A COMPLEXITY THEORY OF CONSTRUCTIBLE FUNCTIONS AND SHEAVES

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Abstract. In this paper we introduce analogues of the discrete complexity classes **VP** and **VNP** of sequences of functions in the Blum-Shub-Smale model. The functions in the new definitions are constructible functions on $\mathbb{R}^n$. We define a class of sequences of constructible functions that play a role analogous to that of **VP** in the discrete theory. The class analogous of **VNP** is defined using Euler integration. We discuss several examples and pose a conjecture analogous to the **VP** vs **VNP** conjecture in the discrete case. In the second part of the paper we extend the notions of complexity classes to sequences of constructible sheaves over $\mathbb{R}^n$ (or its one point compactification). We introduce a class of sequences simple constructible sheaves, that could be seen as the sheaf theoretic analog of the Blum-Shub-Smale class $\mathbf{P}_{\mathbb{R}}$. We also define a hierarchy of complexity classes of sheaves mirroring the polynomial hierarchy, $\mathbf{PH}_{\mathbb{R}}$ in the more classical theory. We prove singly exponential upper bounds on the topological complexity of the sheaves in this hierarchy. We obtain as a result a singly exponential complexity upper bound on the sheaf-theoretic version of quantifier elimination. We pose the natural sheaf theoretic analogues of the classical **P** vs **NP** question and also discuss a connection with Toda's theorem from discrete complexity theory.

1. Introduction

This paper is divided into two parts. The first part is motivated by the problem of finding a proper analog of the complexity classes **VP** and **VNP** introduced by Valiant [45, 43, 44] and which has proved very influential in subsequent development of the theory of computational complexity, in the Blum-Shub-Smale (henceforth abbreviated to "B-S-S") model [15] of real number computation (see [14, 21, 19] for precise definitions of this model). Valiant's definition concerned classes of functions as opposed to sets ([20] for results on the exact relationship between Valiant's conjecture and the classical complexity questions between complexity classes of sets). The class **VP** and its variants (see [19] for many subtle details) are supposed to represent functions that are easy to compute and plays a role analogous to the role of class **P** in the case of complexity classes of sets. While the class of functions, **VNP**, was supposed to play a role analogous to that of the class of languages **NP**. Two remarks are in order. Firstly, the classes **VP** and **VNP** as defined by Valiant are *non-uniform.* The circuits or formulas whose sizes measure the complexity of functions are allowed to be very different for different sizes of the input. Also, as remarked earlier unlike in the classical theory, the elements of the

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classes **VP** and **VNP** are not languages but sequences of functions. In Valiant's work the emphasis was on functions $f : \{0,1\}^n \to \mathbf{k}$ (for some field $\mathbf{k}$). Since any function on $\{0,1\}^n$ can be expressed as a polynomial, it makes sense to consider only polynomial functions. In particular, characteristic functions of subsets of $\{0,1\}^n$ are also expressible as polynomials – and this provides a crucial link between the function viewpoint and the classical question about languages.

Valiant's theory leads to an elegant reduction of the question whether $\mathbf{VP} = \mathbf{VNP}$ to a purely algebraic one – namely, whether the polynomial given by the permanent of an $n \times n$ matrix (with indeterminate entries) can be expressed as the determinant of another (possibly polynomially larger) matrix whose entries are are linear combinations of the entries of the original matrix. Thus, the question of whether $\mathbf{VP} = \mathbf{VNP}$ reduces to a purely mathematical question about polynomials and mathematical tools from representation theory and algebraic geometry can be made to bear on this subject (see [37, 23]).

In this paper we formulate analogues of Valiant's non-uniform algebraic analogues for the B-S-S model. The first point to notice is that unlike in the classical (Boolean) case, when the underlying field is infinite (say $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$), the characteristic function of a definable set (i.e. a constructible set in the case $\mathbf{k} = \mathbb{C}$ and a semi-algebraic set in case $\mathbf{k} = \mathbb{R}$) is no longer expressible as polynomials. In fact, we argue that sequences of polynomial functions are not the right objects to consider in this setting. But a class of functions that appears very naturally in the algebraic geometry over real and complex numbers are the so called *constructible functions*. We will see later that many discrete valued functions that appear in complexity theory including functions such as the characteristic functions of constructible as well as semi-algebraic sets, ranks of matrices and higher dimensional tensors, topological invariants such as the Betti numbers or Euler-Poincaré characteristics, local dimensions of semi-algebraic sets are all examples of such functions. Constructible functions in the place of so called "counting" functions have already appeared in B-S-S style complexity theory over $\mathbb{R}$ and $\mathbb{C}$. For example, in [13] (respectively, [11]) a real (respectively, complex) analogue of Toda's theorem of discrete complexity theory was obtained. The notion of counting the number of satisfying assignments of a Boolean equation was replaced in these papers by the problem of computing the Poincaré polynomial of a semi-algebraic/constructible set (see also [18, 22]). A first goal of this paper is to build up a (non-uniform) complexity theory for constructible functions over real as well as complex numbers that mirror Valiant's theory in the discrete case.

The choice of constructible functions as a "good" class of functions is also motivated from another direction. First recall that in the case of languages, the languages in the class **NP** can be thought of as the images under projection maps of the languages in the class **P** (see Section 3.3 for more precise definitions of these classes). For classes of functions such as the class **VP**, in order to define an analog of the class **NP** one needs a way of "pushing forward" a function under a projection map. It is folklore that functions (or more generally maps) can be pulled-back tautologically, but pushing forward requires some effort. The standard technique in mathematics is to define such a push-forward using "fiber-wise integration". In Valiant's original definition of the class **VNP** this push-forward was implemented by taking the sum of the function to be integrated over the Boolean cube $\{0,1\}^n$. This operation is not very geometric and thus not completely satisfactory in the

B-S-S setting. On the other hand in the B-S-S model over $\mathbb{C}$ or $\mathbb{R}$, integration against most normal measures (other than finite atomic ones such as the one used by Valiant in his definition) will not be computable exactly as the results will not be algebraic. It thus becomes a subtle problem to choose the right class of functions and the corresponding push-forward. It turns out that the class of constructible functions is particularly suited for this purpose, where a discrete notion of integration (with respect to the Euler-Poincaré characteristic) already exists. It makes sense now to put these together and develop an analogue of Valiant theory for this class, which is what we begin to do in this paper. The complexity classes of constructible functions and their corresponding "**P** vs **NP**"-type questions that will arise in these new models, should be considered as the non-uniform versions of the corresponding questions in the B-S-S model (just as the **VP** vs **VNP** is to be considered as a non-uniform version of the **P** vs **NP** question in the classical (i.e. discrete) setting. We define formally these new classes, give some examples, and finally pose a "constructible" analogue of the **VP** vs **VNP** question.

The second part of the paper has no analogue in discrete complexity theory but is strongly motivated by the first part, and prior results on algorithmic complexity of various problems in semi-algebraic geometry. One way that constructible functions appear in various applications is as the fiber-wise Euler-Poincaré characteristic of certain sheaves of complexes with bounded cohomology. The right generality to consider these objects – namely a bounded derived category of sheaves of modules, which are locally constant on the strata of a definable (semi-algebraic) stratification of the ambient manifold – lead naturally to the category of *constructible sheaves*. Constructible sheaves are a particularly simple kind of sheaves arising in algebraic geometry [1] and have found many applications in mathematics (in the theory of linear systems of partial differential equations and micro-local analysis [34], in the study of singularities that appear in linear differential equations with meromorphic coefficients [27, 39], study of local systems in algebraic geometry [26], intersection cohomology theory [17] amongst many others) but to our knowledge they have not been studied yet from the structural complexity point of view. Constructible functions have also being studied by many authors from different perspectives, such as [35, 24, 25]. Recently they have also found applications in more applied areas such as signal processing and data analysis [3], but to our knowledge they have not being studied from the point of view of complexity.

The category of constructible sheaves are closed under the so called "six operations of Grothendieck" – namely $\otimes, \mathrm{RHom}, Rf_*, Rf_!, f^{-1}, f^!$ [1] (see [28, Theorem 4.1.5]). The closure under these operations is reminiscent of the closure of the class of semi-algebraic sets under similar operations – namely, set theoretic operations, direct products, pull-backs and direct images under semi-algebraic maps. Of this the closure under the last operation – that is the fact that the image of a semi-algebraic set is also semi-algebraic – is the most non-trivial property and is equivalent to the Tarski-Seidenberg principle (see for example, [9, Chapter 2] for an exposition). The computational difficulty of this last operation – i.e. elimination of an existential block of quantifiers – is also at the heart of the $\mathbf{P}_{\mathbb{R}}$ vs $\mathbf{NP}_{\mathbb{R}}$ problem in the B-S-S theory [15, 14] .

As mentioned above the category of constructible sheaves is closed under taking direct sums, tensor products, and pull-backs. These should be considered as the "easy" operations. The statement analogous to the Tarski-Seidenberg principle is the stability under taking direct images. These observations hint at a complexity theory of such sheaves that would subsume the ordinary set theoretic complexity classes as special cases. Starting with a properly defined class, $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$, of "simple" sheaves, a conjectural hierarchy can be built up by taking successive direct images followed by truncations, tensor products etc. which resembles the polynomial hierarchy in the B-S-S model. The class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ corresponds roughly to the sequences of constructible sheaves for which there is a compatible stratification of each underlying ambient space (which we will assume to be spheres of various dimensions in this paper) which is singly exponential in size, and where the membership testing can be accomplished in polynomial time (see Definition 3.51 below for a precise definition). In this paper we lay the foundations of such a theory. We give several examples and also prove a result on the topological complexity of sequences of sheaves belonging to such a hierarchy.

Even though constructible sheaves can be defined over any fields – for the purposes of this paper we restrict ourselves to the field of real numbers.

The rest of the paper is organized as follows. In Section 2 we define new complexity classes of constructible functions, give some basic examples and pose a question analogous to the **VP** vs **VNP** conjecture in the discrete case. In Section 3, we extend these notions to the category of constructible sheaves. We begin by giving in Section 3.2 a brief introduction to the basic definitions and results of sheaf theory, especially those related to cohomology of sheaves, and derived category of complexes of sheaves with bounded cohomology, that we will need. The reader is referred to the texts [34, 28, 33, 17] for the missing details. In Section 3.3 we recall the definitions of the main complexity classes in the classical B-S-S setting. In Section 3.4 we define the new sheaf theoretic complexity class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$. In Section 3.5, we extend the definition of $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ to a hierarchy, $\boldsymbol{\mathcal{PH}}_{\mathbb{R}}$, which mirrors the compact polynomial hierarchy $\mathbf{PH}^c_{\mathbb{R}}$. We also formulate the conjectures on separations of sheaf theoretic complexity classes analogous to the classical one and prove a relationship between these conjectures in Section 3.6. In Section 4, we prove a complexity result (Theorem 4.6) bounding from above the *topological complexity* (see Definition 4.4 below) of a sequence in the class $\boldsymbol{\mathcal{PH}}_{\mathbb{R}}$. More precisely, we prove that the topological complexity of sheaves in $\boldsymbol{\mathcal{PH}}_{\mathbb{R}}$ is bounded singly exponentially, mirroring a similar result in the classical case. As a result we also obtain a singly exponential upper bound on the complexity of the "direct image functor" (Theorem 4.14) which is analogous to singly exponential upper bound results for effective quantifier elimination in the first order theory of the reals. This last result might be of interest independent of complexity theory because of its generality. Finally, in Section 5, we revisit Toda's theorem in the discrete as well as B-S-S setting, and conjecture a similar theorem in the sheaf theoretic setting.

2. Constructible functions

2.1. **Main definitions.** Our first goal is to develop a complexity theory for constructible functions on $\mathbb{R}^n$ in the style of Valiant's algebraic complexity theory. In particular, we define two new complexity classes of sequences of such functions which should be considered as "constructible analogues" of the classes defined by Valiant.

We begin with a few definitions.

Definition 2.1 (Real constructible functions)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be a *constructible function* if it is a $\mathbb{R}$-linear combination of the characteristic functions of some constructible subsets of $\mathbb{R}^n$.

Remark 2.2. Since the sum, product and constant multiples of constructible functions are again constructible, the set of constructible functions on $\mathbb{R}^n$ is an (infinite-dimensional) $\mathbb{R}$-algebra.

Example 2.3. The constant function $\mathbf{1}_{\mathbb{R}^n}$ (the characteristic function of $\mathbb{R}^n$), as well as any multiple of it, is constructible.

Example 2.4 (Rank function on matrices and tensors)**.** The function $\mathrm{rk}_{m,n} : \mathbb{R}^{m \times n} \to \mathbb{R}$ which evaluates to the rank of an $m \times n$ matrix with entries in $\mathbb{R}$ is constructible. Similarly, the rank function on higher order tensors is constructible.

We next define a notion of *size* of a formula defining a constructible function that will be used in defining different complexity classes. We first need a notation.

Notation 2.5. Let $\mathcal{P} \subset \mathbb{R}^n$ be a finite family. We call $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$ to be a *sign condition* on $\mathcal{P}$. Given a sign condition $\sigma \in \{0, 1, -1\}$ and any semi-algebraic subset $S \subset \mathbb{R}^n$, we denote by $\mathcal{R}(\sigma, S)$ the semi-algebraic set defined by

$$\mathcal{R}(\sigma, S) = \{\mathbf{x} \in S \mid \mathbf{sign}(P(\mathbf{x})) = \sigma(P), \forall P \in \mathcal{P}\},$$

and call $\mathcal{R}(\sigma, S)$ the *realization* of σ on S. We say that a sign condition σ is *realizable* on S is $\mathcal{R}(\sigma, S) \neq \emptyset$.

More generally, for any first order formula ϕ with atoms of the form $P\{=, >, <\}0, P \in \mathcal{P}$, we denote by $\mathcal{R}(\phi, S)$ (the *realization of* ϕ *on* S) the semi-algebraic set defined by

$$\mathcal{R}(\phi, S) = \{\mathbf{x} \in S \mid \phi(\mathbf{x})\}.$$

Definition 2.6 (Formulas defining constructible functions)**.** Formulas are defined inductively as follows.

(A) If ϕ is a first-order formula in the language of the field k (i.e. either, a first order formula in the language of ordered fields) then $\mathbf{1}_\phi$ is a formula (which defines the characteristic function of the constructible set $\mathcal{R}(\phi) := \{x \in \mathbb{R}^n \mid \phi(x)\}$).
(B) If F_1, F_2 are formulas, and $c \in \mathbb{R}$, then so are $F_1 + F_2, F_1F_2, c \cdot F_1$.

