POLYNOMIAL HIERARCHY, BETTI NUMBERS AND A REAL ANALOGUE OF TODA'S THEOREM

SAUGATA BASU AND THIERRY ZELL

Abstract. We study the relationship between the computational hardness of two well-studied problems in algorithmic semi-algebraic geometry – namely the problem of deciding sentences in the first order theory of reals with a constant number of quantifier alternations, and that of computing Betti numbers of semi-algebraic sets. We obtain a polynomial time reduction of the compact version of the first problem to the second. As a consequence we obtain an analogue of Toda's theorem from discrete complexity theory for real Turing machines (in the sense of Blum, Shub and Smale).

1. Introduction and Main Results

In this paper we study the relationship between the computational hardness of two important classes of problems in algorithmic semi-algebraic geometry. Algorithmic semi-algebraic geometry is concerned with designing efficient algorithms for deciding geometric as well as topological properties of semi-algebraic sets. There is a large body of research in this area (see [4] for background). If we consider the most important algorithmic problems studied in this area (see for instance the survey article [3]), it is possible to classify them into two broad sub-classes. The first class consists of the problem of quantifier elimination, and its special cases such as deciding a sentence in the first order theory of reals, or deciding emptiness of semi-algebraic sets (also often called the existential theory of the reals). The existence of algorithms for solving these problems was first proved by Tarski [24] and later research has aimed at designing algorithms with better complexities [22, 18, 17, 6, 1].

The second class of problems in algorithmic semi-algebraic geometry that has been widely investigated consist of computing topological invariants of semi-algebraic sets, such as counting the number of connected components, computing the Euler-Poincaré characteristic, and more generally all the Betti numbers of semi-algebraic sets [11, 19, 16, 2, 8, 5]. Note that the properties such as connectivity or the vanishing of some Betti number of a semi-algebraic set is not expressible in first-order logic, and thus the existence of algorithms for deciding such properties, is not an immediate consequence of the Tarski's result but usually requires some additional topological ingredients such as semi-algebraic triangulations or Morse theory etc. Even though the most efficient algorithms for computing the Betti numbers of a semi-algebraic set uses efficient algorithms for quantifier elimination in an essential way [5, 7], the exact relationship between these two classes of problems has not

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been clarified from the point of view of computational complexity and doing so is one of the motivations of this paper.

The primary motivation for this paper comes from classical (i.e. discrete) computational complexity theory. In classical complexity theory, there is a seminal result due to Toda [25] linking the complexity of counting with that of deciding sentences with a fixed number of quantifier alternations.

More precisely, Toda's theorem gives the following inclusion (see [21] for precise definitions of the complexity classes appearing in the theorem).

Theorem 1.1 (Toda [25]).

$$\mathbf{PH} \subset \mathbf{P}^{\#\mathbf{P}}.$$

From the point of view of computational complexity theory of real Turing machines (in the sense of Blum-Shub-Smale [9]), the classes **PH** and #**P** appearing in the two sides of the inclusion in Theorem 1.1 can be identified with the two broad classes of problems in algorithmic semi-algebraic geometry discussed previously, viz. the polynomial hierarchy with the problem of deciding sentences with a fixed number of quantifier alternations, and the class #**P** with the problem of computing certain topological invariants of semi-algebraic sets, namely their Betti numbers which generalize the notion of cardinality for finite sets. (This naive intuition is made more precise below.) It is thus quite natural to seek a real analogue of Toda's theorem.

In order to formulate such a result it is first necessary to define precisely real counter-parts of the discrete polynomial time hierarchy **PH** and the discrete complexity class #**P**, and this is what we do next.

1.1. Real counter-parts of PH and #P.

For the rest of the paper R will denote a real closed field (there is no essential loss in assuming that $\mathrm{R} = \mathbb{R}$). By a real Turing machine we will mean a Turing machine in the sense of Blum-Shub-Smale [9] over the ground field R.

1.1.1. *Real analogue of* PH.

The following definitions are well known.

A sequence of semi-algebraic sets $(S_n \subset \mathrm{R}^n)_{n>0}$ is said to be in the complexity class $\mathbf{\Sigma}_{\mathrm{R},\omega}$, if for each n the semi-algebraic set S_n is described by a first order formula

$$(Q_1 X^1) \cdots (Q_\omega X^\omega) \phi_n(X^1, \ldots, X^\omega, Z_1, \ldots, Z_n),$$

with ϕ_n a quantifier free formula in the first order theory of the reals, and for each $i, 1 \le i \le \omega$, $X^i = (X^i_1, \ldots, X^i_{k_i})$ is a block of k_i variables, with

$$\sum_{i=1}^{\omega} k_i = k = n^{O(1)},$$

$Q_i \in \{\exists, \forall\}$, with $Q_j \neq Q_{j+1}, 1 \le j < \omega$, $Q_1 = \exists$, and there exists a polynomial time real Turing machine M testing membership to the semi-algebraic sets $(T_n \subset \mathrm{R}^{k+n})_{n>0}$ defined by the sequence $(\phi_n)_{n>0}$.

Similarly, we call a sequence of semi-algebraic sets $(S_n \subset \mathrm{R}^n)_{n>0}$ to be in the complexity class $\mathbf{\Pi}_{\mathrm{R},\omega}$, if for each n the semi-algebraic set S_n is described by a first order formula

$$(Q_1 X^1) \cdots (Q_\omega X^\omega) \phi_n(X^1, \ldots, X^\omega, Z_1, \ldots, Z_n),$$

with ϕ_n a quantifier free formula in the first order theory of the reals, and for each $i, 1 \leq i \leq \omega$, $X^i = (X^i_1, \ldots, X^i_{k_i})$ is a block of k_i variables, with

$$\sum_{i=1}^{\omega} k_i = k = n^{O(1)},$$

$Q_i \in \{\exists, \forall\}$, with $Q_j \neq Q_{j+1}, 1 \leq j < \omega$, $Q_1 = \forall$, and there exists a polynomial time real Turing machine M testing membership to the semi-algebraic sets $(T_n \subset \mathrm{R}^{k+n})_{n>0}$ defined by the sequence $(\phi_n)_{n>0}$.

Note that by the above definition the class $\mathbf{\Sigma}_{\mathrm{R},0} = \mathbf{\Pi}_{\mathrm{R},0}$ is the familiar class $\mathbf{P}_{\mathrm{R}}$, the class $\mathbf{\Sigma}_{\mathrm{R},1} = \mathbf{NP}_{\mathrm{R}}$ and the class $\mathbf{\Pi}_{\mathrm{R},1} = \text{co-}\mathbf{NP}_{\mathrm{R}}$.

