ASSIGNMENT 7. DUE IN CLASS THUS NOV 15, 2018.

The goal of this multi-part assignment is to give a proof of the existence of $p$-Sylow subgroups. Recall from class, that if $p$ is a prime number, and $G$ a finite group, and $r$ the largest power of $p$ that divides $|G|$, then a subgroup of $G$ of order $p^r$ is called a $p$-Sylow subgroup of $G$. For the rest of the assignment $p$ is a fixed prime number.

1. Let $G$ be a finite group, $H$ a normal subgroup of $G$ of order $p$. Suppose that $G/H$ has a subgroup of order $n$. Prove that $G$ has a subgroup of order $pn$. (Hint. Let $K$ be the subgroup of $G/H$ of order $n$, and $f : G \to G/H$ the canonical surjection. Consider the inverse image of $K$ under $f$.)

2. Prove that if $Z$ is a finite abelian group, and $p$ divides $|Z|$, then $Z$ contains an element whose order is $p$. (Hint. Use induction on the order of $Z$. Start with a non-identity element $z \in Z$, and consider the subgroup $Z'$ generated by $z$. If this subgroup is equal to $Z$, then ... Otherwise, if $p$ divides $|Z'|$, you can use the inductive hypothesis. If $p$ does not divide $|Z'|$, then consider the quotient group $Z/Z'$, but be careful about what you conclude from the induction hypothesis in this case.)

3. Let $G$ be a finite group such that $p$ divides $|G|$. Prove that $G$ has a $p$-Sylow subgroup. (Hint. Use induction on $|G|$. If $|G| = p$, then there is nothing to prove. Otherwise, use the class equation discussed in class (unintended pun here). There are two cases. If there exists a stabilizer subgroup $G_x, x \notin Z(G)$, whose order is divisible by $p^r$ (where $r$ is the largest power of $p$ dividing $G$), then we are done since $|G_x| < |G|$ (why ?). Else, prove that $G$ has a non-trivial center whose order is divisible by $p$. Apply part (2) to deduce that there exists a subgroup of order $p$ contained in $Z(G)$. This subgroup is a normal subgroup of $G$ (why ?). Now proceed again using induction and part (1).)