We now define the size of a formula defining a constructible function. We begin by defining the size of a quantifier-free first- order formula over the reals.

Definition 2.7 (Size of a first-order formula ϕ)**.** For $P \in \mathbb{R}[X_1, \ldots, X_n]$ we define the size(P) as the length of the smallest straight-line program computing P. The size of a first order formula ϕ is defined inductively as follows.

(A) For $P \in \mathbb{R}[X_1, \ldots, X_n]$, the size of the atomic formulas $P = 0, P < 0, P > 0$ are defined to be size$(P) + 1$.
(B) If $\phi = \phi_1 \wedge \phi_2$ or $\phi = \phi_1 \vee \phi_2$, then size$(\phi)$ = size(ϕ_1) + size$(\phi_2) + 1$.
(C) The size of the complement of ϕ is size$(\phi) + 1$.

We can now define a notion of size of a formula defining a constructible function.

Definition 2.8 (Size of a formula defining a constructible function)**.** Size of a formula defining a constructible function is defined inductively as follows.

(A) If ϕ is a first-order formula in the language of the reals, then

$$\text{size}(\mathbf{1}_\phi) = \text{size}(\phi) + 1.$$

(B) If F_1, F_2 are formulas and $c \in \mathbb{R}$ then

$$\text{size}(F_1 + F_2), \text{size}(F_1 F_2) = \text{size}(F_1) + \text{size}(F_2),$$

and

$$\text{size}(c \cdot F_1) = \text{size}(F_1).$$

2.2. **The class $\mathbf{VP}_\mathbb{R}^\dagger$.** We now define sequences of constructible functions that will play a role similar to that of **VP** in the discrete theory.

Definition 2.9 (The class $\mathbf{VP}_\mathbb{R}^\dagger$)**.** Let $m(n) \in \mathbb{Z}[n]$ be any non-negative polynomial. We say that a sequence of constructible functions $(f_n : \mathbb{R}^{m(n)} \to \mathbb{R})_{n>0}$ is in the class $\mathbf{VP}_\mathbb{R}^\dagger$ if for each $n > 0$ there exists a formula F_n defining f_n whose size is bounded polynomially in n.

Remark 2.10. We denote the class by $\mathbf{VP}_\mathbb{R}^\dagger$ (instead of $\mathbf{VP}_\mathbb{R}$) in order to distinguish it from the more commonly considered classes of polynomials $\mathbf{VP}_\mathbb{R}$.

An illustrative example of the power of this class is given by the following example.

Example 2.11. For $0 \leq i \leq n$, let $S_{n,i} \subset \mathbb{R}^n$ be the set of points of $\mathbb{R}^n$ with exactly i non-zero co-ordinates. Then the sequence of functions

$$\left(f_n = \mathbf{1}_{S_{n,\lceil n/2 \rceil}}\right)_{n>0}$$

belongs to the class $\mathbf{VP}_\mathbb{R}^\dagger$. To see this observe that f_n is defined by a short (i.e. polynomial sized) formula, namely,

$$f_n = a_n \prod_{0 \leq i \leq n, i \neq \lceil n/2 \rceil} (g_n - i \cdot \mathbf{1})$$

where

$$g_n = \sum_{i=1}^{n} \mathbf{1}_{(X_i \neq 0)},$$

and

$$a_n = \prod_{0 \leq i \leq n, i \neq \lceil n/2 \rceil} (i - \lceil n/2 \rceil)^{-1}.$$

We next define the B-S-S analogue of the class the **VNP**.

2.3. **The class $\mathbf{VNP}_\mathbb{R}^\dagger$.** We now define a B-S-S analogue of the class **VNP**. In order to do so we first need the notion of the (generalized) Euler-Poincaré characteristic of semi-algebraic sets.

Definition 2.12. The generalized Euler-Poincaré characteristic, $\chi(S)$, of a semi-algebraic set $S \subset \mathbb{R}^k$ is uniquely defined by the following properties [46, Chapter 4]:

(A) χ is invariant under homeomorphisms.
(B)
$$\chi(\{pt\}) = \chi([0,1]) = 1.$$
(C) χ is multiplicative, i.e. $\chi(A \times B) = \chi(A) \cdot \chi(B)$.

(D) χ is additive, i.e. $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$.

Remark 2.13. Note that the generalized Euler-Poincaré characteristic is a homeomorphism (but *not* a homotopy) invariant. For a locally, closed semi-algebraic set X,

$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{Q}} \mathrm{H}_c^i(X, \mathbb{Q}),$$

where $\mathrm{H}_c^i(X, \mathbb{Q})$ is the the i-th co-homology group of X with compact support. Thus, the definition agrees with the usual Euler-Poincaré characteristic as an alternating sum of the Betti numbers for locally closed semi-algebraic sets.

A few illustrative examples are given below.

Notation 2.14. We denote by $\mathbf{B}^n(0, r)$ the open ball in $\mathbb{R}^n$ of radius r centered at the origin. We will denote by $\mathbf{B}^n$ the open unit ball $\mathbf{B}^n(0, 1)$. Similarly, we denote by $\mathbf{S}^{n-1}(0, r)$ the sphere in $\mathbb{R}^n$ of radius r centered at the origin, and by $\mathbf{S}^{n-1}$ the unit sphere $\mathbf{S}^{n-1}(0, 1)$.

Example 2.15. For every $n \geq 0$,

(A)

$$\chi(\overline{\mathbf{B}^n}) = \chi([0, 1]^n) = \chi([0, 1])^n = 1.$$

(B)

$$\chi(\mathbf{B}_n) = \chi((0, 1)^n) = (\chi(0, 1))^n = (\chi([0, 1]) - \chi(0) - \chi(1))^n = (-1)^n.$$

(C)

$$\chi(\mathbf{S}^{n-1}) = \chi(\overline{\mathbf{B}^n}) - \chi(\mathbf{B}^n) = 1 - (-1)^n.$$

As mentioned in the introduction, the generalized Euler-Poincaré characteristic, because of its additive and multiplicative properties, can be used for constructible functions as a discrete measure to integrate against (see [47]).

More precisely:

Definition 2.16 (Integration with respect to the Euler-Poincaré characteristic)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a constructible function defined by

$$f = \sum_{i=1}^{N} a_i \mathbf{1}_{X_i}.$$

We define the *integral of f with respect to the Euler-Poincaré characteristic* (following [47]) to be

$$\int_{\mathbb{R}^n} f d\chi := \sum_{i=1}^{N} a_i \chi(X_i).$$

Remark 2.17. The fact that the definition of $\int_{\mathbb{R}^n} f d\chi$ is independent of the particular representation of the constructible function f (which is far from being unique) is a classical fact [47]. The integral defined above satisfies all the usual properties (of say the Lebesgue integral) such as additivity, Fubini-type theorem etc. [47], and in particular can be used to define "push-forwards" of (constructible) functions via fiber-wise integration.

It was mentioned in the introduction that one difficulty in defining a push-forward of functions in the B-S-S model had to do with the impossibility of computing exactly integrals with respect to usual measures on $\mathbb{R}^n$ (such as the Lebesgue measure), since such integrals could be transcendental numbers or might not converge. In contrast, we have the following effective upper bound on the complexity of computing integrals with respect to the Euler-Poincaré characteristic.

Theorem 2.18. *There exists an algorithm that takes as input a formula F describing a constructible function $f : \mathbb{R}^n \to \mathbb{R}$, and computes*

$$\int_{\mathbb{R}^n} f d\chi.$$

The complexity of the algorithm measured as the number of arithmetic operations over $\mathbb{R}$ as well as comparisons is bounded singly exponentially in n and the size of the formula F.

Proof. It is easy to verify from by an induction on the size of the formula F, that there exists a family of polynomials $\mathcal{P}_F \subset \mathbb{R}[X_1, \ldots, X_n]$, such that $\mathrm{card}(\mathcal{P}_F)$ is bounded by $\mathrm{size}(F)$, and the degrees of the polynomials in $\mathcal{P}_F$ are bounded singly exponentially in $\mathrm{size}(F)$, and moreover that f can be as a linear combination of the characteristic functions of the realizations of the various sign conditions on $\mathcal{P}_F$. More precisely, there is an expression

$$f = \sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}_F}} a_\sigma \mathcal{R}(\sigma, \mathbb{R}^n),$$

where the $a_\sigma \in \mathbb{R}$. Moreover, the set of a_σ's can be computed from F with complexity singly exponential in $\mathrm{size}(F)$. From Definition 2.16 it follows that

$$\int_{\mathbb{R}^n} f d\chi = \sum_{\sigma \in \{0,1,-1\}^{\mathcal{P}_F}} a_\sigma \chi(\mathcal{R}(\sigma, \mathbb{R}^n)).$$

It follows from them main result in [8] (see also [9, Algorithm 13.5]) that the list

$$(\chi(\mathcal{R}(\sigma, \mathbb{R}^n))_{\sigma \in \{0,1,-1\}^{\mathcal{P}_F}, \mathcal{R}(\sigma, \mathbb{R}^n) \neq \emptyset}$$

can be computed with complexity

$$(\mathrm{card}(\mathcal{P}_F) \max_{P \in \mathcal{P}_F} \deg(P))^{O(n)}.$$

Since, $\mathrm{card}(\mathcal{P}_F) \leq \mathrm{size}(F)$ and the degrees of the polynomials in $\mathcal{P}_F$ is bounded singly exponentially in $\mathrm{size}(F)$, the result follows. □

Definition 2.19 (The class $\mathbf{VNP}^\dagger_\mathbb{R}$).
We say that a sequence of constructible functions $(f_n : \mathbb{R}^n \to \mathbb{R})_{n>0}$ is in the class $\mathbf{VNP}^\dagger_\mathbb{R}$ if there exists a a sequence of constructible functions $(g_n : \mathbb{R}^n \to \mathbb{R})_{n>0}$ belonging to the class $\mathbf{VP}^\dagger_\mathbb{R}$, and a polynomial $m(n)$ such that for each $n > 0$ and $x \in \mathbb{R}^n$

$$f_n(x) = \int_{\mathbb{R}^m} g_{m+n}(\cdot, x) d\chi. \tag{2.1}$$

As an example of a sequence in the class $\mathbf{VNP}^\dagger_\mathbb{R}$ we have the following. We first introduce a notation.

Notation 2.20. For $m, n \geq 0$, let $\mathrm{rk}_{m,n} : \mathbb{R}^{m\times n} \to \mathbb{R}$ be the function that maps an $m \times n$ matrix A to its rank. Note that $\mathrm{rk}_{m,n}$ is a constructible function.

Example 2.21.

Theorem 2.22. *The sequence of functions* $(\mathrm{rk}_{n,n} : \mathbb{R}^{n\times n} \to \mathbb{R})_{n>0}$ *belongs to the class* $\mathbf{VNP}_{\mathbb{R}}$.

Proof. Let $A = (a_{i,j})$ be an $n \times n$ matrix with entries in $\mathbb{R}$. Let

$$V_0(A) = \mathbb{R}^n, m_0(A) = n,$$

and for each $i, 1 \leq i \leq n$, let

$$V_i(A) = \ker(A_i), m_i(A) = \dim(V_i(A)),$$

where A_i denotes the sub-matrix consisting of the first i rows of A. Then, for $1 \leq i \leq n$, we have

$$m_{i-1}(A) - 1 \leq m_i(A) \leq m_{i-1}(A).$$

Also notice that $\mathrm{rk}_{n,n}(A)$ is equal to the number of times there is a strict decrease in the sequence $(m_0(A), \ldots, m_n(A))$ i.e.,

$$\mathrm{rk}_{n,n}(A) = \mathrm{card}\{i \mid 1 \leq i \leq n, m_i(A) = m_{i-1}(A) - 1\}.$$

It follows that

$$\begin{aligned}
\mathrm{rk}_{n,n}(A) &= \frac{1}{2}\sum_{i=1}^{n}(1 - (-1)^{m_{i-1}(A)-m_i(A)}) \\
&= \frac{1}{2}\sum_{i=1}^{n}(1 - (-1)^{m_{i-1}(A)+m_i(A)}) \\
&= \frac{1}{2}\sum_{i=1}^{n}(1 - (-1)^{m_{i-1}(A)} \cdot (-1)^{m_i(A)}) \\
&= \frac{1}{2}\sum_{i=1}^{n}(1 - \chi(V_{i-1}(A))\chi(V_i(A))) \\
&= \frac{1}{2}\sum_{i=1}^{n}(1 - \chi(V_{i-1}(A) \times V_i(A))) \\
&= \frac{1}{2}\sum_{i=1}^{n}(\int_{\mathbb{R}^{2n}} \mathbf{1}_{\mathbb{R}^{2n}} d\chi - \int_{\mathbb{R}^{2n}} \mathbf{1}_{V_{i-1}(A)\times V_i(A)} d\chi) \\
&= \int_{\mathbb{R}^{2n}} \frac{1}{2}\left(\sum_{i=1}^{n}(\mathbf{1}_{\mathbb{R}^{2n}} - \mathbf{1}_{V_{i-1}(A)\times V_i(A)})\right) d\chi. \qquad (2.2)
\end{aligned}$$

Defining $g_n : \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$g_n(A, u, v) = \frac{1}{2}\left(\sum_{i=1}^{n}(1 - \mathbf{1}_{V_{i-1}(A)\times V_i(A)}(u, v))\right),$$

it is easy to see that the sequence

$$\left(g_n : \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\right)_{n>0}$$

is in $\mathbf{VP}^{\dagger}_{\mathbb{R}}$. It now follows from Definition 2.19 and (2.2) that the sequence of functions $(\mathrm{rk}_{n,n} : \mathbb{R}^{n\times n} \to \mathbb{R})_{n>0}$ belongs to the class $\mathbf{VNP}^{\dagger}_{\mathbb{R}}$. □

A second example of a sequence in $\mathbf{VNP}^{\dagger}_{\mathbb{R}}$ is as follows.

Notation 2.23. Given any polynomial $P \subset \mathbb{R}[X_1, \ldots, X_n]$ and a semi-algebraic set $S \subset \mathbb{R}^n$, we will denote by $\mathrm{Z}(P, S)$ the set of real zeros of P in S. More generally, for any finite family of polynomials $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_n]$ we will denote by $\mathrm{Z}(\mathcal{P}, S)$ the set of common real zeros of $\mathcal{P}$ in S.