Definition 1.2 (Real polynomial hierarchy)**.** The real polynomial time hierarchy is defined to be the union

$$\mathbf{PH}_{\mathrm{R}} \stackrel{\text{def}}{=} \bigcup_{\omega \geq 0} (\mathbf{\Sigma}_{\mathrm{R},\omega} \cup \mathbf{\Pi}_{\mathrm{R},\omega}) = \bigcup_{\omega \geq 0} \mathbf{\Sigma}_{\mathrm{R},\omega} = \bigcup_{\omega \geq 0} \mathbf{\Pi}_{\mathrm{R},\omega}.$$

For technical reasons (see Remark 2.5) we need to restrict to compact semi-algebraic sets, and for this purpose we define compact analogues of the classes defined above.

Definition 1.3. We call $K \subset \mathrm{R}^n$ a *semi-algebraic compact* if it is a closed and bounded semi-algebraic set. (Note that if $\mathrm{R} \neq \mathbb{R}$, K is not necessarily compact in the order topology.)

Definition 1.4. We call a quantifier-free formula in the first order theory of R to be *closed* if its atoms are of the form $P \geq 0, P \leq 0, P = 0$, and the formula contains only conjunctions and disjunctions (no negations or implications). Given a quantifier free formula $\Phi(X_1, \ldots, X_k)$ we will denote by $\mathcal{R}(\Phi)$ its realization in R^k, that is

$$\mathcal{R}(\Phi) = \{\mathbf{x} = (x_1, \ldots, x_k) \mid \Phi(\mathbf{x}) \text{ is true}\}.$$

If Φ is a closed formula then it is clear that $\mathcal{R}(\Phi)$ is a closed semi-algebraic subset of R^k and $\mathcal{R}(\neg\Phi)$ is an open semi-algebraic subset of R^k.

We also use the following notation.

Notation 1. We denote by $\mathbf{B}^k(0, r)$ the closed ball in R^k of radius r centered at the origin. We will denote by $\mathbf{B}^k$ the closed unit ball $\mathbf{B}^k(0, 1)$. Similarly, we denote by $\mathbf{S}^k(0, r)$ the sphere in R^{k+1} of radius r centered at the origin, and by $\mathbf{S}^k$ the unit sphere $\mathbf{S}^k(0, 1)$.

Notation 2. For any semi-algebraic set $S \subset \mathrm{R}^k$ we denote by $b_i(S)$ the i-th Betti number (that is the rank of the singular homology group $\mathrm{H}_i(S, \mathbb{Z})$) of S. Note that $b_0(S)$ is the number of semi-algebraically connected components of S, and in case S is finite $\#(S) = b_0(S)$.

We now define a *compact* analogue of the real polynomial hierarchy $\mathbf{PH}_{\mathrm{R}}$. Unlike in the non-compact case, we will assume all variables vary over certain compact semi-algebraic sets (namely spheres of varying dimensions). More precisely:

Definition 1.5 (Compact real polynomial hierarchy)**.** We call a sequence of semi-algebraic sets $(S_n \subset \mathbf{S}^n)_{n>0}$ to be in the complexity class $\mathbf{\Sigma}^c_{\mathrm{R},\omega}$, if for each n the semi-algebraic set S_n is described by a first order formula

$$(Q_1 X^1 \in \mathbf{S}^{k_1}) \cdots (Q_\omega X^\omega \in \mathbf{S}^{k_\omega}) \phi_n(X^1, \ldots, X^\omega, Z_0, \ldots, Z_n),$$

with ϕ_n a quantifier-free first order formula defining a *closed* semi-algebraic subset of $\mathbf{S}^{k_1} \times \cdots \times \mathbf{S}^{k_\omega} \times \mathbf{S}^n$ and for each $i, 1 \leq i \leq \omega$, $X^i = (X^i_0, \ldots, X^i_{k_i})$ is a block of $k_i + 1$ variables, with

$$k = \sum_{i=1}^{\omega} (k_i + 1) = n^{O(1)},$$

$Q_i \in \{\exists, \forall\}$, with $Q_j \neq Q_{j+1}, 1 \leq j < \omega$, $Q_1 = \exists$, and there exists a polynomial time real Turing machine M which tests membership in the semi-algebraic sets $(T_n \subset \mathbf{S}^{k_1} \times \cdots \times \mathbf{S}^{k_\omega} \times \mathbf{S}^n)_{n>0}$ defined by the formulas $(\phi_n)_{n>0}$.

We define analogously the class $\mathbf{\Pi}^c_{\mathrm{R},\omega}$, and finally define the *compact real polynomial time hierarchy* to be the union

$$\mathbf{PH}^c_{\mathrm{R}} \stackrel{\text{def}}{=} \bigcup_{\omega \geq 0} (\mathbf{\Sigma}^c_{\mathrm{R},\omega} \cup \mathbf{\Pi}^c_{\mathrm{R},\omega}) = \bigcup_{\omega \geq 0} \mathbf{\Sigma}^c_{\mathrm{R},\omega} = \bigcup_{\omega \geq 0} \mathbf{\Pi}^c_{\mathrm{R},\omega}.$$

Notice that the semi-algebraic sets belonging to any language in $\mathbf{PH}^c_{\mathrm{R}}$ are all semi-algebraic compact (in fact closed semi-algebraic subsets of spheres). Also, note the inclusion

$$\mathbf{PH}^c_{\mathrm{R}} \subset \mathbf{PH}_{\mathrm{R}}.$$

Remark 1.6. Even though the restriction to compact semi-algebraic sets might appear to be only a technicality at first glance, this is actually an important restriction. For instance, it is a long-standing open question in real complexity theory whether there exists an $\mathbf{NP}_{\mathrm{R}}$-complete problem which belongs to the class $\mathbf{\Sigma}^c_1$ (the compact version of the class $\mathbf{NP}_{\mathrm{R}}$). (This distinction between compact and non-compact versions of complexity classes does not arise in discrete complexity theory for obvious reasons.) It is an interesting question whether the main theorem of this paper can be extended to the full class $\mathbf{PH}_{\mathrm{R}}$. For technical reasons which will become clear later in the paper (Remark 2.5) we are unable to achieve this presently.

