Assignment 3, due Oct 23.

October 2, 2018

1. Let $\mathfrak{g}$ be a two dimensional Lie algebra. Prove that $\text{rad}(B_\mathfrak{g}) \neq \text{rad}(\mathfrak{g})$.

2. Prove that if $\mathfrak{g}$ is a finite dimensional nilpotent Lie algebra over $\mathbb{R}$, then the Killing form of $\mathfrak{g}$ is identically 0.

3. Show that if $\mathfrak{g}$ is a solvable Lie algebra over $\mathbb{R}$, and $\mathfrak{n}$ is its largest nilpotent ideal, the $\mathfrak{g}/\mathfrak{n}$ is abelian.

4. Let $\mathfrak{g}$ be a complex Lie algebra of complex matrices, and suppose that $\mathfrak{g}$ is simple over $\mathbb{C}$. Let $C(X,Y) = \text{Tr}(XY)$ for $X,Y \in \mathfrak{g}$. Prove that $C$ is a multiple of the Killing form of $\mathfrak{g}$.

5. Write a complete proof of the PBW theorem.

6. Prove that for a complex finite dimensional semi-simple Lie algebra $\mathfrak{g}$, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

7. Recall that if $A$ is a $\mathbb{C}$-algebra, then a derivation $D : A \to A$ is a linear map satisfying $D(X \cdot Y) = D(X) \cdot Y + X \cdot D(Y)$. The set of derivations, $\text{Der}(A)$ form a Lie subalgebra of $\text{End}(A)$. Now let $A = \mathfrak{g}$ be a complex semi-simple Lie algebra.
   a) Prove that for $X \in \mathfrak{g}$, $\text{ad}X$ is a derivation.
   b) Prove that every derivation of $\mathfrak{g}$ is of the form $\text{ad}X$ for some $X \in \mathfrak{g}$.
   c) Conclude that $\text{ad}$ induces a Lie algebra isomorphism between $\mathfrak{g}$ and $\text{Der}(\mathfrak{g})$.

8. Write a complete proof of the PBW theorem.

9. Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra, $R = U(\mathfrak{g})$, and $V$ an $R$-module. Let $X_n = R \otimes_\mathbb{C} \wedge^n \mathfrak{g}$. Prove that the sequence of homomorphisms $\partial_{n-1} : X_n \to X_{n-1}$ defined in class gives a free resolution of the $R$-module $\mathbb{C}$.