ASSIGNMENT 8. DUE IN CLASS THUS MAR 21, 2019.

The goal of this multi-part assignment is to give a proof of the existence of \( p \)-Sylow subgroups. Recall from class, that if \( p \) is a prime number, and \( G \) a finite group, and \( r \) the largest power of \( p \) that divides \(|G|\), then a subgroup of \( G \) of order \( p^r \) is called a \( p \)-Sylow subgroup of \( G \). For the rest of the assignment \( p \) is a fixed prime number.

1. Let \( G \) be a finite group, \( H \) a normal subgroup of \( G \) of order \( p \). Suppose that \( G/H \) has a subgroup of order \( n \). Prove that \( G \) has a subgroup of order \( pn \). (Hint. Let \( K \) be the subgroup of \( G/H \) of order \( n \), and \( f : G \to G/H \) the canonical surjection. Consider the inverse image of \( K \) under \( f \).)

2. Prove that if \( Z \) is a finite abelian group, and \( p \) divides \(|Z|\), then \( Z \) contains an element whose order is \( p \). (Hint. Use induction on the order of \( Z \). Start with a non-identity element \( z \in Z \), and consider the subgroup \( Z' \) generated by \( z \). If this subgroup is equal to \( Z \), then ... Otherwise, if \( p \) divides \(|Z'|\), you can use the inductive hypothesis. If \( p \) does not divide \(|Z'|\), then consider the quotient group \( Z/Z' \), but be careful about what you conclude from the induction hypothesis in this case.)

3. Let \( G \) be a finite group such that \( p \) divides \(|G|\). Prove that \( G \) has a \( p \)-Sylow subgroup. (Hint. Use induction on \(|G|\). If \(|G| = p \), then there is nothing to prove. Otherwise, use the class equation discussed in class (unintended pun here). There are two cases. If there exists a stabilizer subgroup \( G_x, x \notin Z(G) \), whose order is divisible by \( p^r \) (where \( r \) is the largest power of \( p \) dividing \( G \)), then we are done since \(|G_x| < |G| \) (why ?). Else, prove that \( G \) has a non-trivial center whose order is divisible by \( p \). Apply part (2) to deduce that there exists a subgroup of order \( p \) contained in \( Z(G) \). This subgroup is a normal subgroup of \( G \) (why ?). Now proceed again using induction and part (1).)