1.1.2. *Real Analogue of* $\#\mathbf{P}$. In order to define real analogues of counting complexity classes of discrete complexity theory, it is necessary to identify the proper notion of "counting" in the context of semi-algebraic geometry. Counting complexity classes over the reals have been defined previously by Meer [20], and studied extensively by other authors [10]. These authors used a straightforward generalization to semi-algebraic sets of counting in the case of finite sets – namely the counting function took the value of the cardinality of a semi-algebraic set if it happened to be finite, and ∞ otherwise. This is in our view not a fully satisfactory generalization since the count gives no information when the semi-algebraic set is infinite, and most interesting semi-algebraic sets have infinite cardinality. If one thinks of "counting" a semi-algebraic set $S \subset \mathrm{R}^k$ as computing certain discrete invariants, then a natural well-studied discrete topological invariant of S is its sequence of Betti numbers, $b_0(S), \ldots, b_k(S)$. In case S happens to be finite, $b_0(S)$ is its cardinality, and thus this generalizes the naive notion of counting. The above discussion motivates (see also Remark 1.8 below) the following definition which is different from the definitions of this class considered previously in [20, 10]. (We use the notation $\#\mathbf{P}^\dagger_{\mathrm{R}}$ to denote the class of functions that we define below in order to distinguish it from the class $\#\mathbf{P}_{\mathrm{R}}$ defined by previous authors).

Definition 1.7 (The class $\#\mathbf{P}^\dagger_{\mathrm{R}}$)**.** We call a sequence of functions

$$(f_n : \mathrm{R}^n \to \mathbb{N}^m)_{n>0}$$

with $m = n^{O(1)}$, to be in class $\#\mathbf{P}^{\dagger}_{\mathrm{R}}$ if there exists a sequence of first-order formulas

$$(\Phi_n(Y_1, \ldots, Y_m, Z_1, \ldots, Z_n)_{n>0})$$

and a polynomial time real Turing machine M which tests membership in the semi-algebraic sets $(S_n \subset \mathrm{R}^{m+n})_{n>0}$ defined by the sequence $(\Phi_n)_{n>0}$ such that

$$f_{n,i}(\mathbf{z}) = b_i(S_{n,\mathbf{z}}), 0 \leq i \leq m,$$

and for each $\mathbf{z} \in \mathrm{R}^n$, where $S_{n,\mathbf{z}} = S_n \cap \pi_Y^{-1}(\mathbf{z})$ and $\pi_Y : \mathrm{R}^{m+n} \to \mathrm{R}^n$ is the projection along the Y-coordinates.

Remark 1.8. The connection between counting points of varieties and their Betti numbers is more direct over fields of positive characteristic via the zeta function. The zeta function of a variety defined over $\mathbb{F}_p$ is the exponential generating function of the sequence whose n-th term is the number of points in the variety over $\mathbb{F}_{p^n}$. The zeta function of such a variety turns out to be a rational function in one variable (a deep theorem of algebraic geometry first conjectured by Andre Weil [26] and proved by Dwork[13]), and its numerator and denominator are products of polynomials whose degrees are the Betti numbers of the variety with respect to a certain (ℓ-adic) co-homology theory. The point of this remark is that the problems of "counting" varieties and computing their Betti numbers, are connected at a deeper level, and thus our definition of $\#\mathbf{P}^{\dagger}_{R}$ is not entirely ad hoc.

We can now state the main result of this paper.

Theorem 1.9 (Real analogue of Toda's theorem)**.**

$$\mathbf{PH}^c_{\mathrm{R}} \subset \mathbf{P}_{\mathrm{R}}^{\#\mathbf{P}^{\dagger}_{\mathrm{R}}}.$$

As a consequence of our method we also obtain the following reduction result that might be of independent interest.

We first define the following two problems:

Definition 1.10. (Compact general decision problem with at most ω quantifier alternations ($\mathbf{GDP}^{\mathbf{c}}_{\omega}$))

Input. A sentence Φ in the first order theory of R

$$(Q_1 X^1 \in \mathbf{S}^{k_1}) \cdots (Q_\omega X^\omega \in \mathbf{S}^{k_\omega})\phi(X^1, \ldots, X^\omega),$$

where for each $i, 1 \leq i \leq \omega$, $X^i = (X^i_0, \ldots, X^i_{k_i})$ is a block of k_i+1 variables, $Q_i \in \{\exists, \forall\}$, with $Q_j \neq Q_{j+1}, 1 \leq j < \omega$, and ϕ is a quantifier-free closed formula defining a semi-algebraic subset of $\mathbf{S}^{k_1} \times \cdots \times \mathbf{S}^{k_\omega}$.

Output. True or False depending on whether Φ is true or false in the first order theory of R.

Definition 1.11. (Computing the Betti numbers of semi-algebraic sets (**Betti**))

Input. A quantifier-free formula defining a semi-algebraic set $S \subset \mathrm{R}^k$.

Output. The Betti numbers $b_0(S), \ldots, b_{k-1}(S)$.

Theorem 1.12. *For every $\omega > 0$, there is a deterministic polynomial time reduction in the Blum-Shub-Smale model of* $\mathbf{GDP}^{\mathbf{c}}_{\omega}$ *to* **Betti**.

1.2. **Outline of the proof of the main theorem.** Our main tool is a topological construction described in the next section which given a semi-algebraic map $f : A \to B$ (satisfying a mild hypothesis) and $p \geq 0$, constructs *efficiently* a semi-algebraic set, $D^p_f(A)$, such that

$$b_i(f(A)) = b_i(D^p_f(A)), 0 \leq i < p$$

(in fact, for technical reasons we need two different constructions depending on whether A is an open or a closed semi-algebraic set, but we prefer to ignore this point in this rough outline). An infinitary version of such a construction was described in [15]. However, for it to be useful in our context it is very important that membership in the semi-algebraic set $D^p_f(A)$ should be checkable in polynomial time, given that same is true for A. Notice that even if there exists an efficient (i.e. polynomial time) algorithm for checking membership in A, the same need not be true for the image $f(A)$.

The connection between the decision problems in the compact real polynomial hierarchy and computing Betti numbers of semi-algebraic sets can now be roughly explained as follows.

First consider the class $\mathbf{\Sigma}^c_{\mathrm{R},1}$. Consider a closed semi-algebraic set $S \subset \mathbf{S}^k \times \mathbf{S}^\ell$ defined by a quantifier-free formula $\phi(X,Y)$ and let

$$\pi_Y : \mathbf{S}^k \times \mathbf{S}^\ell \to \mathbf{S}^k$$

be the projection map along the Y variables.

Then the formula $\Phi(X) = \exists Y \phi(X,Y)$ is satisfied by $\mathbf{x} \in \mathbf{S}^k$ if and only if $b_0(S_{\mathbf{x}}) \neq 0$, where $S_{\mathbf{x}} = S \cap \pi_Y^{-1}(\mathbf{x})$. Thus, the problem of deciding the truth of $\Phi(\mathbf{x})$ is reduced to computing a Betti number (the 0-th) of the fiber of S over $\mathbf{x}$.