Example 2.24. For each $n, d > 0$ let $V_{n,d} \cong \mathbb{R}^{\binom{n+d}{d}}$ denote the vector-space of polynomials in $\mathbb{R}[X_1, \ldots, X_n]$ of degree at most d, and let $E_{n,d} : V_{n,d} \to \mathbb{R}$ be the constructible function defined by

$$E_{n,d}(P) = \chi(\mathrm{Z}(P, \mathbb{R}^n)). \tag{2.3}$$

It is an easy exercise to show that the sequence of functions $(E_{n,4} : V_{n,4} \to \mathbb{R})_{n>0}$ is in $\mathbf{VNP}^{\dagger}_{\mathbb{R}}$.

We can now formulate the conjecture analogous to that of Valiant's.

Conjecture 2.25 (B-S-S Analogue of Valiant's Conjecture)**.**

$$\mathbf{VP}^{\dagger}_{\mathbb{R}} \neq \mathbf{VNP}^{\dagger}_{\mathbb{R}}.$$

Remark 2.26. It would be nice to have some complete problems for the classes defined above and reduce Conjecture 2.25 to an algebraic problem in analogy with the famous complexity of the "determinant" vs the "permanent" question in the traditional set-up. Unfortunately, we do not have at present such completeness results.

3. Sheaf theoretic reformulation of complexity questions

3.1. **Motivation.** Constructible functions on a semi-algebraic sets are intimately related to constructible sheaves. In fact, one way constructible functions on a semi-algebraic set S appear is by taking the Euler-Poincaré characteristic of stalks of some constructible sheaf on S (see Proposition 5.4 below). This already hints that the language of sheaves (or more accurately, constructible sheaves) might be useful in formulating certain questions in complexity theory in a more direct and geometric fashion. At the same time such an approach could generalize the existing questions to a more general, geometric setting.

Before delving into sheaf theory (or at least the fragment of it that would be relevant for us) let us consider an example that we formulate more precisely later. Let $\Phi(Y, X)$ be a first order formula in the language of the reals that defines a semi-algebraic subset $S \subset \mathbb{R}^m \times \mathbb{R}^n$. Let $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be the projection map to the second factor. Now consider the two semi- algebraic subsets $T, W \subset \mathbb{R}^n$ defined by the formulas $(\exists Y)\Phi(Y, X)$ and $(\forall Y)\Phi(Y, X)$. The basic problem of separating the complexity classes $\mathbf{P}_{\mathbb{R}}$ from the class $\mathbf{NP}_{\mathbb{R}}$ (respectively, $\mathbf{co\text{-}NP}_{\mathbb{R}}$) is related to proving that while testing membership in S could be easy (in polynomial time), testing membership in T (respectively, W) could be hard. The B-S-S polynomial time hierarchy, $\mathbf{PH}_{\mathbb{R}}$ (whose precise definition appears later in the paper, see Section 3) is built up by taking alternating blocks of existential and universal quantifiers, and it is conjectured that each new quantifier alternation allowed strictly increases the corresponding complexity class. The increase in complexity caused by taking image under a projection (of any easy to compute" semi-algebraic map) is at the heart of these complexity questions.

While the number of quantifier alternation is a well known measure of the complexity of a logical formula, quantifiers are not completely geometric in the following sense. Using the notation of the previous paragraph, for any fixed set $S \subset \mathbb{R}^m \times \mathbb{R}^n$, the existential and universal quantifiers can only characterize points in $\mathbf{x} \in \mathbb{R}^n$, whose corresponding fibers $S_{\mathbf{x}} = \pi^{-1}(\mathbf{x}) \cap S$ are either empty or equal to $\mathbb{R}^m$. However, one might want to characterize the set of points $\mathbf{x} \in \mathbb{R}^n$, whose corresponding fiber $S_{\mathbf{x}}$ has a certain topological property for example, non-vanishing homology groups of a certain dimension. More generally, it would be useful to include in the study of complexity of projection maps, the complexity of the coarsest possible partition of $\mathbb{R}^n$ such that the fibers $S_{\mathbf{x}}$ are locally topologically constant over each element of the partition. The word topological can be used to denote several notions – the strictest notion being that of "semi-algebraic homeomorphism". In the formulation, that we define in this paper we will use the much weaker notion of homological equivalence. Going back to the notions of $\mathbf{P}_{\mathbb{R}}$, $\mathbf{NP}_{\mathbb{R}}$ and $\mathbf{co\text{-}NP}_{\mathbb{R}}$, the sheaf-theoretic interpretation is the following. We can identify a given semi-algebraic set S (which we will assume to be compact in this paper), with a very special kind of *constructible sheaf* – namely, the constant sheaf $\mathbb{Q}_S$ (see Definition 3.2 and Example 3.11). The constructible sheaves $\mathbb{Q}_T$, $\mathbb{Q}_W$ corresponding to the compact sets T and W defined by $(\exists Y)\Phi(Y, X)$ and $(\forall Y)\Phi(Y, X)$ respectively, have sheaf theoretic descriptions. Namely,

$$
\begin{aligned}
\mathbb{Q}_T &= \tau^{\geq 0}\tau^{\leq 0} R\pi_* \mathbb{Q}_{J_\pi(S)}, \\
\mathbb{Q}_W &= \tau^{\geq m}\tau^{\leq m} R\pi_! \mathbb{Q}_S[m].
\end{aligned} \tag{3.1}
$$

The definitions of the, *fibered join* $J_\pi(S)$, the *truncation* functors τ, and the *direct image functors* $R\pi_*, R\pi_!$ in the *derived category of constructible sheaves* is given later (see Section 2.1 below). But Eqn. (3.1) contains the crux of the idea behind defining sheaf-theoretic complexity classes. We first define a class of simple sheaves (an analogue of the set theoretic B-S-S class $\mathbf{P}_{\mathbb{R}}$) which contain in particular the class of constant sheaves $\mathbb{Q}_S$ for all S belonging to the class $\mathbf{P}_{\mathbb{R}}$ (see Definition 3.51 below). We show that this class of constructible sheaves is stable under certain sheaf-theoretic operations mirroring the stability of the class $\mathbf{P}_{\mathbb{R}}$ under operations such as union, intersections, products and pull-backs. We then show how to build (conjecturally) more complicated sheaves using the direct image and truncation functors. We build a hierarchy of sequences of sheaves, mirroring that of $\mathbf{PH}_{\mathbb{R}}$, where the place of a sequence in the hierarchy depends on the number of direct image functors used in the definition of the sheaves in the sequence (in lieu of the number quantifier alternations in the set-theoretic case).

3.2. **Background on sheaves and sheaf cohomology.** We give here a brief redux of the definitions and main results from sheaf theory that would be necessary for the rest of the paper. We refer the reader to [31] for more details concerning sheafs including the basic definitions, and to the books [33, 30, 17, 34] for details regarding derived categories and hypercohomology. In particular, the book [34] is a good reference for constructible sheaves in the context of the current paper.

Let A be a fixed commutative ring. Later on for simplicity we will specialize to the case when $A = \mathbb{Q}$.

Definition 3.1. (Pre-sheaf of A-modules) A *pre-sheaf* $\mathcal{F}$ of A-modules over a topological space X associates to each open subset $U \subset X$ an A-module $\mathcal{F}(U)$, such that that for all pairs of open subsets U, V of X, with $V \subset U$, there exists a *restriction* homomorphism $r_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ satisfying:

(A) $r_{U,U} = \mathrm{Id}_{\mathcal{F}(U)}$,
(B) for U, V, W open subsets of X, with $W \subset V \subset U$,

$$r_{U,W} = r_{V,W} \circ r_{U,V}.$$

(For open subsets $U, V \subset X$, $V \subset U$, and $s \in \mathcal{F}(U)$, we will sometimes denote the element $r_{U,V}(s) \in \mathcal{F}(V)$ simply by $s|_V$.)

Definition 3.2. (Sheaf of A-modules) A pre-sheaf $\mathcal{F}$ of A-modules on X is said to be a *sheaf* if it satisfies the following two axioms. For any collection of open subsets $\{U_i\}_{i\in I}$ of X with $U = \bigcup_{i\in I} U_i$;

(A) if $s \in \mathcal{F}(U)$ and $s|_{U_i} = 0$ for all $i \in I$, then $s = 0$;
(B) if for all $i \in I$ there exists $s_i \in \mathcal{F}(U_i)$ such that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

for all $i, j \in I$, then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each $i \in I$.

Definition 3.3. Let $\mathcal{F}$ be a (pre)-sheaf of A-modules on X and $\mathbf{x} \in X$. The *stalk* $\mathcal{F}_{\mathbf{x}}$ of $\mathcal{F}$ at x is defined as the inductive limit

$$\mathcal{F}_{\mathbf{x}} = \varinjlim_{U \ni \mathbf{x}} \mathcal{F}(U).$$

For any $U \ni \mathbf{x}$, there exists a canonical homomorphism $\mathcal{F}(U) \to \mathcal{F}_{\mathbf{x}}$, and the image of $s \in \mathcal{F}(U)$ under this homomorphism is denoted by s_x. For $s_1, s_2 \in \mathcal{F}(U)$, we have $(s_1)_{\mathbf{x}} = (s_2)_{\mathbf{x}}$ if and only if there exists an open $V \subset U$ with $\mathbf{x} \in V$, such that $s_1|_V = s_2|_V$.

The subset of X defined by $\{\mathbf{x} \in X \mid \mathcal{F}_{\mathbf{x}} \neq 0\}$ is called the *support* of the sheaf $\mathcal{F}$ and denoted $\mathrm{Supp}(\mathcal{F})$. It is easy to check that this is always a closed subset of X. In particular, if X is a compact semi-algebraic set, then so is $\mathrm{Supp}(\mathcal{F})$ for any sheaf $\mathcal{F}$ on X.

There is a canonically defined sheaf associated to any pre-sheaf. This is important since certain operations on a sheaf such as taking co-kernels of a sheaf morphism or the inverse image (see below) produces only a pre-sheaf which then needs to sheafified if we are to stay within the category of sheaves.

Definition 3.4. (Sheaf associated to a pre-sheaf) Let $\mathcal{F}$ be a pre-sheaf of A-modules over X. Then, the sheaf $\mathcal{F}'$ associated to $\mathcal{F}$ (the *sheafification* of $\mathcal{F}$) is defined by associating to each open subset $U \subset X$, the A-module $\mathcal{F}'(U)$ consisting of all maps

$$s : U \to \bigcup_{\mathbf{x}\in U} \mathcal{F}_{\mathbf{x}}$$

satisfying the condition that for every $x \in U$, and any open neighborhood U' of $\mathbf{x}$ in U, there exists $s' \in \mathcal{F}(U')$, such that for all $\mathbf{x}' \in U'$, $s(\mathbf{x}') = s'_{\mathbf{x}'}$.

Notation 3.5. We will denote by $\mathbf{Sh}(X, A)$ the category whose objects are sheaves of A-modules on X. For each open subset $U \subset X$, we denote by $\Gamma(U, \cdot)$ the functor from $\mathbf{Sh}(X, A)$ to the category A**-mod** of A-modules, obtained by setting $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$.

More generally, for any locally closed subset $Z \subset X$ we set $\Gamma_Z(U, \cdot)$ to be the functor from $\mathbf{Sh}(X, A)$ to the category A-mod of A-modules, obtained by setting $\Gamma(U, \mathcal{F}) = \ker(\mathcal{F}(U) \to \mathcal{F}(U \setminus Z))$, i.e. $\Gamma_Z(U, \mathcal{F})$ is the A-module of sections of $\mathcal{F}(U)$ whose supports are contained in $Z \cap U$. Notice that $\Gamma(U, \cdot)$ defined earlier coincides with $\Gamma_X(U, \cdot)$.

Moreover, we denote by $\Gamma_Z(\mathcal{F})$ the sheaf on X defined by $\Gamma_Z(\mathcal{F})(U) = \Gamma_Z(U, \mathcal{F})$ for each open subset $U \subset X$.

Example 3.6. The simplest sheaf of A-modules on a topological space X is the constant sheaf (denoted A_X) defined by setting $A_X(U) = A$ for every open subset $U \subset X$. Each stalk $(A_X)_{\mathbf{x}}$ of A_X is then isomorphic to A.

Definition 3.7. Let X, Y be topological spaces and $f : X \to Y$ a continuous map. Let $\mathcal{F}$ (respectively, $\mathcal{G}$) be a sheaf of A-modules on X (respectively, Y).

- The association to each open set $U \subset Y$, the A-module $\mathcal{F}(f^{-1}(U)$ defines a sheaf of A-modules on Y, and this sheaf is denoted by $f_*\mathcal{F}$ and is called the *direct image* of $\mathcal{F}$ under f.
- A sub-sheaf of $f_*\mathcal{F}$, namely the *direct image under f with proper support*, denoted $f_!\mathcal{F}$, will sometimes also be important for us. It is defined by associating to each open set $U \subset Y$, the sub-module of the A-module $\mathcal{F}(f^{-1}(U)$, consisting only of those elements $s \in \mathcal{F}(f^{-1}(U)$, such that $f|_{\mathrm{Supp}(s)} : \mathrm{Supp}(s) \to U$ is a proper map (i.e. inverse image of any compact set is compact). It is clear from the definition that $f_!\mathcal{F}$ is a sub-sheaf of $f_*\mathcal{F}$.
- The association to each open set U of X, the A-module
$$\varinjlim_{V \supset f(U)} \mathcal{G}(V),$$
defines a *pre-sheaf* on X. The sheaf associated to this pre-sheaf (Definition 3.4) is denoted by $f^{-1}(\mathcal{G})$ and is called the *inverse image* of $\mathcal{G}$ under f.

Definition 3.8. (Morphisms between sheaves) Let $\mathcal{F}, \mathcal{G}$ be two sheaves on X. Then a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is given by associating to each open subset U of X, an element $\phi(U) \in \hom_A(\mathcal{F}(U), \mathcal{G}(U))$ such that for all pairs of open subsets U, V of X with $V \subset U$ the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{r_{U,V}} & \mathcal{F}(V) \\ \downarrow{\scriptstyle \phi(U)} & & \downarrow{\scriptstyle \phi(V)} \\ \mathcal{G}(U) & \xrightarrow{r_{U,V}} & \mathcal{G}(V) \end{array}$$

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism, then for each $\mathbf{x} \in X$ there is an induced homomorphism $\phi_{\mathbf{x}} : \mathcal{F}_{\mathbf{x}} \to \mathcal{G}_{\mathbf{x}}$.

