Throughout, $A$ denotes a commutative ring with an identity element, and unless specified otherwise all modules referred to below are $A$-modules.

1. Let $k$ be a field, $E, F$ two finite-dimensional vector spaces over $k$, and $u \in \text{Hom}_k(E, F)$. Identifying $u$ with an element of $E^* \otimes F$, prove that $\text{rk}(u) = \dim \text{Im}(u)$.

2. Let $E, F, G, H$ be four finite dimensional vector spaces over a field $k$ and $u : E \to F, v : F \to G, w : G \to H$ be linear mappings. Prove that

$$\text{rk}(v \circ u) + \text{rk}(w \circ v) \leq \text{rk}(v) + \text{rk}(w \circ v \circ u).$$

3. (Cauchy determinant). If $a_i, b_j \in k$ are such that $a_i + b_j \neq 0$ for every ordered pair $(i, j)$, prove that

$$\det\left(\frac{1}{a_i + b_j}\right) = \prod_{i < j}(a_j - a_i)(b_j - b_i) / \prod_{i,j}(a_i + b_j).$$

4. Let $X$ be an $n \times m$ matrix, and $Y$ a $p \times n$ matrix with entries in a field $k$, and let $Z = YX$. Using notation for minors used in class prove that the minors of $Z$ is given by

$$\det(Z_{L,H}) = \sum_K \det(Y_{L,K}) \det(X_{K,H}),$$

where $L, H$ have cardinality $q \leq n$, and $K$ runs through subsets of $[1, n]$ of cardinality $q$.

5. Let $\Delta$ be the determinant of a square matrix $X$ of order $n$, and $\Delta_p$ the determinant of the square matrix $\wedge^p X$ (of order $\binom{n}{p}$). Prove that,

$$\Delta_p \Delta_{n-p} = \Delta_{\binom{n}{p}}.$$

6. Let $X$ be a matrix over a field $k$. For $X$ to be of rank $p$, it suffices that there exists a minor of $X$ of order $p$ which is $\neq 0$ and such that all the minors of order $p+1$ containing this minor of order $p$ are zero. (Note that the rank of a matrix is defined to be the rank of the corresponding linear map with respect to the standard bases.)

7. (Cayley’s formula for the determinant) Let $M$ be an $n \times n$ matrix with entries in $A$. Prove that $\det(M)$ equals the sum of the following terms. The determinant of the matrix $M$ with the diagonal entries replaced by $0$, for each $n$ choice of a diagonal entry $m_{ii}$, the product of $m_{ii}$ with the determinant of the complementary $(n-1) \times (n-1)$ minor with the diagonal entries replaced by $0$, for each of the $\binom{n}{2}$ choices of $m_{i,i}, m_{jj}, i \neq j$, the product of $m_{ii}m_{jj}$ with the determinant of the complementary minor with diagonal
entries replaced by 0 and so on. In the case $n = 3$, the formula is the following:

\[
\begin{align*}
\det \begin{bmatrix}
m_{11} & m_{12} & m_{13} \\
m_{21} & m_{22} & m_{23} \\
m_{31} & m_{32} & m_{33} \\
\end{bmatrix} &= \\
\det \begin{bmatrix}
0 & m_{12} & m_{13} \\
m_{21} & 0 & m_{23} \\
m_{31} & m_{32} & 0 \\
\end{bmatrix} + \\
m_{11} \det \begin{bmatrix}
0 & m_{31} \\
m_{32} & 0 \\
\end{bmatrix} + m_{22} \det \begin{bmatrix}
0 & m_{13} \\
m_{31} & 0 \\
\end{bmatrix} + m_{33} \det \begin{bmatrix}
0 & m_{12} \\
m_{21} & 0 \\
\end{bmatrix} + \\
+m_{11}m_{22}m_{33}.
\end{align*}
\]

(Note that there is a break in the sequence while passing from $(n - 2)$-tuples of diagonal elements directly to the $n$-tuple (the terms involving $(n - 1)$-tuples is equal to 0).