Now consider the class $\mathbf{\Pi}^c_{\mathrm{R},1}$. Using the same notation as above we have that the formula $\Psi(X) = \forall Y \phi(X,Y)$ is satisfied by $\mathbf{x} \in \mathbf{S}^k$ if and only if $b_0(\mathbf{S}^\ell \setminus S_{\mathbf{x}}) = 0$ which is equivalent to $b_\ell(S_{\mathbf{x}}) = 1$. Notice, that as before the problem of deciding the truth of $\Psi(\mathbf{x})$ is reduced to computing a Betti number (the ℓ-th) of the fiber of S over $\mathbf{x}$.

Proceeding to a slightly more non-trivial case, consider the class $\mathbf{\Pi}^c_{\mathrm{R},2}$ and let $S \subset \mathbf{S}^k \times \mathbf{S}^\ell \times \mathbf{S}^m$ be a closed semi-algebraic set defined by a quantifier-free formula $\phi(X,Y,Z)$ and let

$$\pi_Z : \mathbf{S}^k \times \mathbf{S}^\ell \times \mathbf{S}^m \to \mathbf{S}^k \times \mathbf{S}^\ell$$

be the projection map along the Z variables, and

$$\pi_Y : \mathbf{S}^k \times \mathbf{S}^\ell \to \mathbf{S}^k$$

be the projection map along the Y variables as before. Consider the formula $\Phi(X) = \forall Y \exists Z \phi(X,Y,Z)$. It is easy to see that for $\mathbf{x} \in \mathbf{S}^k$, $\Phi(\mathbf{x})$ is true if and only if $\pi_Z(S)_{\mathbf{x}} = \mathbf{S}^\ell$, which is equivalent to $b_\ell(D^{\ell+1}_{\pi_Z}(S)_{\mathbf{x}}) = 1$. Thus for any $\mathbf{x} \in \mathbf{S}^k$, the truth or falsity of $\Phi(\mathbf{x})$ is determined by a certain Betti number of the fiber $D^{\ell+1}_{\pi_Z}(S)_{\mathbf{x}}$ over $\mathbf{x}$ of a certain semi-algebraic set $D^{\ell+1}_{\pi_Z}(S)$ which can be constructed efficiently in terms of the set S. The idea behind the proof of the main theorem is a recursive application of the above argument in case when the number of quantifier alternations is larger (but still bounded by some constant) while keeping track of the growth in the sizes of the intermediate formulas and also the number of quantified variables.

The rest of the paper is organized as follows. In Section 2 we fix notation, and prove the topological results needed for the proof of the two main theorems. We prove the main theorems in Section 3.

2. Ingredients

We first fix some notation.

Notation 3. For each $p \geq 0$ we denote

$$\Delta^p = \{(t_0, \ldots, t_p) \mid t_i \geq 0, 0 \leq i \leq p, \sum_{i=0}^{p} t_i = 1\}$$

the standard p-simplex.

Notation 4. Let $f : A \to B$ be a map between topological spaces A and B. For each $p \geq 0$, We denote by $W_f^p(A)$ the $(p+1)$-fold fiber product of A over f. In other words

$$W_f^p(A) = \{(x_0, \ldots, x_p) \in A^{p+1} \mid f(x_0) = \cdots = f(x_p)\}.$$

Definition 2.1 (Topological join over a map)**.** Let $f : A \to B$ be a map between topological spaces A and B. For $p \geq 0$ the $(p+1)$-fold join $J_f^p(A)$ of A over f is

$$J_f^p(A) \stackrel{\text{def}}{=} W_f^p(A) \times \Delta^p / \sim, \tag{2.1}$$

where

$$(x_0, \ldots, x_p, t_0, \ldots, t_p) \sim (y_0, \ldots, y_p, t_0, \ldots, t_p)$$

if for each i with $t_i \neq 0$, $x_i = y_i$.

We now impose certain conditions on the map f.

2.1. **Compact Coverings.** Recall that we call $K \subset \mathrm{R}^n$ a semi-algebraic compact if it is a closed and bounded semi-algebraic set.

Notation 5. For any semi-algebraic $A \subset \mathrm{R}^n$, we denote by $\mathrm{K}(A)$ the collection of all semi-algebraic compact subsets of A.

Definition 2.2. Let $f : A \to B$ be a semi-algebraic map. We say that f *covers semi-algebraic compacts* if for any $L \in \mathrm{K}(f(A))$, there exists $K \in \mathrm{K}(A)$ such that $f(K) = L$.

Definition 2.3 (p-equivalence)**.** A map $f : A \to B$ between two topological spaces is called a p-equivalence if the induced homomorphism

$$f_* : \pi_i(A) \to \pi_i(B)$$

is an isomorphism for all $0 \leq i < p$, and an epimorphism for $i = p$, and we say that A is p-equivalent to B. (Note that p-equivalence is not an equivalence relation).

The following theorem relates the topology of $J^p(A)$ to that of the image of f in the case when f covers semi-algebraic compacts and is crucial for what follows.

Theorem 2.4. *Let $f : A \to B$ be a semi-algebraic map that covers semi-algebraic compacts. Then for every $p \geq 0$, the map f induces a p-equivalence $J(f) : J_f^p(A) \to f(A)$.*

Proof. We begin with the case $A \in \mathrm{K}(\mathrm{R}^n)$. Let $J(f) : J^p_f(A) \to f(A)$ be the map given by

$$J(f)(x_0, \ldots, x_p, t_0, \ldots, t_p) = f(x_0).$$

The map $J(f)$ is well defined since $(x_0, \ldots, x_p) \in W^p_f(A)$, and is closed since $J^p_f(A)$ is a semi-algebraic compact. Moreover, the fibers of $J(f)$ are p-equivalent to a point.

Thus, by the Vietoris-Begle theorem, the map $J(f)$ induces isomorphisms

$$J(f)_* : H_i(J^p_f(A)) \to H_i(f(A));$$

for $0 \leq i < p$. Note that in the case $\mathrm{R} \neq \mathbb{R}$, the validity of the Vietoris-Begle theorem can be seen as a corollary of the existence of a semi-algebraic co-homology that satisfies the Eilenberg-Steenrod axioms for a Čech theory (see [14]).