Definition 3.9. (Kernel, co-kernel of sheaf morphisms) If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X, then the *kernel* of ϕ, denoted $\ker(\phi)$, is the sheaf which associates to each open subset U of X, the A-module $\ker(\phi)(U) = \ker(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))$. In particular, note that for each $x \in X$, the stalk $(\ker(\phi))_{\mathbf{x}} = \ker(\phi_{\mathbf{x}} : \mathcal{F}_{\mathbf{x}} \to \mathcal{G}_{\mathbf{x}})$.

The *co-kernel* of ϕ, denoted $\mathrm{coker}(\phi)$ is the sheaf associated to the pre-sheaf (this pre-sheaf in general is not a sheaf) which associates to each open $U \subset X$, the A-module $\mathcal{G}(U)/\mathrm{Im}(\phi(U))$.

Definition 3.10. (Direct sum, tensor product and hom of two sheaves) If $\mathcal{F}, \mathcal{G}$ are sheaves on X, then the sheaves $\mathcal{F} \oplus \mathcal{G}$,(respectively, $\mathcal{F} \otimes \mathcal{G}$, $\hom_A(\mathcal{F}, \mathcal{G})$) is the sheaf obtained by associating to any open $U \subset X$, the A-module $\mathcal{F}(U) \oplus \mathcal{G}(U)$ (respectively, $\mathcal{F}(U) \otimes_A \mathcal{G}(U)$, $\hom_A(\mathcal{F}(U), \mathcal{G}(U))$).

Example 3.11. (The sheaf A_X) The sheaf on X that associates to each open subset U of X, the module A is denoted by A_X and is called the *constant sheaf on* X . More generally, if Z is a closed subset of X, and $i : Z \hookrightarrow X$ the corresponding inclusion map, then will denote by A_Z the sheaf on X defined by

$$A_Z = i_* i^{-1}(A_X).$$

Definition 3.12. A sheaf $\mathcal{F}$ on a semi-algebraic set X is called locally constant if there exists an open cover of X by open semi-algebraic sets $(U_i)_{i \in I}$ such that for each $i \in I$, $\mathcal{F}|_{U_i}$ is a constant sheaf.

We will need the following proposition later in the paper.

Proposition 3.13. *If X is contractible then any locally constant sheaf on X is isomorphic to a constant sheaf.*

Proof. See [33, Proposition 4.20] or [34, page 132]. □

Notation 3.14. We denote by $\Gamma(Z, \cdot)$ the functor from $\mathbf{Sh}(X, A)$ to A-**mod** defined by $\mathcal{F} \mapsto A_Z(X)$. This agrees with the prior definition of $\Gamma(U, \cdot)$ when U is an open subset of Z.

Warning 3.15. Note that the functor $\Gamma_Z(\cdot)$ defined earlier in Notation 3.5 takes objects in $\mathbf{Sh}(X, A)$ to objects in $\mathbf{Sh}(X, A)$, whereas the functor $\Gamma(Z, \cdot)$ defined above in Notation 3.14 takes objects in $\mathbf{Sh}(X, A)$ to A-modules.

Remark 3.16. The category $\mathbf{Sh}(X, A)$ is an *abelian category* [34, Proposition 2.2.4]. Roughly, this means that every morphism between $\phi : \mathcal{F} \to \mathcal{G}$ between sheaves admit a kernel and a co-kernel, and give rise to the following diagram in which the induced morphism u is an isomorphism (see [34, Definition 1.2.1]).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\phi) & \longrightarrow & \mathcal{F} & \xrightarrow{\phi} & \mathcal{G} & \longrightarrow & \mathrm{coker}(\phi) \longrightarrow 0 \\ & & & & \downarrow & & \uparrow & & \\ & & & & \mathrm{coimage}(\phi) & \xrightarrow{u} & \mathrm{Im}(\phi) & & \end{array}$$

In particular, this means we have the notion of exactness in the category $\mathbf{Sh}(X, A)$. Recall that a sequence of morphisms $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ is exact if $\ker(g) = \mathrm{Im}(f)$.

Remark 3.17. Since the category $\mathbf{Sh}(X, A)$ is an abelian category, it makes sense to talk about *exactness* of the functors $f^{-1}, f_*, f_!, \Gamma(Z, \cdot), \Gamma_Z$ defined earlier. It turns out that f^{-1} is an exact functor (i.e. it takes an exact sequence of sheaves to an exact sequence), while the functors $f_*, f_!, \Gamma(Z, \cdot), \Gamma_Z$ are only *left exact* (see [31, page 15] for definition of exactness as well as left exactness of functors).

3.2.1. *Sheaf cohomology.* The functors $\Gamma(Z,\cdot),\Gamma_Z$ defined above are left-exact but not exact – i.e. they do not preserve exactness when applied to an exact sequence of morphism on sheaves. Sheaf cohomology is a measure of this deviation from exactness, and is defined by applying one of these (non-exact) functor to an *injective resolution* ([31] of a given sheaf (which always exists since the category $\mathbf{Sh}(X,A)$ has enough *injective objects* [34, II, 2.6]), and then taking the homology of the resulting (not necessarily exact complex).

In this way we obtain functors $\mathrm{H}^i(Z,\cdot),\mathrm{H}^i_Z(\cdot)$ from the category of $\mathbf{Sh}(X,A)$ to the categories A-**mod** and $\mathbf{Sh}(X,A)$ respectively. More accurately, $\mathrm{H}^i(Z,\cdot) = R^i\Gamma(Z,\cdot)$ is the i-th *right derived functor* [31] of $\Gamma(Z,\cdot)$, and $\mathrm{H}^i_Z(\cdot) = R\Gamma^i_Z(\cdot)$ is the i-th right derived functor of Γ_Z. It is very important to remember that $\mathrm{H}^i(Z,\mathcal{F})$ is an A-module, whereas $\mathrm{H}^i_Z(\mathcal{F})$ is an object of $\mathbf{Sh}(X,A)$.

Now suppose that $f : X \to Y$ is a continuous semi-algebraic map. Then, $f_*, f_!$ is a functor from $\mathbf{Sh}(X,A)$ to $\mathbf{Sh}(Y,A)$ which is left exact, and carries injectives to injectives, and thus one obtains for any $\mathcal{F} \in \mathrm{Ob}(\mathbf{Sh}(X,A))$ and $i \geq 0$, the *higher direct image sheaves* $R^i f_*(\mathcal{F}), R^i f_!(\mathcal{F}) \in \mathrm{Ob}(\mathbf{Sh}(Y,A))$. The following example is instructive.

Example 3.18. Let S be a compact semi-algebraic subset of $\mathbb{R}^m \times \mathbb{R}^n$ defined by a first order formula $\Phi(Y,X)$, where $Y = (Y_1,\ldots,Y_m)$ and $X = (X_1,\ldots,X_n)$. Let T be the semi-algebraic subset of $\mathbb{R}^n$ defined by the formula $(\forall Y)\Phi(Y,X)$. Let $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ denote the projection map to the second factor, and for each $\mathbf{x} \in \mathbb{R}^n$, let $S_{\mathbf{x}} = \pi^{-1}(\mathbf{x}) \cap S$. Then, notice that $\mathbf{x} \in T$ if and only if $S_{\mathbf{x}} = \mathbb{R}^n$, if and only if $\mathrm{H}^m_c(S_{\mathbf{x}},\mathbb{Q}) \cong \mathbb{Q}$. This implies that the sheaf $\mathbb{Q}_T$ can also be expressed as $R^m\pi_!\mathbb{Q}_S$. Notice that the universal quantifier has been replaced by the higher direct image functor $R^m\pi_!$.

It turns out that for our purposes we need to enlarge the category $\mathbf{Sh}(X,A)$ by letting for each open $U \subset X$, $\mathcal{F}(U)$ to be not just an A-module, but a *complex* of A-modules.

Recall that:

Definition 3.19. A complex $C^\bullet$ of A-modules is a sequence of homomorphisms $(C^i \xrightarrow{\phi^i} C^{i+1})_{i\in\mathbb{Z}}$ such that $\phi^{i+1} \circ \phi^i = 0$ for each i. Given a complex $C^\bullet$, the i-th cohomology module $\mathrm{H}^i(\mathbb{C}^\bullet)$ is defined as the quotient $\ker(\phi^i)/\mathrm{Im}(\phi^{i-1})$.

Definition 3.20. (Truncation and shift operators). Given a complex

$$C^\bullet = \left(\cdots \xrightarrow{\delta^{i-1}} C^i \xrightarrow{\delta^i} C^{i+1} \xrightarrow{\delta i+1} \cdots\right)$$

of A-modules, we define for any $n \in \mathbb{Z}$ the truncated complex $\tau^{\leq n}(C^\bullet)$ to be the complex

$$\tau^{\leq n}(C^\bullet) = \left(\cdots \xrightarrow{\delta^{n-3}} C^{n-2} \xrightarrow{\delta^{n-2}} C^{n-1} \xrightarrow{\delta^{n-1}} \ker(\delta^n) \to 0 \to 0 \cdots\right).$$

Similarly, we define the truncated complex $\tau^{\geq n}(C^\bullet)$ to be the complex

$$\tau^{\geq n}(C^\bullet) = \left(\cdots 0 \to 0 \to \mathrm{coker}(\delta^n) \xrightarrow{\delta^n} C^{n+1} \xrightarrow{\delta^{n+1}} C^{n+2} \xrightarrow{\delta^{n+2}} \cdots\right).$$

We define the shifted complex

$$C[n]^\bullet = \left(\cdots \xrightarrow{\delta[n]^{i-1}} C[n]^i \xrightarrow{\delta[n]^i} C[n]^{i+1} \xrightarrow{\delta[n]^{i+1}} \cdots \right),$$

by setting $C[n]^i = C^{i+n}$ and $\delta[n]^i = \delta^{i+n}$.

Definition 3.21. The category $\mathbf{Kom}(A\textbf{-mod})$ is defined as the category whose objects are complexes of A-module and whose morphisms are morphisms of complexes. We say that a morphism $f^\bullet : C^\bullet \to D^\bullet$ (i.e.$f^\bullet \in \hom_{\mathbf{Kom}(A\textbf{-mod})}(C^\bullet, D^\bullet)$) is a *quasi-isomorphism* if the induced homomorphism, $f^i : \mathrm{H}^i(C^\bullet) \to \mathrm{H}^i(D^\bullet)$ is an isomorphism for all $i \in \mathbb{Z}$.

Definition 3.22. Two morphisms of complexes $f^\bullet, g^\bullet : C^\bullet \to D^\bullet$ are said to be *homotopically equivalent* ($f^\bullet \sim g^\bullet$) if there exists a collection of homomorphisms $h^i : C^i \to D^{i-1}, i \in \mathbb{Z}$, such that

$$f^i - g^i = h^{i+1} \circ \delta^i_{C^\bullet} + \delta^{i-1}_{D^\bullet} \circ h^i.$$

The *homotopy category of complexes* $\mathbf{K}(A\textbf{-mod})$ is the category whose objects are the same as the objects of $\mathbf{Kom}(A\textbf{-mod})$, but whose morphisms are defined by

$$\hom_{\mathbf{K}(A\textbf{-mod})}(C^\bullet, D^\bullet) = \hom_{\mathbf{Kom}(A\textbf{-mod})}(C^\bullet, D^\bullet)/\sim .$$

Finally (we are being slightly imprecise here in the interest of space and readability; the reader should refer to [30, Chapter III] for a more precise definition):

Definition 3.23. The *derived category* of A-modules, $\mathbf{D}(A\textbf{-mod})$, is the category whose objects are the same as those of $\mathbf{K}(A\textbf{-mod})$, but whose set of morphisms $\hom_{\mathbf{D}(A\textbf{-mod})}(C^\bullet, D^\bullet)$ are equivalence classes of diagrams of the form

$$\begin{array}{ccc} & E^\bullet & \\ {\scriptstyle qis}\swarrow & & \searrow \\ C^\bullet & & D^\bullet \end{array}$$

where the left arrow is an quasi-isomorphism. The derived category $\mathbf{D}(A\textbf{-mod})$ is no longer an abelian category, but is an example of what is known as a triangulated category [30, Chapter IV]. Finally, restricting to complexes with bounded cohomology (i.e. complexes $C^\bullet$ such that $\mathrm{H}^i(\mathbb{C}^\bullet) = 0$ for $|i| \gg 0$) we obtain the corresponding categories, $\mathbf{Kom}^b(A\textbf{-mod}), \mathbf{K}^b(A\textbf{-mod}), \mathbf{D}^b(A\textbf{-mod})$.

Proposition 3.24. *If $F \in \mathbf{D}^b(A\textbf{-mod})$, then we have the isomorphism (in the derived category)*

$$F \cong \bigoplus_{m \in \mathbb{Z}} \mathrm{H}^m(F)[-m]$$

Proof. See [34, Ex. 1.18]. □

3.2.2. *The categories* $\mathbf{Kom}^b(X, A)$*,* $\mathbf{K}^b(X, A)$ *and* $\mathbf{D}^b(X, A)$*.* Passing to sheaves over a semi-algebraic set X, the definitions of the last section gives successively the category $\mathbf{Kom}^b(X, A)$ of sheaves of complexes of A-modules on X whose stalks have bounded cohomology and which contains the category of $\mathbf{Sh}(X, A)$ as a subcategory of sheaves of complexes concentrated in degree 0. Furthermore, considering complexes only up to homotopy gives rise to the homotopy category $\mathbf{K}^b(X, A)$ of sheaves of complexes on X whose stalks have bounded cohomology, and finally by a localization process, needed to formally invert arrows which are homotopy equivalences, we arrive at the derived category $\mathbf{D}^b(X, A)$ of sheaves on X. This passage from $\mathbf{Sh}(X, A)$ to $\mathbf{D}^b(X, A)$ while fairly standard, takes up a couple of chapters in textbooks and we refer the reader to [34] for the details. While the definition of the derived category might seem too cumbersome and unnecessary at first glance, once its existence is taken for granted, it provides a very useful and concise geometric language to express relationships especially pertaining to cohomology of sheaves which is very useful in many applications

3.2.3. *Extension of operations on sheaves to the derived category.* The sheaf operations of taking direct sums, tensor products, hom, and direct and inverse images under maps $f : X \to Y$ extend to the category $\mathbf{D}^b(X, A)$ as follows. Since the functors $\Gamma(Z, \cdot)$, $\Gamma_Z(\cdot)$ are left exact (see 3.17) and take injective objects to injective objects, they induce corresponding derived functors $R\Gamma(Z, \cdot)$, $R\Gamma_Z(\cdot)$ which take objects in $\mathbf{D}^b(X, A)$ to objects in $\mathbf{D}^b(A)$ and $\mathbf{D}^b(X, A)$ respectively. Similarly, for any map $f : X \to Y$, we have the derived functors $Rf_!, Rf_* : \mathbf{D}^b(X, A) \to \mathbf{D}^b(Y, A)$. The functor f^{-1} being exact extends directly to a functor $f^{-1} : \mathbf{D}^b(Y, A) \to \mathbf{D}^b(X, A)$ in the derived category.

