Assignment 4

In the following $R$ is a real closed field.

1. Let $S$ be a semi-algebraic subset of $R\langle \varepsilon \rangle^k$ that is semi-algebraically connected and bounded over $R$. Prove that $\lim_{\varepsilon} S$ is is a semi-algebraically connected, closed and bounded semi-algebraic subset of $R^k$.

2. If $S \subset R^k$ is a semi-algebraically connected semi-algebraic set, prove that $\text{Ext}(S, R\langle \varepsilon \rangle)$ is again semi-algebraically connected.

3. Let $H_{n,k} \subset R^{n \times n}$ denote the set of matrices $A$ with
   \[ A = A^t, A^2 = A, \text{tr}(A) = k. \]

4. Let $\Psi: \text{Gr}_R(n, k) \to H_{n,k}$ denote the map sending a $k$-dimensional subspace $V$ of $R^n$, to the matrix $\Psi(V)$ corresponding to the orthogonal projection onto $V$ (with respect to the standard basis and inner product in $R^n$). Prove that $\Psi$ is an isomorphism of real affine algebraic varieties.

5. Prove that $\text{Gr}_R(n, k)$ is semi-algebraically connected.

6. Recall that two real affine algebraic varieties, $X$ and $Y$ are isomorphic if their is a bi-regular isomorphism between them i.e. there exists a bijective regular map $f: X \to Y$, whose inverse is also regular. Give an example where $f$ is a bijective regular map, but whose inverse is not regular (hence, $f$ is not an isomorphism).

7. Prove that $\text{Gr}_R(n, k)$ and $\text{Gr}_R(n, n-k)$ are isomorphic as real affine algebraic varieties.

8. Let $C = R[i]$ and let $\text{Gr}_C(n, k)$ denote the Grassmannian of (complex) $k$ dimensional subspaces of $C^n$. Prove that:
   i. $\text{Gr}_C(n, k)$ is a complex projective variety (use Plucker embedding);
   ii. $\text{Gr}_C(n, k)$ also has a structure of a real affine algebraic variety.

9. A semi-algebraically connected component $C$ of a semi-algebraic set $S$ is a maximal, non-empty semi-algebraic subset of $S$. Now let $S \subset R^k$, be a semi-algebraic subset defined by a quantifier-free $\mathcal{P}$-formula, where the number of polynomials in $\mathcal{P}$ is bounded by $s$, and their degrees bounded by $d$. Suppose $k = 1$. What is the maximum possible number of distinct semi-algebraically connected components of $S$? What about in the case $k = 2$?