In the general case, consider $K_1 \subset K_2$ two semi-algebraic compacts in $\mathrm{K}(A)$. The inclusion gives rise to the following diagram,

$$\begin{tikzcd} J^p_f(K_1) \arrow[r, hook, "i"] \arrow[d, "J(f|_{K_1})"] & J^p_f(K_2) \arrow[d, "J(f|_{K_2})"] \\ f(K_1) \arrow[r, hook, "j"] & f(K_2) \end{tikzcd}$$

where the vertical maps are p-equivalence by the previous case. We have a similar diagram at the homology level; if we take the direct limit as K ranges in $\mathrm{K}(A)$, we obtain the following:

$$\begin{tikzcd} \varinjlim H(J^p_f(K)) \arrow[r, "\cong"] \arrow[d, "\varinjlim J(f|_K)"] & H(J^p_f(A)) \arrow[d, "J(f)"] \\ \varinjlim H(f(K)) \arrow[r, "\cong"] & H(f(A)) \end{tikzcd}$$

The isomorphism on the top level comes from the fact that homology and direct limit commute [23], along with the fact that for a semi-algebraic set, one can compute the homology using chains supported exclusively on semi-algebraic compacts [12]. For the bottom isomorphism, we need the additional fact that since we assume that f covers semi-algebraic compacts, we have

$$\varinjlim\{H(f(K)) \mid K \in \mathrm{K}(A)\} = \varinjlim\{H(L) \mid L \in \mathrm{K}(B)\}.$$

Since each $J(f|_K)$ was a p-equivalence, the vertical maps in the limit are p-equivalences too. $\square$

Remark 2.5. Note that the condition on the map in Theorem 2.4 is satisfied in the case the set A is either open or compact and the map f is the projection along certain co-ordinates. Note also that Theorem 2.4 is not true without the assumption that f covers semi-algebraic compacts, and this is the reason why we restrict attention to open or closed semi-algebraic subsets of spheres in this paper.

Let $S \subset \mathbf{S}^k \times \mathbf{S}^\ell$ be a closed subset defined by a first-order formula $\Phi(X, Y)$, and let π_Y denote the projection along the Y co-ordinates. We now define semi-algebraic sets having the same homotopy type as the join space $J_{\pi_Y}(S)$ in the case when S is a closed (respectively open) semi-algebraic subset of $\mathbf{S}^k \times \mathbf{S}^\ell$.

Notation 6. Let $S \subset \mathbf{S}^k \times \mathbf{S}^\ell$ be a closed subset defined by a closed first-order formula $\Phi(X,Y)$, and let π_Y denote the projection along the Y co-ordinates.

We denote by $D^p_{\pi_Y,c}(S)$ the semi-algebraic set defined by

$$
\begin{aligned}
D^p_{\pi_Y,c}(S) \quad &\stackrel{\text{def}}{=} \quad \{(u,\mathbf{x},\mathbf{y}^0,\ldots,\mathbf{y}^p,\mathbf{t}) \mid \mathbf{x}\in\mathbf{S}^k, \mathbf{t}\in\Delta^p, \\
&\qquad \text{for each } i,\ 0\le i\le p, \mathbf{y}^i\in\mathbf{B}^{\ell+1}, (t_i=0)\vee\Phi(\mathbf{x},\mathbf{y}^i), \\
&\qquad u^2+|\mathbf{x}|^2+\sum_{i=0}^p|\mathbf{y}^i|^2+|\mathbf{t}|^2=p+4, \text{and } u\ge 0\}.
\end{aligned}
\tag{2.2}
$$

Notice that $D^p_{\pi_Y,c}(S)$ is a closed semi-algebraic subset of the upper hemisphere of the sphere $\mathbf{S}^N(0,p+4)$, where $N=(k+1)+(p+1)(\ell+2)$.

We will denote by $D^p_{\pi_Y,c}(\Phi)$ the following first-order formula defining the semi-algebraic set $D^p_{\pi_Y,c}(S)$, namely

$$
D^p_{\pi_Y,c}(\Phi) \stackrel{\text{def}}{=} \Theta_1(T)\wedge\Theta_2(X,Y^0,\ldots,Y^p,T)\wedge\Theta_2(U_0,X,Y^0,\ldots,Y^p,T)
\tag{2.3}
$$

where

$$
\begin{aligned}
\Theta_1(T) \quad &:= \quad (\bigwedge_{i=0}^p T_i\ge 0)\wedge(\sum_{i=0}^p T_i=1), \\
\Theta_2(X,Y^0,\ldots,Y^p,T) \quad &:= \quad ((|X|^2=1)\bigwedge_{i=0}^p(((|Y^i|^2\le 1)\wedge((T_i=0)\vee\Phi(X,Y^i))), \\
\Theta_3(U_0,X,Y^0,\ldots,Y^p,T) \quad &:= \quad (U_0^2+|X|^2+\sum_{i=0}^p|Y^i|^2+|T|^2=p+4)\wedge(U_0\ge 0).
\end{aligned}
$$

We have a similar construction in case S is an open subset of $\mathbf{S}^k\times\mathbf{S}^\ell$. In this case we thicken the various faces of the standard simplex Δ^p so that they become convex open subsets of R^{p+1}, but maintaining the property that a subset of these thickened faces have a non-empty intersection if and only if the closures of the corresponding faces in Δ^p had a non-empty intersection. In this way we ensure that our construction produces an open subset of a sphere, while having again the homotopy type of the join space.

Notation 7. Let $S \subset \mathbf{S}^k \times \mathbf{S}^\ell$ be an open subset defined by an open first-order formula $\Phi(X,Y)$, and let π_Y denote the projection along the Y co-ordinates.

We will denote by $D^p_{\pi_Y,o}(\Phi)$ the following first-order formula.

$$
D^p_{\pi_Y,o}(\Phi) \stackrel{\text{def}}{=} \Theta_1(T)\wedge\Theta_2(X,Y^0,\ldots,Y^p,T)\wedge\Theta_3(U_0,X,Y^0,\ldots,Y^p,T)
\tag{2.4}
$$

where

$$\begin{aligned}\Theta_1(T) &:= (\bigwedge_{i=0}^{p} T_i > 0) \wedge (1 - 1/p+1 < \sum_{i=0}^{p} T_i < 1 + 1/p+1)\\ \Theta_2(X, Y^0, \ldots, Y^p, T) &:= \bigwedge_{i=0}^{p} ((|Y^i|^2 < 3/2) \wedge ((T_i < 1/(2(p+1))) \vee \Phi_+(X, Y^i)))\\ \Theta_3(U_0, X, Y^0, \ldots, Y^p, T) &:= (U_0^2 + |X|^2 + \sum_{i=0}^{p} |Y^i|^2 + |T|^2 = 2p+4) \wedge (U_0 > 0)\end{aligned}$$

and

$$\Phi_+(X, Y) \stackrel{\text{def}}{=} (1/2 < |X|^2 < 3/2) \wedge (1/2 < |Y|^2 < 3/2) \wedge \Phi(X/|X|, Y/|Y|).$$

We will denote by $D^p_{\pi_Y,o}(S)$ the semi-algebraic set defined by $D^p_{\pi_Y,o}(\Phi)$. Notice that $D^p_{\pi_Y,o}(S)$ is an open subset of the upper hemisphere of the sphere $\mathbf{S}^N(0, 2p+4)$, where $N = (k+1) + (p+1)(\ell+2)$.

