of a projective A-module is free.
   (b) Show that the $\mathbb{Z}$-module $\mathbb{Q}$ is not a projective module $\mathbb{Z}$-module.
5. An A-module I is called *injective* is for every exact sequence $E' \to E \to E''$, the sequence $\mathrm{Hom}_A(E'', I) \to \mathrm{Hom}_A(E, I) \to \mathrm{Hom}_A(E', I)$ is exact. Show that the following are equivalent.
   (a) I is injective.
   (b) For every exact sequence $0 \to E' \to E \to E'' \to 0$, the sequence $0 \to \mathrm{Hom}_A(E'', I) \to \mathrm{Hom}_A(E, I) \to \mathrm{Hom}_A(E', I) \to 0$ is exact.
   (c) For every A-module M and every sub-module N of M, every linear map from N to I can be *extended* to a linear map from M to I.
   (d) (Optional) I is a direct factor of every A-module containing it as a submodule.
   (e) (Optional) For every ideal $\mathfrak{a}$ of A, every linear map from $\mathfrak{a}$ to I can be extended to a linear map from A to I.