PRACTICE MIDTERM 2, SPRING 2019

Please write your name in the top left corner. Attempt all questions. Time 50 min.

1. (20 pts)
   (i) Let $G$ be a group, $H$ be a subgroup of $G$ and $g \in G$. Suppose that $gHg^{-1} \subseteq H$.
      Is it always true that in this case $gHg^{-1} = H$?
   (ii) Is the group $GL(3, \mathbb{R})$ abelian?
   (iii) Is the group $GL(3, \mathbb{R})/SL(3, \mathbb{R})$ abelian?
   (iv) How many distinct subgroups does the group $Z_{24}$ have?
   (v) How many elements of $Z_{1000000}$ have order 20?
   (vi) Let $p$ be a prime number. Can a group of order $p^3$ be non-abelian?
   (vii) Is it always true that the center, $Z(G)$, of a group $G$ is a normal subgroup of $G$?
   (viii) Is it true that every non-abelian group has a non-trivial center?
   (ix) Let $p$ be a prime. Is it true that every group of order $p^5$ has a non-trivial center?
   (x) Let $G$ be a finite abelian group. Is it true that the number of conjugacy classes of $G$ is always strictly smaller than the order of $G$?
   (xi) Is the image of a group homomorphism $f : G \rightarrow H$ always a normal subgroup of $H$?
   (xii) Is the kernel of a group homomorphism always a normal subgroup of $H$?
   (xiii) Is it true that every normal subgroup of a group $G$ is the kernel of some homomorphism $f : G \rightarrow H$?
   (xiv) Is the quotient of a group of $G$ by its center always an abelian group?
   (xv) Is the quotient of a group of $G$ by its commutator subgroup always an abelian group?
   (xvi) Is the group of inner automorphisms of a group always an abelian group?
   (xvii) Let $G$ be a acting on a set $X$, and $x, x' \in X$ such that $x \in \text{orbit}(x')$. Is it true that $G_x = G_{x'}$?
   (xviii) Let $G$ be a acting on a set $X$, and $x, x' \in X$ such that $G_x = G_{x'}$?. Is it true that then $x = x'$?
   (xix) Let $G$ be a finite group acting on a set $X$. Is it true that the number of orbits has to always divide the order of the group.
   (xx) Let $G$ be a acting on a set $X$, and $x \in X$. Is it always true that $G_x$ is a normal subgroup of $G$?
   (xxi) How many conjugacy classes does the symmetric group $S_5$ have?
   (xxii) For each $n \geq 2$, is the alternating group $A_n$ always a normal subgroup of $S_n$?
   (xxiii) Is the element $(1, 2, 3)(4, 5) \in S_n$ an even permutation?
   (xxiv) How many elements are there in the conjugacy class of the element $(1, 2, 3)(4, 5) \in S_5$?
(xxv) How many elements are there in the conjugacy class of the element $(1, 2, 3)(4, 5) \in S_6$?

2. (2+8 pts)
   (a) State the class equation.
   (b) Using the class equation prove that if $p$ is a prime, then every group whose order is a positive power of $p$ has a nontrivial center.

3. (10 pts) Let $G$ be a finite group acting on a set $X$. Prove that the number of orbits equals the quantity $\frac{1}{|G|} \sum_{g \in G} |X^g|$, where for $g \in G$, $X^g$ denotes the number of fixed points of $G$.

4. (10 pts) Prove that for every $n \geq 1$,

   $$n = \sum_{k|n} \phi(k),$$

   where $\phi$ denotes the Euler totient function.

5. (2+8 pts) Let $G$ be a group.
   (a) Define the groups of automorphisms and inner automorphisms of $G$.
   (b) Prove that the group of inner automorphisms of $G$ is isomorphic to the quotient of $G$ by its center.

6. (10 pts)
   (a) Define even and odd permutations in $S_n$.
   (b) Prove that the number of even permutations equal to the number of odd permutations in $S_n$. 