We now prove some important properties of the sets $D^p_{\pi_Y,c}(S), D^p_{\pi_Y,o}(S)$ defined above as well as of the formulas $D^p_{\pi_Y,c}(\Phi), D^p_{\pi_Y,o}(\Phi)$ defining them.

Proposition 2.6 (Polynomial time computability)**.** *Suppose there exists a polynomial time real Turing machine M which recognizes the sequence of semi-algebraic sets $(S_n)_{n>0}$ defined by the sequence of first order formulas*

$$(\Phi_n(X_0, \ldots, X_{k(n)}, Y_0, \ldots, Y_{\ell(n)})_{n>0}, k, \ell = n^{O(1)})$$

where for each $n > 0$, Φ_n defines a closed (respectively open) semi-algebraic subset S_n of $\mathbf{S}^{k(n)} \times \mathbf{S}^{\ell(n)}$. Then there exists a polynomial time real Turing machine M' recognizing the semi-algebraic sets defined by $(D^p_{\pi_Y,c}(\Phi_n))_{n>0}$ (respectively $(D^p_{\pi_Y,o}(\Phi_n))_{n>0}$).

Proof. Clear from the construction of the formulas $(D^p_{\pi_Y,c}(\Phi_n))_{n>0}$ (respectively $(D^p_{\pi_Y,o}(\Phi_n))_{n>0}$). □

We now prove an important topological property of the semi-algebraic sets $D^p_{\pi_Y,c}(S), D^p_{\pi_Y,o}(S)$ defined above.

Proposition 2.7 (Homotopy equivalence to the join)**.** *Let $S \subset \mathbf{S}^k \times \mathbf{S}^\ell$ be a closed (respectively, open) subset of $\mathbf{S}^k \times \mathbf{S}^\ell$ defined by a first-order formula $\Phi(X, Y)$, and let π_Y denote the projection along the Y co-ordinates. Then for all $p \geq 0$, $J^p_{\pi_Y}(S)$ is homotopy equivalent to $D^p_{\pi_Y,c}(S)$ (respectively, $D^p_{\pi_Y,o}(S)$).*

Proof. Suppose S is a closed subset of $\mathbf{S}^k \times \mathbf{S}^\ell$ and

$$g : D^p_{\pi_Y,c}(S) \to J^p_{\pi_Y}(S)$$

be the map which takes a point $(u, \mathbf{x}, \mathbf{y}^0, \ldots, \mathbf{y}^p, \mathbf{t}) \in D^p_{\pi_Y,c}(S)$ to the equivalence class represented by the point $((\mathbf{x}, \mathbf{y}^0), \ldots, (\mathbf{x}, \mathbf{y}^p), \mathbf{t})$ in $J^p_{\pi_Y}(S)$. From the definition of the spaces $D^p_{\pi_Y,c}(S)$ and $J^p_{\pi_Y}(S)$, we have that the inverse image under g of a point represented by $((\mathbf{x}, \mathbf{y}^0), \ldots, (\mathbf{x}, \mathbf{y}^p), \mathbf{t})$ in $J^p_{\pi_Y}(S)$ is given by

$$\begin{aligned}&g^{-1}(((\mathbf{x}, \mathbf{y}^0), \ldots, (\mathbf{x}, \mathbf{y}^p), \mathbf{t})) = \{(u, \mathbf{x}, \mathbf{z}^0, \ldots, \mathbf{z}^p, \mathbf{t}) \mid \text{ for each } i, 0 \leq i \leq p,\\ &\mathbf{z}^i \in \mathbf{B}^{\ell+1} \text{ and } \mathbf{z}^i = \mathbf{y}^i \text{ if } t_i \neq 0, u^2 + |\mathbf{x}|^2 + \sum_{i=0}^{p} |\mathbf{z}^i|^2 + |\mathbf{t}^2| = p+4, u \geq 0\}.\end{aligned}$$

It is easy to see from the above formula that the inverse image under g of each point of $J^p_{\pi_Y}(S)$ is homeomorphic to a product of balls and hence contractible. The proposition now follows from the Vietoris-Begle theorem.

The open case is proved analogously. □

As an immediate corollary we obtain

Corollary 2.8. *Let $S \subset \mathbf{S}^k \times \mathbf{S}^\ell$ be a closed (respectively, open) subset of $\mathbf{S}^k \times \mathbf{S}^\ell$ defined by a first-order formula $\Phi(X,Y)$, and let π_Y denote the projection along the Y co-ordinates. Then for all $p \geq 0$,*

Then for all $p \geq 0$, $D^p_{\pi_Y,c}(S)$ (respectively, $D^p_{\pi_Y,o}(S)$) is p-equivalent to $\pi_Y(S)$, and

$$b_i(D^p_{\pi_Y,c}(S)) = b_i(\pi_Y(S))$$

(respectively, $b_i(D^p_{\pi_Y,o}(S)) = b_i(\pi_Y(S))$) for $0 \leq i < p$.

Proof. Since S is either an open or closed subset of $\mathbf{S}^k \times \mathbf{S}^\ell$ it is clear that the projection map π_Y covers semi-algebraic compacts. Now apply Theorem 2.4. □

Lemma 2.9. *Let $\Phi(X_0,\ldots,X_k,Y_0,\ldots,Y_\ell)$ be the following first-order formula*

$$\Phi := (Q_1 Z^1 \in \mathbf{S}^{k_1})(Q_2 Z^2 \in \mathbf{S}^{k_2})\ldots(Q_\omega Z^\omega \in \mathbf{S}^{k_\omega})\Psi(Y,X,Z^1,\ldots,Z^\omega)$$

with $Q_i \in \{\exists,\forall\}$, and Ψ a quantifier-free first order formula.