Definition 3.25. The images of the higher derived functors, $R^i\Gamma$ of the global section functor $\Gamma : \mathbf{Sh}(X, A) \to A\textbf{-mod}$, will be denoted by $\mathbb{H}^i(X, \mathcal{F})$ for $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(X, A))$ and call $\mathbb{H}^i(X, \mathcal{F})$ the *i-th hypercohomology group* of $\mathcal{F}$.

Given $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(X, A))$, and $\mathbf{x} \in X$, the stalk $\mathcal{F}_{\mathbf{x}}$ is represented by complex of A-modules (i.e. an object of the category $\mathbf{C}^b(A)$). Thus, the cohomology groups $\mathrm{H}^i(\mathcal{F}_{\mathbf{x}})$ are A-modules and vanish for $|i| \gg 0$. Furthermore, recalling the definitions of the truncation and shift operations on complexes of A-modules (Definition 3.20), these operations extend naturally to $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(X, A))$, and we obtain for each $n \in \mathbb{Z}$, $\tau^{\leq n}\mathcal{F}, \tau^{\geq n}\mathcal{F}, \mathcal{F}[n]$ respectively. It is an immediate consequence of the definitions that:

Proposition 3.26. *Suppose $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}(X, A))$. Then, for $n \in \mathbb{Z}$ and $\mathbf{x} \in X$, we have:*

$$\begin{aligned} \mathrm{H}^i((\tau^{\leq n}\mathcal{F})_{\mathbf{x}}) &= \mathrm{H}^i(\mathcal{F}_{\mathbf{x}}) \text{ for } i \leq n, \\ \mathrm{H}^i((\tau^{\leq n}\mathcal{F})_{\mathbf{x}}) &= 0 \text{ for } i > n, \\ \mathrm{H}^i((\tau^{\geq n}\mathcal{F})_{\mathbf{x}}) &= \mathrm{H}^i(\mathcal{F}_{\mathbf{x}}) \text{ for } i \geq n, \\ \mathrm{H}^i((\tau^{\leq n}\mathcal{F})_{\mathbf{x}}) &= 0 \text{ for } i < n, \\ \mathrm{H}^i((\mathcal{F}[n])_{\mathbf{x}}) &= \mathrm{H}^{i+n}(\mathcal{F}_{\mathbf{x}}) \text{ for all } i \in \mathbb{Z}. \end{aligned}$$

Proof. Follows immediately from Definition 3.20. □

There are two important isomorphisms in the derived category that will play a role later in the paper. We record them here for convenience.

Let $f : X \to Y$ be a continuous semi-algebraic map, and suppose that X, Y are compact. In particular, this means that f is proper. Suppose also that, Let $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(X, A)), \mathcal{G} \in \mathrm{Ob}(\mathbf{D}^b(Y, A))$, $\mathbf{x} \in X, \mathbf{y} = f(\mathbf{x})$.

Proposition 3.27. *(Adjunction isomorphism) For every $i \in Z$*

$$\mathrm{H}^i((f^{-1}(\mathcal{G})_{\mathbf{x}})) \cong \mathrm{H}^i((\mathcal{G})_{\mathbf{y}}).$$

Proof. See [33, II.4]. □

Proposition 3.28. *(Proper base change theorem) For every $i \in \mathbb{Z}$,*

$$\mathrm{H}^i((Rf_*(\mathcal{F}))_{\mathbf{y}}) \cong \mathrm{H}^i(f^{-1}(\mathbf{y}), \mathcal{F}).$$

Proof. See [33, Theorem 6.2]. □

We also have the following proposition about composition of maps and the direct image functor.

Proposition 3.29. *If $f : X \to Y$, and $g : Y \to Z$ are continuous semi-algebraic maps between compact semi-algebraic sets, and $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(X, A))$, then*

$$R(g \circ f)_*\mathcal{F} \cong Rg_*(Rf_*\mathcal{F}).$$

Proof. See [34, Section 2.6, page 111]. □

Finally, we are going to use a Mayer-Vietoris spectral sequence for bounding the dimensions of the hypercohomology groups. We assume familiarity with spectral sequences in the following.

Proposition 3.30. *(Leray-Serre spectral sequence) Let A be a field and let X be a locally closed semi-algebraic set, and $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(X, A)$. Then, there is a spectral sequence abutting to to $\mathbb{H}^*(X, \mathcal{F})$ whose E_2 term is given by*

$$E_2^{p,q} = \mathrm{H}^p(X, \mathcal{H}^q(\mathcal{F})),$$

where $\mathcal{H}^q(\mathcal{F}) \in \mathrm{Ob}(\mathbf{Sh}(X, A))$ is the sheaf defined by: $\Gamma(U, \mathcal{H}^q(\mathcal{F})) = \mathrm{H}^q(\mathcal{F}(U))$ for every open subset U of X. In particular, if $\mathrm{H}^*((\mathcal{F}_{\mathbf{x}})$ is constant for $\mathbf{x} \in X$, then this spectral sequence degenerates at its E_2-term, and we have

$$\mathbb{H}^m(X, \mathcal{F}) \cong \bigoplus_{p+q=m} \mathrm{H}^p(X, A) \otimes \mathrm{H}^q((\mathcal{F}_{\mathbf{x}})),$$

for all $\mathbf{x} \in X$.

Proof. This is a standard fact. See for example [31, Theorem 4.6.1] or [28, Corollary 2.3.4]. □

Proposition 3.31. *(Mayer-Vietoris spectral sequence) Let $\mathbf{K}$ be a compact semi-algebraic set and $\mathcal{C} = \{C_i\}_{i \in I}$ a finite covering of $\mathbf{K}$ by semi-algebraic closed subsets. Let $\mathcal{F} \in \mathbf{Sh}(\mathbf{K}, A)$. Then, there is a spectral sequence abutting to $\mathrm{H}^*(X, \mathcal{F})$ whose E_2-term is give by*

$$E_2^{p,q} = \bigoplus_{(i_0,\ldots,i_p)} \mathrm{H}^q(C_{i_0,\ldots,i_p}, \mathcal{F}|_{C_{i_0,\ldots,i_p}}),$$

where $C_{i_0,\ldots,i_p} = C_{i_0} \cap \cdots \cap C_{i_p}$.

Proof. Follows from [34, Lemma 2.8.2] and standard arguments using spectral sequences arising from double complexes. □

3.2.4. *Constructible Sheaves.*

Definition 3.32. Following [34], we call a complex $C^\bullet$ of A-modules to be a *perfect* complex if it is isomorphic (in the derived category) to a bounded complex of finitely generated projective A-modules.

Definition 3.33 (Constructible Sheaves)**.** Let X be a locally closed semi-algebraic set. Following [34], an object $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(X, A))$ is said to be *constructible* if it satisfies the following two conditions:

(A) There exists a finite partition $X = \cup_{i \in I} C_i$ of X by locally closed semi-algebraic subsets such that for $j \in \mathbb{Z}$ and $i \in I$, the $\mathrm{H}^j(\mathcal{F})|_{C_i}$ are locally constant. This means that for each $i \in I$, and $j \in \mathbb{Z}$, the sheaf on C_i associated to the pre-sheaf defined by $U \mapsto \mathrm{H}^j(\mathcal{F}|_U)$ is a constant sheaf (cf. Definition 3.12). This is equivalent to saying that for each $i \in I$, and $\mathbf{x} \in C_i$, there exists an open neighborhood U of $\mathbf{x}$ in C, such that for every $\mathbf{x}' \in U$, the restriction map r induces isomorphisms $r_* : \mathrm{H}^*(\mathcal{F}(U)) \to \mathrm{H}^*((\mathcal{F})_{\mathbf{x}'})$. We call such a partition to be *subordinate* to $\mathcal{F}$.

For each $\mathbf{x} \in X$ we denote by $P_{\mathcal{F}_\mathbf{x}} \in \mathbb{Z}[T]$ the Poincaré polynomial of the bounded complex $\mathrm{H}^*(\mathcal{F}_\mathbf{x})$ (which is in fact a Laurent polynomial in this case) defined by

$$P_{\mathcal{F}_\mathbf{x}}(T) = \sum_{i \in \mathbb{Z}} (\dim_A \mathrm{H}^i(\mathcal{F}_x)) T^i.$$

(B) For each $x \in X$, the stalk $\mathcal{F}_\mathbf{x}$ is a perfect complex. In particular, this means that for $j \in \mathbb{Z}$, the cohomology groups $\mathrm{H}^j(\mathcal{F}_\mathbf{x})$ are finitely generated, and and there exists N such that $\mathrm{H}^j(\mathcal{F}_\mathbf{x}) = 0$ for all $\mathbf{x} \in X$ and $|j| > N$.

We will denote the category of constructible sheaves on X by $\mathbf{D}^b_{\mathrm{sa}}(X, A)$.

From now one we will fix $A = \mathbb{Q}$ for convenience. Then all A-modules are projective, and in fact vector spaces over $\mathbb{Q}$. We will henceforth drop in all the notation the reference to the ring A, taking $A = \mathbb{Q}$.

Example 3.34. Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic subset. Then the sheaf $\mathbb{Q}_X$ is a constructible sheaf on $\mathbb{R}^n$.

3.2.5. *Closure of the category of constructible sheaves under sheaf operations.*

Lemma 3.35. *Let X, Y be semi-algebraic sets, $f : X \to Y$ a semi-algebraic map, and $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(X)), \mathcal{G} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(Y))$ constructible. Then,*

(A) *Then $f^{-1}\mathcal{G} \in \mathbf{D}^b_{\mathrm{sa}}(X)$.*

(B) *Suppose that f is proper restricted to* $\mathrm{Supp}(\mathcal{F})$*. Then, $Rf_*(\mathcal{F}), Rf_!(\mathcal{F}) \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(Y))$.*

Proof. See [34, Propositions 8.4.8, 8.4.10]. □

The following is a key example.

Example 3.36. Let $S \subset \mathbb{R}^m \times \mathbb{R}^n$ be a compact semi-algebraic set and $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ the projection map to the second factor. Clearly the map π is proper restricted to $\mathrm{Supp}(\mathbb{Q}_S) = S$. Then, for each $\mathbf{x} \in \mathbb{R}^n$, we have using Propositions 3.28 and 3.24 the following isomorphism (in the derived category):

$$(R\pi_*\mathbb{Q}_S)_\mathbf{x} \cong (R\pi_!\mathbb{Q}_S)_\mathbf{x} \cong \bigoplus_n \mathrm{H}^n(S_\mathbf{x}, \mathbb{Q})[-n],$$

where $S_{\mathbf{x}} = \pi^{-1}(\mathbf{x}) \cap S$. In other words, the stalks of $R\pi_*\mathbb{Q}_S$ are isomorphic to the cohomology groups (with coefficients in $\mathbb{Q}$) of the fiber of S over x under the map π. It follows from Hardts theorem of triviality of semi-algebraic map [16, Theorem 9.3.2.], that there is a finite semi-algebraic partition of $\mathbb{R}^n$ into connected, locally closed semi-algebraic sets, such that for each set C of the partition, the homeomorphism type of the fibers $S_{\mathbf{x}}, \mathbf{x} \in C$ and hence the stalks $(R\pi_*\mathbb{Q}_S)_{\mathbf{x}}$ stays invariant. This is the sheaf theoretic analogue of *quantifier elimination* in the first order theory of the reals. Later in the paper we prove an effective version of this result (see Theorem 4.14 below).

3.3. **Real analogue of PH.** Our ultimate goal is to generalize the set theoretic complexity classes to a more general sheaf theoretic context. Before doing so we recall the definitions of the standard B-S-S complexity classes. In fact, we will need to use the "compact versions" of these classes which were introduced in [13] in order to avoid certain difficulties arising from non-proper projection maps.

We first recall the definition of the B-S-S the polynomial hierarchy for the reals. It mirrors the discrete case very closely (see [41]).

Definition 3.37 (The class $\mathbf{P}_{\mathbb{R}}$)**.** Let $k(n)$ be any polynomial in n. A sequence of semi-algebraic sets $(T_n \subset \mathbb{R}^{k(n)})_{n>0}$ is said to belong to the class $\mathbf{P}_{\mathbb{R}}$ if there exists a machine M over $\mathbb{R}$ (see [15] or [14, §3.2]), such that for all $\mathbf{x} \in \mathbb{R}^{k(n)}$, the machine M tests membership of $\mathbf{x}$ in T_n in time bounded by a polynomial in n.

Definition 3.38. Let $k(n), k_1(n), \ldots, k_p(n)$ be polynomials in n. A sequence of semi-algebraic sets $(S_n \subset \mathbb{R}^{k(n)})_{n>0}$ is said to be in the complexity class $\mathbf{\Sigma}_{\mathbb{R},p}$, if for each $n > 0$ the semi-algebraic set S_n is described by a first order formula

$$(Q_1\mathbf{Y}^1)\cdots(Q_p\mathbf{Y}^p)\phi_n(X_1,\ldots,X_{k(n)},\mathbf{Y}^1,\ldots,\mathbf{Y}^p), \tag{3.2}$$

with ϕ_n a quantifier free formula in the first order theory of the reals, and for each $i, 1 \le i \le p$, $\mathbf{Y}^i = (Y^i_1, \ldots, Y^i_{k_i(n)})$ is a block of $k_i(n)$ variables, $Q_i \in \{\exists, \forall\}$, with $Q_j \neq Q_{j+1}, 1 \le j < p$, $Q_1 = \exists$, and the sequence of semi-algebraic sets $(T_n \subset \mathbb{R}^{k(n)+k_1(n)+\cdots+k_p(n)})_{n>0}$ defined by the quantifier-free formulas $(\phi_n)_{n>0}$ belongs to the class $\mathbf{P}_{\mathbb{R}}$.

