THEOREM 1.1.1 *The distinct equivalence classes of an equivalence relation on A provide us with a decomposition of A as a union of mutually disjoint subsets. Conversely, given a decomposition of A as a union of mutually disjoint, nonempty subsets, we can define an equivalence relation on A for which these subsets are the distinct equivalence classes.*

Proof. Let the equivalence relation on A be denoted by $\sim$.

We first note that since for any $a \in A$, $a \sim a$, a must be in $\mathrm{cl}(a)$, whence the union of the $\mathrm{cl}(a)$'s is all of A. We now assert that given two equivalence classes they are either equal or disjoint. For, suppose that $\mathrm{cl}(a)$ and $\mathrm{cl}(b)$ are not disjoint; then there is an element $x \in \mathrm{cl}(a) \cap \mathrm{cl}(b)$. Since $x \in \mathrm{cl}(a)$, $a \sim x$; since $x \in \mathrm{cl}(b)$, $b \sim x$, whence by the symmetry of the relation, $x \sim b$. However, $a \sim x$ and $x \sim b$ by the transitivity of the relation forces $a \sim b$. Suppose, now that $y \in \mathrm{cl}(b)$; thus $b \sim y$. However, from $a \sim b$ and $b \sim y$, we deduce that $a \sim y$, that is, that $y \in \mathrm{cl}(a)$. Therefore, every element in $\mathrm{cl}(b)$ is in $\mathrm{cl}(a)$, which proves that $\mathrm{cl}(b) \subset \mathrm{cl}(a)$. The argument is clearly symmetric, whence we conclude that $\mathrm{cl}(a) \subset \mathrm{cl}(b)$. The two opposite containing relations imply that $\mathrm{cl}(a) = \mathrm{cl}(b)$.

We have thus shown that the distinct $\mathrm{cl}(a)$'s are mutually disjoint and that their union is A. This proves the first half of the theorem. Now for the other half!

Suppose that $A = \bigcup A_\alpha$ where the A_α are mutually disjoint, nonempty sets (α is in some index set T). How shall we use them to define an equivalence relation? The way is clear; given an element a in A it is in *exactly one* A_α. We define for $a, b \in A$, $a \sim b$ if a and b are in the same A_α. We leave it as an exercise to prove that this is an equivalence relation on A and that the distinct equivalence classes are the A_α's.

Problems

1. (a) If A is a subset of B and B is a subset of C, prove that A is a subset of C.
   (b) If $B \subset A$, prove that $A \cup B = A$, and conversely.
   (c) If $B \subset A$, prove that for any set C both $B \cup C \subset A \cup C$ and $B \cap C \subset A \cap C$.
2. (a) Prove that $A \cap B = B \cap A$ and $A \cup B = B \cup A$.
   (b) Prove that $(A \cap B) \cap C = A \cap (B \cap C)$.
3. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
4. For a subset C of S let C' denote the complement of C in S. For any two subsets A, B of S prove the *De Morgan rules*:
   (a) $(A \cap B)' = A' \cup B'$.
   (b) $(A \cup B)' = A' \cap B'$.
5. For a finite set C let $o(C)$ indicate the number of elements in C. If A and B are finite sets prove $o(A \cup B) = o(A) + o(B) - o(A \cap B)$.

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ate the number of elements in C. If A $\cup B) = o(A) + o(B) - o(A \cap B)$.

6. If A is a finite set having n elements, prove that A has exactly 2^n distinct subsets.
7. A survey shows that 63% of the American people like cheese whereas 76% like apples. What can you say about the percentage of the American people that like both cheese and apples? (The given statistics are not meant to be accurate.)
8. Given two sets A and B their *symmetric difference* is defined to be $(A - B) \cup (B - A)$. Prove that the symmetric difference of A and B equals $(A \cup B) - (A \cap B)$.
9. Let S be a set and let S^* be the set whose elements are the various subsets of S. In S^* we define an addition and multiplication as follows: If $A, B \in S^*$ (remember, this means that they are subsets of S):
   (1) $A + B = (A - B) \cup (B - A)$.
   (2) $A \cdot B = A \cap B$.
   Prove the following laws that govern these operations:
   (a) $(A + B) + C = A + (B + C)$.
   (b) $A \cdot (B + C) = A \cdot B + A \cdot C$.
   (c) $A \cdot A = A$.
   (d) $A + A =$ null set.
   (e) If $A + B = A + C$ then $B = C$.
   (The system just described is an example of a *Boolean algebra*.)
10. For the given set and relation below determine which define equivalence relations.
    (a) S is the set of all people in the world today, $a \sim b$ if a and b have an ancestor in common.
    (b) S is the set of all people in the world today, $a \sim b$ if a lives within 100 miles of b.
    (c) S is the set of all people in the world today, $a \sim b$ if a and b have the same father.
    (d) S is the set of real numbers, $a \sim b$ if $a = \pm b$.
    (e) S is the set of integers, $a \sim b$ if both $a > b$ and $b > a$.
    (f) S is the set of all straight lines in the plane, $a \sim b$ if a is parallel to b.
11. (a) Property 2 of an equivalence relation states that if $a \sim b$ then $b \sim a$; property 3 states that if $a \sim b$ and $b \sim c$ then $a \sim c$. What is wrong with the following proof that properties 2 and 3 imply property 1? Let $a \sim b$; then $b \sim a$, whence, by property 3 (using $a = c$), $a \sim a$.
    (b) Can you suggest an alternative of property 1 which will insure us that properties 2 and 3 do imply property 1?
12. In Example 1.1.3 of an equivalence relation given in the text, prove that the relation defined is an equivalence relation and that there are exactly n distinct equivalence classes, namely, cl(0), cl(1), . . . , cl($n - 1$).
13. Complete the proof of the second half of Theorem 1.1.1.