Let π_Y denote the projection along the Y coordinates. Then, for each $p \geq 0$ the formula

$$D^p_{\pi_Y,*}(\Phi)(X,Y^0,\ldots,Y^p,T)$$

(where $*$ denotes either c or o) is equivalent to the formula

$$\begin{aligned}\bar{D}^p_{\pi_Y,*}(\Phi) := &(Q_1 Z^{1,0} \in \mathbf{S}^{k_1},\ldots,Q_1 Z^{1,p} \in \mathbf{S}^{k_1})\\ &(Q_2 Z^{2,0} \in \mathbf{S}^{k_2},\ldots,Q_2 Z^{2,p} \in \mathbf{S}^{k_2})\\ &\vdots\\ &(Q_\omega Z^{\omega,0} \in \mathbf{S}^{k_\omega},\ldots,Q_\omega Z^{\omega,p} \in \mathbf{S}^{k_\omega})\\ &(D^p_{\pi_Y,*}(\Psi)(X,Y^0,\ldots,Y^p,Z^{1,0},\ldots,Z^{\omega,p},T_0,\ldots,T_p)),\end{aligned}$$

where $Y^i = (Y^i_0,\ldots,Y^i_\ell)$ and $Z^{j,i} = (Z^j_0,\ldots,Z^j_{k_j})$ for $0 \leq i \leq p, 1 \leq j \leq \omega$, and π_Y is the projection along the Y co-ordinates.

Proof. It follows from the structure of the formula $D^p_{\pi_Y,*}(\Phi)(X,Y^0,\ldots,Y^p,T)$ that the inner most quantifiers can be pulled outside at the cost of introducing $(p+1)$ copies of the quantified variables. □

Theorem 2.10 (Alexander-Lefshetz duality)**.** *Let $K \subset \mathbf{S}^n$ be a compact semi-algebraic subset with $n \geq 2$. Then*

$$\begin{aligned} b_0(K) &= 1 + b_{n-1}(\mathbf{S}^n - K) - b_n(\mathbf{S}^n - K),\\ b_i(K) &= b_{n-i-1}(\mathbf{S} - K),\ \ 1 \leq i \leq n-2,\\ b_{n-1}(K) &= b_0(\mathbf{S}^n - K) - 1 + \max(1 - b_0(\mathbf{S}^n - K), 0),\\ b_n(K) &= 1 - \min(1, b_0(\mathbf{S}^n - K)).\end{aligned}$$

Proof. Lefshetz duality theorem [23] gives for each $i, 0 \leq i \leq n$,

$$b_i(\mathbf{S}^n - K) = b_{n-i}(\mathbf{S}^n, K).$$

The theorem now follows from the long exact sequence of homology,

$$\cdots \to \mathrm{H}_i(K) \to \mathrm{H}_i(\mathbf{S}^n) \to \mathrm{H}_i(\mathbf{S}^n, K) \to \mathrm{H}_{i-1}(K) \to \cdots$$

after noting that $\mathrm{H}_i(\mathbf{S}^n) = 0, i \neq 0, n$ and $\mathrm{H}_0(\mathbf{S}^n) = \mathrm{H}_n(\mathbf{S}^n) = \mathbb{Z}$. □

3. Proof of the main theorem

We are now in a position to prove Theorem 1.9.

The proof of Theorem 1.9 depends on the following key proposition.

Proposition 3.1. *Suppose there exists a real Turing machine M, and a sequence of formulas*

$$\Phi_n(Y_0, \ldots, Y_{m-1}, Z_0, \ldots, Z_n) := \\ (Q_1 X^1 \in \mathbf{S}^{k_1}) \cdots (Q_\omega X^\omega \in \mathbf{S}^{k_\omega}) \phi_n(X^1, \ldots, X^\omega, Y, Z),$$

having free variables $(Y, Z) = (Y_0, \ldots, Y_{m-1}, Z_0, \ldots, Z_n)$, with

$$Q_1, \ldots, Q_\omega \in \{\exists, \forall\}, Q_i \neq Q_{i+1},$$

where ϕ_n a quantifier-free formula defining a closed (respectively open) semi-algebraic subset of $\mathbf{S}^n$, and such that M recognizes the semi-algebraic sets defined by ϕ_n in polynomial time. Then, there exists a polynomial time real Turing machine M' which recognizes the semi-algebraic sets defined by a sequence of quantifier-free first order formulas $(\Theta_n(Z, V_0, \ldots, V_N))_{n>0}$ such that for each $\mathbf{z} \in \mathbf{S}^m$, where $\Theta_n(\mathbf{z}, V)$ describes a closed (respectively open) semi-algebraic subset T_n of $\mathbf{S}^N$, with $N = n^{O(1)}$, and polynomial-time computable functions

$$F_n : \mathbb{N}^{N+1} \to \mathbb{N}^{m+1}$$

such that for each $i, 0 \leq i \leq m$,

$$b_i(\mathcal{R}(\Phi_n(Y, \mathbf{z}))) = F_{n,i}(b_0(\mathcal{R}(\Theta_n(\mathbf{z}, V)), \ldots, b_N(\mathcal{R}(\Theta_n(\mathbf{z}, V)))))).$$

Proof. The proof is by an induction on ω. We assume that the formula ϕ_n defines a closed semi-algebraic set. The open case can be handled analogously.

If $\omega = 0$ then we let $\Theta_n = \Phi_n$ and $M' = M$, $N = m$, and

$$F_{n,i}(j_0, \ldots, j_N) = j_i, 0 \leq i \leq m.$$

Since there are no quantifiers, for each $n \geq 0$ the semi-algebraic set recognized by M and M' are the same and thus the Betti numbers of the sets recognized by M and M' agree.

If $\omega > 0$, we have the following two cases.

(A) Case 1, $Q_1 = \exists$: In this case consider the sequence of formulas $\Phi'_n := \bar{D}^n_{\pi_{X^1}, c}(\Psi_n)$ (cf. Lemma 2.9), where Ψ_n is the following formula with free variables Y, X^1

$$\Psi_n(Y, Z, X^1) := (Q_2 X^2 \in \mathbf{S}^{k_2}) \cdots (Q_\omega X^\omega \in \mathbf{S}^{k_\omega}) \phi_n(X^1, \ldots, X^\omega, Y, Z).$$

The formula $\bar{D}^m_{\pi_{X^1},c}(\Psi_n)$ is given by

$$\begin{aligned}\bar{D}^m_{\pi_{X^1},c}(\Psi_n) := (Q_2 X^{2,0} \in \mathbf{S}^{k_2}, \ldots, Q_2 X^{2,m} \in \mathbf{S}^{k_2})\\ (Q_3 X^{3,0} \in \mathbf{S}^{k_3}, \ldots, Q_3 X^{3,m} \in \mathbf{S}^{k_3})\\ \vdots\\ (Q_\omega X^{\omega,0} \in \mathbf{S}^{k_\omega}, \ldots, Q_\omega X^{\omega,m} \in \mathbf{S}^{k_\omega})\\ (D^m_{\pi_{X^1},c}(\phi_n)(X^{1,0}, \ldots, X^{1,m}, X^{2,0}, \ldots, X^{\omega,n}, Y, Z, T_0, \ldots, T_m)).\end{aligned}$$

