Throughout, $A$ denotes a commutative ring with an identity element, and unless specified otherwise all modules referred to below are $A$-modules.

1. Let $A = k$ be a field and $E$ a $k$-vector space. Let $A, B$ be two finite bases of $E$. Prove that $\text{card}(A) = \text{card}(B)$.

2. Suppose that $0 \longrightarrow E' \xrightarrow{u} E \xrightarrow{v} E'' \longrightarrow 0$ is an exact sequence of $A$-modules. Prove that the sequence splits if and only if the homomorphism $v$ admits a section. (Recall that a section $s$ of $v$ is an element of $\text{Hom}_A(E'', E)$ such that $v \circ s = 1_{E''}$.)

3. Recall that an $A$-module $E$ is called cyclic (or monogeneous) if it is generated by a single element. Prove the following.
   (a) Every simple module $M$ (i.e. a module not having any submodules other than 0 and itself) is cyclic.
   (b) Every sub-module of a cyclic module is isomorphic to a quotient module $b/a$ where $a, b$ are two left ideals of $A$ with $a \subset b$.
   (c) Every quotient module of a cyclic module is cyclic.
   (d) Sub-modules of cyclic modules are not necessarily cyclic.

4. Let $(M_i)_{i \in I}, (N_j)_{j \in J}$ be two families of $A$-modules. Prove that:
   (a) There is a canonical isomorphism
   $$\text{Hom}_A(\bigoplus_{i \in I} M_i, \prod_{j \in J} N_j) \cong \prod_{(i,j) \in I \times J} \text{Hom}_A(M_i, N_j).$$
   (b) If $(M'_i)_{i \in I}, (N'_j)_{j \in J}$ are another two families of $A$-modules, and $(u_i : M_i \to M'_i)_{i \in I}, (v_j : N_j \to N'_j)_{j \in J}$ families of linear maps, prove that the following diagram (where the horizontal arrows are the canonical isomorphisms from part (a)) is commutative:

\[
\begin{array}{ccc}
\text{Hom}_A(\bigoplus_{i \in I} M_i, \prod_{j \in J} N'_j) & \longrightarrow & \prod_{(i,j) \in I \times J} \text{Hom}_A(M'_i, N'_j) \\
\text{Hom}_A(\bigoplus_{i \in I} M_i, \prod_{j \in J} N_j) & \longrightarrow & \prod_{(i,j) \in I \times J} \text{Hom}_A(M_i, N_j)
\end{array}
\]