Assignment 2, due Tues, Sep 18.

September 10, 2018

1. The purpose of this exercise is to construct invariant volume forms on concrete Lie groups.

   a) A differential $k$-form $\omega$ on a Lie Group $G$ is said to be left invariant if for each $g, g_0 \in G$, if under the pull back map $L_{g_0}^* : \bigwedge^k T_{g_0}^* \rightarrow \bigwedge^k T_g^*$, $\omega(g_0 g)$ maps to $\omega(g)$. Let $G = \text{GL}(n, \mathbb{R})$. Consider the $n \times n$ matrix of 1-forms $\Omega = g^{-1} dg$ for $g \in \text{GL}(n, \mathbb{R})$. Prove that the entries of $\Omega$ are left invariant 1-forms.

   b) Now let $n = 2$. Prove that $\frac{da_{11}\wedge da_{12}\wedge da_{21}\wedge da_{22}}{(a_{11}a_{22} - a_{21}a_{12})^2}$ is a left invariant volume form on $\text{GL}(2, \mathbb{R})$.

   c) Define what it means for a volume form to be right invariant and prove that the volume form constructed in the previous exercise is also right invariant. (A Lie group is called unimodular if its left invariant volume form is also right invariant. Compact Lie groups and also commutative Lie groups are unimodular. The above calculation shows that $\text{GL}(2, \mathbb{R})$ (and in fact $\text{GL}(n, \mathbb{R})$) is also unimodular, though it is neither compact or commutative).

   d) Let $G \subset \text{GL}(2, \mathbb{R})$ be the subgroup of matrices $g = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$. Find the invariant volume of $G$ using the above method. (Being a commutative group $G$ is unimodular).

   e) Let $G$ be the subgroup of upper-triangular matrices of $\text{GL}(n, \mathbb{R})$ (you can take $n = 2$). Compute the left and right invariant volume forms on $G$. Is $G$ unimodular?

2. Write out with full details the classification of all finite dimensional irreducible representations of $\mathfrak{so}(2, \mathbb{C})$ discussed in class.

3. (Extra credit) Consider the group $\text{SO}(3, \mathbb{R})$.

   a) Prove that every element of $g \in \text{SO}(3, \mathbb{R})$ can be represented as a product $g(\phi, \theta, \psi) = R_z(\phi)R_y(\theta)R_z(\psi)$,
where $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$, and

$$R_z(\phi) = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_y(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix},$$

are the rotations around the $z$ and $y$ axes respectively. The parameters $\phi, \theta, \psi$ are called the Euler angles.

b) Prove that

$$\sin(\theta) d\phi \wedge d\theta \wedge d\psi$$

is an invariant volume form on $SO(3, \mathbb{R})$. 