Note that the quantifier-free inner formula in the above expression,

$$D^m_{\pi_{X^1},c}(\phi_n)(X^{1,0}, \ldots, X^{1,m}, X^{2,0}, \ldots, X^{\omega,n}, Y, Z, T_0, \ldots, T_m)$$

has

$$N = \sum_{j=1}^{\omega} (k_j + 1)(m + 1) + 2(m + 1) = n^{O(1)}$$

free variables, and it is clear from the definition of the formula $D^m_{\pi_{X^1},c}(\phi_n)$ (cf. Eqn. 2.3), that there exists a polynomial time Turing machine (say M_1) to evaluate it since we have a polynomial time Turing machine M for evaluating ϕ_n.

Moreover, the formula $\bar{D}^m_{\pi_{X^1},c}(\Psi_n)$ has one less quantifier alternation than the formula Φ_n.

We now apply the proposition inductively to obtain a machine M_2 evaluating $(\Theta_n)_{n>0}$, and a polynomial time computable functions $(F'_n)_{n>0}$. By inductive hypothesis we can suppose that for each $i, 0 \leq i \leq m$ we have for each $\mathbf{z} \in \mathbf{S}^n$

$$b_i(\mathcal{R}(\bar{D}^m_{\pi_{X^1},c}(\Psi_n(Y, \mathbf{z}, X^1)) = F'_{n,i}(b_0(\mathcal{R}(\Theta_n(\mathbf{z}, \cdot))), \ldots, b_N(\mathcal{R}(\Theta_n(\mathbf{z}, \cdot)))).$$

But by Corollary 2.8, we have that for $0 \leq i \leq m$,

$$b_i(\mathcal{R}(\Phi_n(Y, \mathbf{z}))) = b_i(\pi_{X^1}(\mathcal{R}(\Psi_n(Y, \mathbf{z}, X^1)))) = b_i(\mathcal{R}(\bar{D}^n_{\pi_{X^1},c}(\Psi_n(Y, \mathbf{z}, X^1)))).$$

We set $M' = M_2$ and

$$F_{n,i} = F'_{n,i}, 0 \leq i \leq m,$$

which completes the induction in this case.

(B) Case 2, $Q_1 = \forall$: In this case consider the sequence of formulas $\Phi'_n := \bar{D}^m_{\pi_{X^1},o}(\neg\Psi_n)$ (cf. Lemma 2.9), where Ψ_n is the following formula with free variables Y, X^1

$$\Psi_n(Y, Z, X^1) := (Q_2 X^2 \in \mathbf{S}^{k_2}) \cdots (Q_\omega X^\omega \in \mathbf{S}^{k_\omega})\phi_n(X^1, \ldots, X^\omega, Y).$$

We now apply the proposition inductively as above to obtain a machine M_2 evaluating $(\Theta_n)_{n>0}$, and functions $(F'_n)_{n>0}$. By inductive hypothesis we can suppose that for each $\mathbf{z} \in \mathbf{S}^n$ and $i, 0 \leq i \leq m$ we have

$$b_i(\mathcal{R}(\bar{D}^m_{\pi_{X^1},o}(\neg\Psi_n(Y, \mathbf{z}, X^1)) = F'_{n,i}(b_0(\mathcal{R}(\Theta_n(\mathbf{z}, \cdot))), \ldots, b_N(\mathcal{R}(\Theta_n(\mathbf{z}, \cdot)))).$$

But by Corollary 2.8, we have that for $0 \leq i \leq m$,

$$b_i(\mathbf{S}^m \setminus \mathcal{R}(\Phi_n(Y, \mathbf{z})) = b_i(\pi_{X^1}(\mathcal{R}(\neg\Psi_n(Y, \mathbf{z}, X^1))) = b_i(\mathcal{R}(\bar{D}^m_{\pi_{X^1},o}(\neg\Psi_n(Y, \mathbf{z}, X^1))).$$

But by Alexander-Lefshetz duality (cf. Theorem 2.10) we have, setting $K = \mathcal{R}(\Phi_n(Y,\mathbf{z}))$,

$$\begin{aligned} b_0(K) &= 1 + b_{m-1}(\mathbf{S}^m - K) - b_m(\mathbf{S}^m - K), \\ b_i(K) &= b_{m-i-1}(\mathbf{S}^m - K), \;\; 1 \leq i \leq m-2, \\ b_{m-1}(K) &= b_0(\mathbf{S}^m - K) - 1 + \max(1 - b_0(\mathbf{S}^n - K), 0), \\ b_m(K) &= 1 - \min(1, b_0(\mathbf{S}^m - K)). \end{aligned}$$

We set $M' = M_2$ and

$$\begin{aligned} F_{n,0} &= 1 + F'_{n,m-1} - F'_{n,m}, \\ F_{n,i} &= F'_{n,m-i-1}, \;\; 1 \leq i \leq m-2, \\ F_{n,n-1} &= F'_{n,0} - 1 + \max(1 - F'_{n,0}, 0), \\ F_{n,n} &= 1 - \min(1, F'_{n,0}). \end{aligned}$$

which completes the induction in this case as well.

□

Proof of Theorem 1.9. Follows immediately from Proposition 3.1 in the special case when the set consisting of the variables Y is empty. In this case the sequence of formulas $(\Phi_n)_{n>0}$ correspond to a language in the polynomial hierarchy and for each n, $\mathbf{z} = (z_0, \ldots, z_n) \in S_n \subset \mathbf{S}^n$ if and only if

$$F_{n,0}(b_0(\mathcal{R}(\Theta_n(\mathbf{z}, \cdot))), \ldots, b_N(\mathcal{R}(\Theta_n(\mathbf{z}, \cdot))) \neq 0$$

and this last condition can be checked in polynomial time with advice from the class $\#P_R^\dagger$. □

Proof of Theorem 1.12. Follows from Proposition 3.1 since the semi-algebraic the formula Θ_n is clearly computable in polynomial time from the given formula Φ_n as long as the number of quantifier alternations ω is bounded by a constant. □

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Department of Mathematics, Purdue University, West Lafayette, IN 47906, U.S.A.
E-mail address: `sbasu@math.purdue.edu`

Department of Mathematics and Computing Sciences, Lenoir-Rhyne University, Hickory, NC 28603
E-mail address: `thierry.zell@lr.edu`