Throughout, $A$ denotes a commutative ring with an identity element, and unless specified otherwise all modules referred to below are $A$-modules.

1. Let $a_1, \ldots, a_p$ be ideals, $a = \sum_{i=1}^p a_i$, and for $1 \leq i \leq p$, $E_i = A/a_i$. Prove that,
\[ E_1 \otimes_A \cdots \otimes_A E_p \cong A/a \]
as $A$-modules.

2. Suppose that $a_1 \subset \cdots \subset a_n$ is an ascending sequence of ideals of $A$, $E$ the $A$-module $A/a_1 \oplus \cdots \oplus A/a_n$, and $1 \leq p \leq n$. Using Part (1) prove that, $a_p$ is the annihilator ideal of $\wedge^p E$.

3. Let $A$ be a PID, and let $U = (a_{ij}) \in A^{m \times n}$.
   (a) Suppose that the entries of $U$ are set-wise coprime. Show that there exists two invertible matrices $P$ and $Q$ with entries in $A$ such that one of the entries in the matrix $PUQ$ is equal to $1$.
   (b) If $\delta_1$ is a gcd of the entries of $U$, show that there exist two invertible matrices $P_1$ and $Q_1$ such that
   \[ P_1 U Q_1 = \begin{bmatrix} \delta_1 & 0 \\ 0 & U_1 \end{bmatrix}, \]
   where all entries of $U_1$ are divisible by $\delta_1$ (use Part (3a)).
   (c) For $A = \mathbb{Z}$, use Part (3b) to obtain a method of calculating the invariant factors of an explicit matrix with entries in $\mathbb{Z}$. Apply this method to the matrix,
   \[
   \begin{bmatrix}
   6 & 8 & 4 & 24 \\
   12 & 12 & 18 & 30 \\
   18 & 4 & 4 & 10 
   \end{bmatrix}.
   \]

4. Let $C$ be a PID. Consider a finite system of linear equations $Ax = b$, where $A \in C^{m \times n}$. Prove that the system has at least one solution in $C^n$ if and only if the following conditions are satisfied.
   (i) The matrices $A$ and $B = [A|b]$ have the same rank $p$.
   (ii) The gcd of all minors of order $p$ of $A$ is equal to the gcd of all minors of order $p$ of $B$. 