1. (10 pts) Let $G$ be a group $H,N$ subgroups of $G$ and let $N$ be a normal subgroup $G$.
   (a) Prove that $H \cap N$ is a normal subgroup of $H$.
   (b) Let $NH = \{nh | n \in N, h \in H\}$ and $HN = \{hn | h \in H, n \in N\}$. Prove that $NH = HN$ and that $HN$ is a subgroup of $G$.
   (c) Observe that $N$ is a subgroup of $HN$. Prove that $N$ is a normal subgroup of $HN$. (Notice that if $G'$ is a subgroup of $G$ containing $N$, then $N$ is also a subgroup of $G'$, but not necessarily a normal subgroup of $G'$. Why ?)
   (d) (Optional) Prove that $HN/N$ is isomorphic to $H/H \cap N$. (This often goes by the name “Second Isomorphism Theorem”).

2. (10 pts) Let $G$ be a group. For $a,b \in G$, we denote by $[a,b]$ the element $aba^{-1}b^{-1}$ (called the commutator of $a$ and $b$) of $G$. Let $[G,G]$ denote the set of elements of $G$ which are each a product of a finite number of commutators. Thus, every element of $[G,G]$ is of the form $[a_1,b_1] \cdots [a_m,b_m]$ for some $m \geq 0$, and $a_1,b_1,\ldots,a_m,b_m \in G$.
   (a) Prove that the inverse of a commutator is again a commutator.
   (b) Prove that $[G,G]$ is a normal subgroup of $G$ (this subgroup is called the commutator subgroup of $G$, or sometimes the first derived group, $D^0(G)$, of $G$).
   (c) What is the subgroup $[G,G]$ in the case $G$ is abelian ?
   (d) Prove that the quotient group $G/[G,G]$ is always abelian. (The group $G/[G,G]$ is often called the abelianization of $G$. The next two exercises show that $[G,G]$ is the smallest normal subgroup of $G$ such that quotienting by it gives an abelian group.)
   (e) Prove that if $\phi : G \to A$ is a group homomorphism of $G$ to an abelian group $A$, then $[G,G] \subseteq \ker(\phi)$.
   (f) Suppose that $N$ is a normal subgroup of $G$ such that $G/N$ is abelian. Prove that $[G,G] \subseteq N$.
   (g) Let $G$ be the dihedral group, $D_8$, of order 8. Compute $[G,G]$ and $G/[G,G]$.

3. (10 pts) Let $U$ be the subset of $GL(3,\mathbb{R})$ consisting of all elements which are upper triangular and with 1’s on the diagonal. Thus,

$$ U = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a,b,c \in \mathbb{R} \right\}, $$
and let

\[ V = \left\{ \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid b \in \mathbb{R} \right\}. \]

(a) Prove that \( U \) is a subgroup of \( \text{GL}(3, \mathbb{R}) \).

(b) Prove that \( V \) is a normal subgroup of \( U \), and the quotient group \( U/V \) is isomorphic to the additive group \( \mathbb{R}^2 \). (Hint. Use the first isomorphism theorem).

4. (10 pts) Consider the action by conjugation of the group \( D_8 \) on itself. Thus, using the notation used in class, \( G = D_8 \), \( X = D_8 \), and the action is defined by \( g \cdot x = gxg^{-1} \) for all \( g \in G, x \in x \). List all the orbits of this action.