Similarly, the complexity class $\mathbf{\Pi}_{\mathbb{R},p}$ is defined as in Definition 3.38, with the exception that the alternating quantifiers in (4.1) start with $Q_1 = \forall$. Since, adding an additional block of quantifiers on the outside (with new variables) does not change the set defined by a quantified formula we have the following inclusions:

$$\mathbf{\Sigma}_{\mathbb{R},p} \subset \mathbf{\Pi}_{\mathbb{R},p+1}, \text{ and } \mathbf{\Pi}_{\mathbb{R},p} \subset \mathbf{\Sigma}_{\mathbb{R},p+1}.$$

Note that by the above definition the class $\mathbf{\Sigma}_{\mathbb{R},0} = \mathbf{\Pi}_{\mathbb{R},0}$ is the familiar class $\mathbf{P}_{\mathbb{R}}$, the class $\mathbf{\Sigma}_{\mathbb{R},1} = \mathbf{NP}_{\mathbb{R}}$ and the class $\mathbf{\Pi}_{\mathbb{R},1} = \textbf{co-NP}_{\mathbb{R}}$.

Definition 3.39 (Real polynomial hierarchy)**.** The real polynomial time hierarchy is defined to be the union

$$\mathbf{PH}_{\mathbb{R}} \stackrel{\text{def}}{=} \bigcup_{p\ge 0}(\mathbf{\Sigma}_{\mathbb{R},p} \cup \mathbf{\Pi}_{\mathbb{R},p}) = \bigcup_{p\ge 0}\mathbf{\Sigma}_{\mathbb{R},p} = \bigcup_{p\ge 0}\mathbf{\Pi}_{\mathbb{R},p}.$$

As mentioned before, in order to get around certain difficulties caused by non-locally-closed sets and non-proper maps, a restricted polynomial hierarchy was defined in [13]. We now recall the definition of this compact analogue, $\mathbf{PH}^c_{\mathbb{R}}$, of

$\mathbf{PH}_{\mathbb{R}}$. Unlike in the non-compact case, we will assume all variables vary over certain compact semi-algebraic sets (namely spheres of varying dimensions).

Definition 3.40 (Compact real polynomial hierarchy [13])**.** Let

$$k(n), k_1(n), \ldots, k_p(n)$$

be polynomials in n. A sequence of semi-algebraic sets $(S_n \subset \mathbf{S}^{k(n)})_{n>0}$ is in the complexity class $\mathbf{\Sigma}^c_{\mathbb{R},p}$, if for each $n > 0$ the semi-algebraic set S_n is described by a first order formula

$$(Q_1 \mathbf{Y}^1 \in \mathbf{S}^{k_1(n)}) \cdots (Q_p \mathbf{Y}^p \in \mathbf{S}^{k_p(n)}) \phi_n(X_0, \ldots, X_{k(n)}, \mathbf{Y}^1, \ldots, \mathbf{Y}^p),$$

with ϕ_n a quantifier-free first order formula defining a *closed* semi-algebraic subset of $\mathbf{S}^{k_1(n)} \times \cdots \times \mathbf{S}^{k_p(n)} \times \mathbf{S}^{k(n)}$ and for each $i, 1 \leq i \leq p$, $\mathbf{Y}^i = (Y^i_0, \ldots, Y^i_{k_i})$ is a block of $k_i(n) + 1$ variables, $Q_i \in \{\exists, \forall\}$, with $Q_j \neq Q_{j+1}, 1 \leq j < p$, $Q_1 = \exists$, and the sequence of semi-algebraic sets $(T_n \subset \mathbf{S}^{k_1(n)} \times \cdots \times \mathbf{S}^{k_p(n)} \times \mathbf{S}^{k(n)})_{n>0}$ defined by the formulas $(\phi_n)_{n>0}$ belongs to the class $\mathbf{P}_{\mathbb{R}}$.

Example 3.41. The following example that appears in [13] is an example of a language in $\mathbf{\Sigma}^c_{\mathbb{R},1}$ (i.e., of the compact version of $\mathbf{NP}_{\mathbb{R}}$).

Let $k(n) = \binom{n+4}{4} - 1$ and identify $\mathbb{R}^{k(n)+1}$ with the space of *homogeneous* polynomials in $\mathbb{R}[X_0, \ldots, X_n]$ of degree 4. Let $S_n \subset \mathbf{S}^{k(n)} \subset \mathbb{R}^{k(n)+1}$ be defined by

$$S_n = \{P \in \mathbf{S}^{k(n)} \mid \exists \mathbf{x} = (x_0 : \cdots : x_n) \in \mathbb{P}^n_{\mathbb{R}} \text{ with } P(\mathbf{x}) = 0\};$$

in other words S_n is the set of (normalized) real forms of degree 4 which have a zero in the real projective space $\mathbb{P}^n_{\mathbb{R}}$. Then

$$(S_n \subset \mathbf{S}^{k(n)})_{n>0} \in \mathbf{\Sigma}^c_{\mathbb{R},1},$$

since it is easy to see that S_n also admits the description:

$$S_n = \{P \in \mathbf{S}^{k(n)} \mid \exists \mathbf{x} \in \mathbf{S}^n \text{ with } P(\mathbf{x}) = 0\}.$$

Note that it is *not known* if $(S_n \subset \mathbf{S}^{k(n)})_{n>0}$ is $\mathbf{NP}_{\mathbb{R}}$-complete (see Remark 3.42), while the non-compact version of this language, i.e., the language consisting of (possibly non-homogeneous) polynomials of degree at most four having a zero in $\mathbb{A}^n_{\mathbb{R}}$ (instead of $\mathbb{P}^n_{\mathbb{R}}$), has been shown to be $\mathbf{NP}_{\mathbb{R}}$-complete [14].

We define analogously the class $\mathbf{\Pi}^c_{\mathbb{R},p}$, and finally define the *compact real polynomial time hierarchy* to be the union

$$\mathbf{PH}^c_{\mathbb{R}} \stackrel{\text{def}}{=} \bigcup_{p \geq 0} (\mathbf{\Sigma}^c_{\mathbb{R},p} \cup \mathbf{\Pi}^c_{\mathbb{R},p}) = \bigcup_{p \geq 0} \mathbf{\Sigma}^c_{\mathbb{R},p} = \bigcup_{p \geq 0} \mathbf{\Pi}^c_{\mathbb{R},p}.$$

Notice that the semi-algebraic sets belonging to any language in $\mathbf{PH}^c_{\mathbb{R}}$ are all semi-algebraic compact (in fact closed semi-algebraic subsets of spheres). Also, note the inclusion

$$\mathbf{PH}^c_{\mathbb{R}} \subset \mathbf{PH}_{\mathbb{R}}.$$

Remark 3.42. The restriction to compact sets in [13] was necessitated by the fact that certain topological results used their needed certain maps to be proper, and assuming compactness was an easy way to ensure properness of these maps. Similarly, in this paper it will be convenient to assume that certain maps are proper restricted to supports of some given sheaves on a semi-algebraic set X. Since the support of a sheaf is always closed, the properness is ensured if we assume X is compact. In the absence of the compactness assumption, one would have to use the derived functors $Rf_!$ and $Rf^!$, instead of Rf_* and f^{-1}, and always consider cohomology groups with compact supports. While this might indeed be worthwhile to do in the future to have the fullest generality, we avoid complications in this paper by making the compactness assumption.

However, note that even though the restriction to compact semi-algebraic sets might appear to be only a technicality at first glance, this is actually an important restriction. For instance, it is a long-standing open question in real complexity theory whether there exists an $\mathbf{NP}_{\mathbb{R}}$-complete problem which belongs to the class $\mathbf{\Sigma}^c_{\mathbb{R},1}$ (the compact version of the class $\mathbf{NP}_{\mathbb{R}}$, see Example 3.41). See also [13] for natural examples of sequences in the class $\mathbf{\Sigma}^c_{\mathbb{R},1}$.

Remark 3.43. The topological methods used in this paper only require the sets to be compact. Using spheres to achieve this compact situation is a natural choice in the context of real algebraic geometry, since the inclusion of the space $\mathbb{R}^n$ into its one-point compactification $\mathbf{S}^n$ is a continuous semi-algebraic map that sends semi-algebraic subsets of $\mathbb{R}^n$ to their own one-point compactification (see [16, Definition 2.5.11]).

3.3.1. *Stability of the classes $\mathbf{P}_{\mathbb{R}}$ and $\mathbf{P}^c_{\mathbb{R}}$ under certain operations.* It is important to note that the the B-S-S complexity class $\mathbf{P}_{\mathbb{R}}$ (as well as $\mathbf{P}^c_{\mathbb{R}}$) is stable under certain operations. In fact, many results (such as the analogue of Toda's theorem in the B-S-S model proved in [13] as an illustrative example) depend only on these stability properties of the class $\mathbf{P}_{\mathbb{R}}$ and not on its actual definition involving B-S-S machines. We will formulate similar stability properties for the sheaf theoretic generalization of the class $\mathbf{P}^c_{\mathbb{R}}$.

Remark 3.44. We note here that we will sometimes identify a compact subset of $S \subset \mathbb{R}^{n+1}$, with the corresponding subset of the one-point compactification of $\mathbb{R}n+1$ which is homeomorphic to $\mathbf{S}^n$ and write $S \subset \mathbf{S}^n$. For example, we write $\mathbf{S}^m \times \mathbf{S}^n \subset \mathbf{S}^{m+n+1}$, the implied embedding is obtained by taking the product of the standard embeddings $\mathbf{S}^m \hookrightarrow \mathbb{R}^{m+1}$, $\mathbf{S}^n \hookrightarrow \mathbb{R}^{n+1}$, and then taking the one-point compactification of $\mathbb{R}^{m+n+2}$.

We omit the proofs of the following two propositions which follow immediately from the definition of the classes $\mathbf{P}_{\mathbb{R}}$ and $\mathbf{P}^c_{\mathbb{R}}$.

Proposition 3.45. *Let $m(n) \in \mathbb{Z}[n]$ be a fixed non-negative polynomial. Let $\left(X_n \subset \mathbb{R}^{m(n)}\right)_{n>0}$ and $\left(Y_n \subset \mathbb{R}^{m(n)}\right)_{n>0}$ both belong to $\mathbf{P}_{\mathbb{R}}$. Then, $(X_n \cup Y_n)_{n>0}$, $(X_n \cap Y_n)_{n>0}$, $(X_n \times Y_n)_{n>0}$, $\left(\mathbb{R}^{m(n)} \setminus X_n\right)_{n>0}$ all belong to the class $\mathbf{P}_{\mathbb{R}}$. Moreover, if $\left(X_n \subset \mathbf{S}^{m(n)}\right)_{n>0}$ and $\left(Y_n \subset \mathbf{S}^{m(n)}\right)_{n>0}$ both belong to $\mathbf{P}^c_{\mathbb{R}}$, then $(X_n \cup Y_n)_{n>0}$, $(X_n \cap Y_n)_{n>0}$, $(X_n \times Y_n)_{n>0}$ all belong to the class $\mathbf{P}^c_{\mathbb{R}}$ as well.*

Even though the B-S-S complexity class $\mathbf{P}_{\mathbb{R}}$ is by definition a sequence of semi-algebraic sets, sometimes it is also convenient to have a notion of a polynomial time computable maps. We will use the following slight abuse of notation.

Notation 3.46. Let $m_1(n), m_2(n) \in \mathbb{Z}[n]$ two fixed non-negative polynomials and let $\left(f_n : \mathbb{R}^{m_1(n)} \to \mathbb{R}^{m_2(n)}\right)_{n>0}$ be a sequence of maps. We say that the sequence $\left(f_n : \mathbb{R}^{m_1(n)} \to \mathbb{R}^{m_2(n)}\right)_{n>0} \in \mathbf{P}_{\mathbb{R}}$ if the maps $\left(f_n : \mathbb{R}^{m_1(n)} \to \mathbb{R}^{m_2(n)}\right)_{n>0}$ are computable by a B-S-S machine in polynomial time.

Proposition 3.47. *Let $m_1, m_2 \in \mathbb{Z}[n]$ two fixed non-negative polynomials and let $\left(f_n : \mathbb{R}^{m_1(n)} \to \mathbb{R}^{m_2(n)}\right)_{n>0} \in \mathbf{P}_{\mathbb{R}}$.*

(A) *For any sequence $\left(X_n \subset \mathbb{R}^{m_2(n)}\right)_{n>0}$ belonging to $\mathbf{P}_{\mathbb{R}}$, the sequence*

$$\left(f_n^{-1}(X_n) \subset \mathbb{R}^{m_1(n)}\right)_{n>0}$$

also belongs to the class $\mathbf{P}_{\mathbb{R}}$.

(B) *For any two sequences $\left(X_n \subset \mathbb{R}^{m_1(n)}\right)_{n>0}$ and $\left(Y_n \subset \mathbb{R}^{m_1(n)}\right)_{n>0}$ belonging to $\mathbf{P}_{\mathbb{R}}$, the sequence as well as $\left(X_n \times_{f_n} Y_n \subset \mathbb{R}^{m_1(n)} \times \mathbb{R}^{m_1(n)}\right)_{n>0}$ also belongs to the class $\mathbf{P}_{\mathbb{R}}$.*

Similar statements hold for $\mathbf{P}^c_{\mathbb{R}}$ as well.

Proof. Immediate. □

Remark 3.48. One important special case of Proposition 3.47 is when the semi-algebraic maps f_n are just projection maps forgetting the first n co-ordinates. Notice also that Propositions 3.45 and 3.47 together imply that the B-S-S classes $\mathbf{P}_{\mathbb{R}}$ and $\mathbf{P}^c_{\mathbb{R}}$ are closed under usual set theoretic operations, as well under taking inverse images under and fiber-products over polynomially computable semi-algebraic maps.

While the classes $\mathbf{P}_{\mathbb{R}}$ and $\mathbf{P}^c_{\mathbb{R}}$ are closed under inverse images under polynomially computable maps, the question whether the same is true under taking direct images is equivalent to the famous $\mathbf{P}_{\mathbb{R}}$ vs $\mathbf{NP}_{\mathbb{R}}$ (respectively, $\mathbf{P}^c_{\mathbb{R}}$ vs $\mathbf{NP}^c_{\mathbb{R}}$) question and the prevailing belief in fact is that this is not the case.

More formally:

Conjecture 3.49. [14] *The class $\mathbf{P}_{\mathbb{R}}$ is strictly included in the class $\mathbf{NP}_{\mathbb{R}}$.*

We also make the compact version of the above conjecture.

Conjecture 3.50. *The class $\mathbf{P}^c_{\mathbb{R}}$ is strictly included in the class $\mathbf{NP}^c_{\mathbb{R}}$.*

3.4. **Definition of the class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ of constructible sheaves.** As noted previously for reasons of expediency we are going to restrict to compact complexity classes from now on. For set theoretic classes this means we only consider sequences of compact subsets of spheres, and in the sheaf theoretic generalizations we will only consider sequences of sheaves supported on spheres. Note that in this case the supports of such sheaves are necessarily compact.

We now define the sheaf theoretic analogue of the complexity class $\mathbf{P}^c_{\mathbb{R}}$.

Definition 3.51. (The class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$) The class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ of constructible sheaves consists of sequences $\left(\mathcal{F}_n \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})\right)_{n>0}$, where $m(n) \in \mathbb{Z}[n]$ is a non-negative polynomial satisfying the following conditions. There exists a non-negative polynomial $m_1(n) \in \mathbb{Z}[n]$ such that:

(A) For each $n > 0$, there is an index set I_n of cardinality $2^{m_1(n)}$, and a semi-algebraic partition, $(S_{n,i})_{i \in I_n}$, of $\mathbf{S}^{m(n)}$ into locally closed semi-algebraic sets $S_{n,i}$ indexed by I_n, which is subordinate to $\mathcal{F}_n$.

(B) For each $n > 0$ and each $\mathbf{x} \in \mathbf{S}^{m(n)}$,
   1. The dimensions $\dim_{\mathbb{Q}} \mathrm{H}^j((\mathcal{F}_n)_{\mathbf{x}})$ are bounded by $2^{m_1(n)}$;
   2. $\mathrm{H}^j((\mathcal{F}_n)_{\mathbf{x}}) = 0$ for all j with $|j| > m_1(n)$.

The two sequences of functions $(i_n : \mathbf{S}^{(m(n)} \to I_n)_{n>0}$, and $(p_n : \mathbf{S}^{m(n)} \to \mathbb{Z}[T])$ defined by

$$\begin{aligned} i_n(\mathbf{x}) &= i \in I_n, \text{ such that, } \mathbf{x} \in S_{n,i} \\ p_n(\mathbf{x}) &= P_{(\mathcal{F}_n)_\mathbf{x}} \end{aligned}$$

are computable by B-S-S machines in time polynomial in n (recall from Definition 3.33 that $P_{\mathcal{F}_\mathbf{x}}$ denotes the Poincaré polynomial of the stalk of $\mathcal{F}$ at $\mathbf{x}$) (Notice that the number of bits needed to represent elements of I_n, and the coefficients of $P_{(\mathcal{F}_n)_\mathbf{x}}$ are bounded polynomially in n.)

One immediate property of the class $\boldsymbol{\mathcal{P}}_\mathbb{R}$ is the following.

In what follows it will be convenient to have the following notation.

Notation 3.52. For any finite family $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_n]$ and a semi-algebraic set $S \subset \mathbb{R}^n$, we will denote by $\Pi(\mathcal{P}, X)$ the partition of X into the connected components of $\mathcal{R}(\sigma, S)$ for each realizable sign condition $\sigma \in \{0, 1, -1\}^\mathcal{P}$ on S.

Proposition 3.53. *Let $\left(\mathcal{F}_n \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})\right)_{n>0}$ belong to the class $\boldsymbol{\mathcal{P}}_\mathbb{R}$. Then, there exists for each $n > 0$ a family of polynomials $\mathcal{P}_n \subset \mathbb{R}[X_0, \ldots, X_{m(n)}]$, such that the semi-algebraic partition $\Pi(\mathcal{P}_n, \mathbf{S}^{m(n)})$ is subordinate to $\mathcal{F}_n$, and moreover* $\mathrm{card}(\mathcal{P}_n)$ *as well as the degrees of the polynomials in $\mathcal{P}_n$ are bounded singly exponentially as a function of n.*

Proof. This is an immediate consequence of the fact that the sequence of functions i_n is computable in polynomial time. □

One connection of $\boldsymbol{\mathcal{P}}_\mathbb{R}$ with the standard B-S-S complexity class $\mathbf{P}^c_\mathbb{R}$ is as follows.

Proposition 3.54. *Let $\left(X_n \subset \mathbf{S}^{m(n)}\right)_{n>0}$ be a sequence of compact semi-algebraic sets. Then, $\left(X_n \in \mathbf{S}^{m(n)}\right)_{n>0} \in \mathbf{P}^c_\mathbb{R}$ if and only if the sequence of constructible sheaves $(\mathbb{Q}_{X_n} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})))_{n>0} \in \boldsymbol{\mathcal{P}}_\mathbb{R}$.*

Proof. For any compact semi-algebraic set $X \subset \mathbf{S}^{m(n)}$, the stalks $(\mathbb{Q}_X)_\mathbf{x} = 0$ for $\mathbf{x} \notin X$. For $\mathbf{x} \in X$,

$$\begin{aligned} \mathrm{H}^j((\mathbb{Q}_X)_\mathbf{x}) &= 0 \text{ for } j \neq 0, \\ &= \mathbb{Q} \text{ otherwise.} \end{aligned}$$

Now suppose that $\left(X_n \in \mathbf{S}^{m(n)}\right)_{n>0} \in \mathbf{P}^c_\mathbb{R}$. Then, letting for each $n > 0$, $I_n = \{0, 1\}$, and

$$\begin{aligned} S_{n,0} &= \mathbf{S}^{m(n)} \setminus X_n, \\ S_{n,1} &= X_n, \end{aligned}$$

it is easy to verify that $(\mathbb{Q}_{X_n} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})))_{n>0} \in \boldsymbol{\mathcal{P}}_\mathbb{R}$.

Conversely, if $(\mathbb{Q}_{X_n} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})))_{n>0} \in \boldsymbol{\mathcal{P}}_\mathbb{R}$, then for each $n > 0$, there is an index set I_n of cardinality $2^{m_1(n)}$, and a semi-algebraic partition, $(S_{n,i})_{i \in I_n}$, of $\mathbf{S}^{m(n)}$ into locally closed semi-algebraic sets $S_{n,i}$ indexed by I_n, which is subordinate to $\mathcal{F}_n$ satisfying the properties listed in Definition 3.51. Now, from the fact that the sequence of functions $(p_n : \mathbf{S}^{m(n)} \to \mathbb{Z}[T])$ defined by

$$p_n(\mathbf{x}) = P_{(\mathcal{F}_n)_\mathbf{x}}$$

is computable by B-S-S machines in time polynomial in n, it follows immediately that $\left(X_n \in \mathbf{S}^{m(n)}\right)_{n>0} \in \mathbf{P}^c_{\mathbb{R}}$. □

The class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ is stable under standard sheaf-theoretic operations. These closure properties are reminiscent of the stability properties of the classes $\mathbf{P}_{\mathbb{R}}$ and $\mathbf{P}^c_{\mathbb{R}}$ (cf. Propositions 3.45 and 3.47).

Proposition 3.55. *(Stability properties of the class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$) Let $m(n) \in \mathbb{Z}[n]$ be a fixed non-negative polynomial.*

(A) *(Closure under direct sums and tensor products.) If*

$$\left(\mathcal{F}_n \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})\right)_{n>0}, \left(\mathcal{G}_n \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})\right)_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}},$$

then $(\mathcal{F}_n \oplus \mathcal{G}_n)_{n>0}, (\mathcal{F}_n \otimes \mathcal{G}_n)_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}$.

(B) *(Closure under pull-backs.) For any fixed non-negative polynomial $m_1(n) \in \mathbb{Z}[n]$, let $\pi_n : \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)} \to \mathbf{S}^{m(n)}$ be the projection map on the second factor, and*

$$\left(\mathcal{F}_n \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})\right)_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}.$$

Then, $\left(\pi_n^{-1}(\mathcal{F}_n)\right)_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}$.

(C) *(Closure under truncations.) For any non-negative polynomial $m_1(n) \in \mathbb{Z}[n]$, and a sequence*

$$\left(\mathcal{F}_n \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})\right)_{n>0}$$

belonging to the class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$, we have

$$\left(\tau^{\leq m_1(n)}\mathcal{F}_n\right)_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}},$$

$$\left(\tau^{\geq m_1(n)}\mathcal{F}_n\right)_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}.$$

(D) *(Closure under shifts.) For any non-negative polynomial $m_1(n) \in \mathbb{Z}[n]$, and a sequence*

$$\left(\mathcal{F}_n \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})\right)_{n>0}$$

belonging to the class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$, we have

$$(\mathcal{F}_n[m_1(n)])_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}},$$

$$(\mathcal{F}_n[-m_1(n)])_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}.$$

Proof. (A) Let for each $n > 0$, $(S'_{n,i})_{i' \in I'_n}$ and $(S''_{n,i})_{i'' \in I''_n}$ be semi-algebraic partitions of $\mathbf{S}^{m(n)}$ subordinate to the $\mathcal{F}_n$ and $\mathcal{G}_n$, and i'_n, p'_n, i''_n, p''_n the corresponding functions (cf. Definition 3.51). Also, let $m'_1(n)$ (respectively, $m''_1(n)$) be the polynomial appearing in the definition of the class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ for the sequence $(\mathcal{F}_n)_{n>0}$ (respectively, $(\mathcal{G}_n)_{n>0}$). Let $I_n = I'_n \times I''_n$, and for each $i = (i', i'') \in I_n$, let $S_n = S'_{n,i'} \cap S''_{n,i''}$. Then,

(1) For each $i = (i', i'') \in I_n$, $j \in \mathbb{Z}$, and $\mathbf{x} \in S_{n,i}$

$$\mathrm{H}^j((\mathcal{F}_n \oplus \mathcal{G}_n)_{\mathbf{x}}) \cong \mathrm{H}^j((\mathcal{F}_n)_{\mathbf{x}}) \oplus \mathrm{H}^j((\mathcal{G}_n)_{\mathbf{x}})$$

and is clearly locally are locally constant for $\mathbf{x} \in S_{n,i}$, since $\mathrm{H}^*((\mathcal{F}_n)_{\mathbf{x}})$ is locally constant for $\mathbf{x} \in S'_{n,i'}$ and $\mathrm{H}^*((\mathcal{G}_n)_{\mathbf{x}})$ is locally constant for $\mathbf{x} \in S''_{n,i''}$;

(2) We also have

$$\begin{aligned}\dim_{\mathbb{Q}} \mathrm{H}^j((\mathcal{F}_n \oplus \mathcal{G}_n)_{\mathbf{x}}) &= \dim_{\mathbb{Q}} \mathrm{H}^j((\mathcal{F}_n)_{\mathbf{x}}) + \dim_{\mathbb{Q}} \mathrm{H}^j((\mathcal{G}_n)_{\mathbf{x}}) \\ &\leq 2^{m_1'(n)} + 2^{m_1''(n)};\end{aligned}$$

Also using the Kunneth formula , and the definition of tensor products of constructible sheaves, we have

$$\begin{aligned}\dim_{\mathbb{Q}} \mathrm{H}^j((\mathcal{F}_n \otimes \mathcal{G}_n)_{\mathbf{x}}) &= \dim_{\mathbb{Q}} \left(\bigoplus_{p+q=j} \mathrm{H}^p((\mathcal{F}_n)_{\mathbf{x}}) \otimes \mathrm{H}^q((\mathcal{G}_n)_{\mathbf{x}}) \right) \\ &= \sum_{p+q=j} (\dim_{\mathbb{Q}} \mathrm{H}^p((\mathcal{F}_n)_{\mathbf{x}}))(\dim_{\mathbb{Q}} \mathrm{H}^q((\mathcal{G}_n)_{\mathbf{x}}) \\ &\leq \sum_{p+q=j} 2^{m_1'(n)+m_1''(n)} \\ &\leq 2(m_1(n)+m_2(n)+1)2^{m_1'(n)+m_1''(n)},\end{aligned}$$

noting that

$$\mathrm{H}^j((\mathcal{F}_n)_{\mathbf{x}}), \mathrm{H}^j((\mathcal{G}_n)_{\mathbf{x}}) = 0$$

for all j with $|j| > m_1'(n) + m_1''(n)$.

(3) from the fact that the functions $i_n' : \mathbf{S}^{m(n)} \to I_n'$ and $i_n'' : \mathbf{S}^{m(n)} \to I_n''$ are computable in polynomial time it follows that it follows that the function $i_n : \mathbf{S}^{m(n)} \to I_n$, defined by $i_n(\mathbf{x}) = (i_n'(\mathbf{x}), i_n''(\mathbf{x}))$ is also computable in polynomial time.

(4) from the fact that the functions $p_n' : \mathbf{S}^{m(n)} \to \mathbb{Z}[T]$ and $p_n'' : \mathbf{S}^{m(n)} \to \mathbb{Z}[T]$ are computable in polynomial time it follows that it follows that the functions $p_n^{\oplus} : \mathbf{S}^{m(n)} \to \mathbb{Z}[T]$, $p_n^{\otimes} : \mathbf{S}^{m(n)} \to \mathbb{Z}[T]$, defined by

$$\begin{aligned}p_n^{\oplus}(\mathbf{x}) &= p_n'(\mathbf{x}) + p_n''(\mathbf{x}), \\ p_n^{\otimes}(\mathbf{x}) &= p_n'(\mathbf{x}) \cdot p_n''(\mathbf{x}),\end{aligned}$$

are also computable in polynomial time.

It follows from the above that both sequences $(\mathcal{F}_n \oplus \mathcal{G}_n)_{n>0}, (\mathcal{F}_n \otimes \mathcal{G}_n)_{n>0}$ belong to the class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.

(B) Let for each $n > 0$, $(S_{n,i}')_{i \in I_n'}$ be the semi-algebraic partition of $\mathbf{S}^{m(n)}$ corresponding to the $\mathcal{F}_n$, and i_n', p_n' the corresponding functions (cf. Definition 3.51). Also, let $m_1'(n)$ be the polynomial appearing in the definition of the class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ for the sequence $(\mathcal{F}_n)_{n>0}$.

Let $I_n = I_n'$. Defining $i_n : \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)} \to I_n$ and $p_n : \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)} \to \mathbb{Z}[T]$ by

$$\begin{aligned}i_n(\mathbf{y}, \mathbf{x}) &= i_n'(\mathbf{x}) \\ p_n(\mathbf{y}, \mathbf{x}) &= p_n'(\mathbf{x}),\end{aligned}$$

it follows from the fact that the sequences $(i_n')_{n>0}, (p_n')_{n>0}$ are computable in polynomial time, that so are the sequences $(i_n)_{n>0}$ and $(p_n)_{n>0}$.

Moreover, for each $(\mathbf{y}, \mathbf{x}) \in \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)}$ we have for each $j \in \mathbb{Z}$, using Proposition 3.27 an isomorphism

$$\mathrm{H}^j(\pi_n^{-1}(\mathcal{F}_n)_{(\mathbf{y},\mathbf{x})} \cong \mathrm{H}^j((\mathcal{F}_n)_{\mathbf{x}}.$$

This shows that $\left(\pi_n^{-1}(\mathcal{F}_n)\right)_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}$.

The remaining parts of the proposition are immediate. ☐

We now give several illustrative examples of sequences of constructible sheaves in the class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ other than those coming directly from a B-S-S complexity class $\mathbf{P}^c_{\mathbb{R}}$.

We first need a notation.

Notation 3.56. If X is a locally closed semi-algebraic set then we denote

$$\begin{aligned} b_i(X) &= \dim_{\mathbb{Q}} \mathrm{H}^i(X,\mathbb{Q}), \\ b_i^{BM}(X) &= \dim_{\mathbb{Q}} \mathrm{H}^i_c(X,\mathbb{Q}), \\ b(X) &= \sum_i b_i(X), \\ b^{BM}(X) &= \sum_i b_i^{BM}(X). \end{aligned}$$

We will also denote by $P_X \in \mathbb{Z}[T]$ the Poincaré polynomial of X defined by

$$P_X = \sum_{i\geq 0} b_i^{BM}(X) T^i.$$

Example 3.57. (Rank stratification sheaf) For each $n > 0$, let $V_n \subset \mathbf{S}^{n-1} \times \mathbf{S}^{n^2-1}$ be the semi-algebraic set (an incidence variety) defined by

$$V_n = \{(\mathbf{x}, A) \mid \mathbf{x} \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}, A\cdot \mathbf{x} = 0, ||A||^2 = 1, ||\mathbf{x}||^2 = 1\}.$$

Let $\pi_n : \mathbf{S}^{n-1} \times \mathbf{S}^{n^2-1} \to \mathbf{S}^{n^2-1}$ denote the projection to the second factor.

Proposition 3.58. *The sequence of constructible sheaves*

$$\left(R\pi_{n,*}\mathbb{Q}_{V_n} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{n^2-1}))\right)_{n>0}$$

belongs to the class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$*.*

Proof. It is clear that the semi-algebraic partition by rank of the matrices $A \in \mathbf{S}^{n^2-1}$ is subordinate to the constructible sheaf $R\pi_{n,*}\mathbb{Q}_{V_n}$. Moreover, for each A $\in \mathbf{S}^{n^2-1}$, we have that

$$\mathrm{H}^*((R\pi_{n,*}\mathbb{Q}_{V_n})_A) \cong \mathrm{H}^*(\mathbf{S}^{n-1-\mathrm{rk}_{n,n}(A)}, \mathbb{Q}).$$

The claim is now clear from the fact that the rank of a matrix is computable in polynomial time by a B-S-S machine. ☐

This example should be compared with Example 2.21 in the previous Section 2.

The next example might look a little artificial at first glance but shows how index sets with exponential cardinality, as well as Poincaré polynomials with coefficients which are exponentially large in n could arise for a sequence in $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.

Example 3.59. For $n > 0$ let $P_n \in \mathbb{R}[X_0, X_1, \ldots, X_n, Y_0, \mathbf{Y}_1, \ldots, \mathbf{Y}_n]$, where each $\mathbf{Y}_i = (Y_{i,1}, \ldots, Y_{i,i})$ is a block of i variables, be defined by

$$P_n = \sum_{i=1}^{n} \left((X_i^2 - 1)^2 + \frac{1}{N}(\sum_{j=1}^{i} Y_{i,j}^2 + (1 - X_i))^2 \right),$$

with
$$N = \sum_{i=1}^{n} i = \binom{n+1}{2},$$
and let $V_n = \mathrm{Z}(P_n, \mathbf{S}^n \times \mathbf{S}^N_{Y_0 \geq 0})$ (cf. Notation 2.23) (recall that $\mathbf{S}^n$ and $\mathbf{S}^N$ are the *unit* spheres centered at the origin in $\mathbb{R}^{n+1}$ and $\mathbb{R}^{N+1}$ respectively). Let $\pi_n : \mathbf{S}^n \times \mathbf{S}^N \to \mathbf{S}^n$ be the projection to the first factor. Observe that
$$\pi_n(V_n) = \{(0, x_1, \ldots, x_n) \mid x_i \in \{-1, +1\}, 1 \leq i \leq n\},$$
and for $\mathbf{x} = (0, x_1, \ldots, x_n) \in \pi_n(V_n)$, $(V_n)_{\mathbf{x}} = \pi_n^{-1}(\mathbf{x}) \cap V_n$ is described by
$$(V_n)_{\mathbf{x}} = \mathbf{S}^N_{Y_0 \geq 0} \cap (T_1 \times \cdots \times T_n),$$
where for each i, $1 \leq i \leq n$,
$$\begin{aligned} T_i &\cong \mathbf{S}^i \text{ if } x_i = -1, \\ &\cong \{pt\} \text{ if } x_i = 1. \end{aligned}$$

Proposition 3.60. *The sequence* $\left(R\pi_{n,*}\mathbb{Q}_{V_n} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^n))\right)_{n>0}$ *belongs to the class* $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.

Proof. Let for $I_n = \{+1, -1\}^n \cup \{0\}$, and for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{-1, +1\}^n$
$$S_{n,\alpha} = \{\mathbf{x} = (x_0, x_1, \ldots, x_n) \in \mathbf{S}^n \mid x_i = \alpha_i, 1 \leq i \leq n\},$$
and let
$$S_0 = \mathbf{S}^n \setminus \bigcup_{\alpha \in \{-1,+1\}^n} S_{n,\alpha}.$$

Then the family $(S_{n,\alpha})_{\alpha \in I_n}$ is a semi-algebraic partition of $\mathbf{S}^n$ into $\mathrm{card}(I_n) = 2^n + 1$ locally closed semi-algebraic sets. It is easy to check that $\mathrm{H}^*((R\pi_{n,*}\mathbb{Q}_{V_n})_{\mathbf{x}})$ is locally constant over each element of the partition, and

$$\begin{aligned} \mathrm{H}^*((R\pi_{n,*}\mathbb{Q}_{V_n})_{\mathbf{x}}) &= 0 \text{ if } \mathbf{x} \in S_{n,0}, \\ &\cong \mathrm{H}^*(T_1 \times \cdots \times T_n, \mathbb{Q}) \text{ for } \mathbf{x} \in S_{n,\alpha} \text{ with } \alpha \in \{-1,+1\}^n, \end{aligned}$$
where for each i, $1 \leq i \leq n$,
$$\begin{aligned} T_i &\cong \mathbf{S}^i \text{ if } \alpha_i = -1, \\ &\cong \{pt\} \text{ if } \alpha_i = 1. \end{aligned}$$

By Kunneth formula we have that
$$\begin{aligned} P_{T_1 \times \cdots \times T_n}(T) &= \prod_{i=1}^{n} P_{T_i} \\ &= \prod_{i=1}^{n} \left(\frac{(1-\alpha_i)}{2}(1+T^i) \right) \end{aligned} \tag{3.3}$$
(recall from Notation 3.56 that for any locally closed semi-algebraic set X we denote by $P_X(T)$ the Poincaré polynomial of X).

It follows from Eqn. (3.3) that the dimensions $\dim_{\mathbb{Q}} \mathrm{H}^*((R\pi_{n,*}\mathbb{Q}_{V_n})_{\mathbf{x}})$ are bounded singly exponentially in n.

It is also clear that the sequence of maps $(i_n : \mathbf{S}^n \to I_n)_{n>0}$ and $(p_n : \mathbf{S}^n \to \mathbb{Z}[T])$ are computable in polynomial time. Together with the above this shows that $\left(R\pi_{n,*}\mathbb{Q}_{V_n} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^n))\right)_{n>0}$ belongs to the class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$. □

Remark 3.61. Notice that in the above example, it follows from the unique factorization property of the polynomial ring $\mathbb{Z}[T]$, that $\mathrm{H}^*((R\pi_{n,*}\mathbb{Q}_{V_n})_{\mathbf{x}})$ is distinct over each element, $S_{n,\alpha}$, of the partition $(S_{n,\alpha})_{\alpha\in I_n}$, and this is the *coarsest possible* partition which is subordinate to $R\pi_{n,*}\mathbb{Q}_{V_n}$, and this partition has size 2^n+1 which is singly exponential in n. Moreover, the dimension $\sum_{i\in\mathbb{Z}}\dim_{\mathbb{Q}}\mathrm{H}^i((R\pi_{n,*}\mathbb{Q}_{V_n})_{\mathbf{x}})$ can be as large as 2^n, and in fact $\sum_{i\in\mathbb{Z}}\dim_{\mathbb{Q}}\mathrm{H}^i((R\pi_{n,*}\mathbb{Q}_{V_n})_{\mathbf{x}}) = 2^n$ for $\mathbf{x} = (0,-1,-1,\ldots,-1)$ (seen by substituting $T=1$ in the corresponding Poincaré polynomial).

Example 3.62. Let $d,t>0$ be fixed integers. Let for each $n>0$, $P_n \in \mathbb{R}[X_1,\ldots,X_n]$ be a polynomial with $\deg(P_n)\le d$, and such that the sequence of functions $(P_n:\mathbb{R}^n\to\mathbb{R})_{n>0}\in\mathbf{P}_{\mathbb{R}}$ (see Notation 3.46). For example, we could take for P_n the elementary symmetric polynomial $e_{d,n}$ of degree d in n variables. Let $V_n = \mathrm{Z}(P_n,\mathbf{S}^{n-1})\subset\mathbb{R}^n$, and it is immediate that the sequence $\left(V_n\subset\mathbf{S}^{n-1}\right)_{n>0}\in\mathbf{P}^c_{\mathbb{R}}$. Now for each $n>t$, $\pi_{n,t}$ denote the projection map from $\mathbf{S}^{n+t-1}\subset\mathbb{R}^{n+t}$ to $\mathbf{S}^n$ which forgets that last t variables, and where we have replaced $\mathbb{R}^n$ by its one-point compactification (refer to Remark 3.44).

Proposition 3.63. *The sequence* $(R\pi_{n,t,*}\mathbb{Q}_{V_n})_{n>0}$ *belongs to the class* $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.

Proof. First observe that the number of coefficients of the polynomial P_{n+t} is bounded polynomially in n (for constant t and d), and more over from the fact that the sequence of functions $(P_n:\mathbb{R}^n\to\mathbb{R})_{n>0}\in\mathbf{P}_{\mathbb{R}}$ it follows that these coefficients can be computed in polynomially many steps by a B-S-S machine. Notice that for each $x\in\mathbb{R}^n$, the fiber $(V_{n+t})_{\mathbf{x}} = V_{n+t}\cap\pi^{-1}_{n,t}(x)$ can be identified with the set of zeros of the polynomials $P_{n+t}(\mathbf{x},Y_1,\ldots,Y_t)$ intersected with the sphere having the equation $Y_1^2+\cdots+Y_t^2+|\mathbf{x}|^2-1$. Moreover the degree of the polynomial $P_{n+t}(\mathbf{x},Y_1,\ldots,Y_t)$ in $Y_1,\ldots,Y_t$ is at most d. Now, it is a standard fact (see for example [9]) that there exists a family of polynomials $\mathcal{Q}_n\subset\mathbb{R}[X_1,\ldots,X_n]$, such that $\mathrm{card}(\mathcal{Q}_n)$ as well as the degrees of the polynomials in $\mathcal{Q}_n$ are bounded by $d^{2^{O(t)}} = O(1)$, such that for each $\mathbf{x}\in\mathbb{R}^n$ the signs of the polynomials in $\mathcal{Q}_n$ at x determines the topological type of the fiber $(V_{n,t})_{\mathbf{x}}$. The number of of realizable sign conditions of this family $\mathcal{Q}_n$ is bounded singly exponentially in n. Moreover, for each $i\in\mathbb{Z}$ and $\mathbf{x}\in\mathbb{R}^n$,

$$\mathrm{H}^i((R\pi_{n,t,*}\mathbb{Q}_{V_n})_{\mathbf{x}}) = \mathrm{H}^i(V_{n,t})_{\mathbf{x}},\mathbb{Q}).$$

From dimension considerations it follows that $\mathrm{H}^i(V_{n,t})_{\mathbf{x}},\mathbb{Q}) = 0$ for $i\notin[0,t]$. Moreover, it follows from standard bounds on the Betti numbers of real varieties [38] that, $\sum_i\dim_{\mathbb{Q}}(\mathrm{H}^i(V_{n,t})_{\mathbf{x}},\mathbb{Q}))\le O(d)^t$. It is also clear that given $\mathbf{x}\in\mathbb{R}^n$, the signs of the polynomials in $\mathcal{Q}_n$ at $\mathbf{x}$ as well as the dimensions, $\dim_{\mathbb{Q}}(\mathrm{H}^i(V_{n,t})_{\mathbf{x}},\mathbb{Q}))$ of can be computed in polynomial time. This completes the proof that

$$(R\pi_{n,t,*}\mathbb{Q}_{V_n})_{n>0}\in\boldsymbol{\mathcal{P}}_{\mathbb{R}}.$$

□

In order to describe the next example we need a new notation.

Notation 3.64. We denote by $\mathrm{Sym}_{n,d}(\mathbb{R})$ the $\mathbb{R}$-vector space of forms over $\mathbb{R}$ of degree d in $n+1$ variables. We will denote by $N_{n,d} = \binom{n+d}{d} = \dim_{\mathbb{R}}\mathrm{Sym}n,d$. Note that each form $P\in\mathrm{Sym}_{n,d}$ can be identified uniquely with a point on the sphere $a_P\in\mathbf{S}^{N_{n,d}-1}$ obtained by intersecting the sphere $\mathbf{S}^{N_{n,d}-1}$ with the half-ray consisting of the positive multiple of P. We will identify P with the point a_P in what follows.

Example 3.65.

Proposition 3.66. *Let $s > 0$ be fixed, and consider the for each $n > 0$, the compact set $V_n \subset (\mathbf{S}^{N_{n,d}-1})^s \times \mathbf{S}^n$ defined by*

$$V_n = \{(P_1, \ldots, P_s, x) \mid x \in \mathbf{S}^n, P_1(x) = \cdots = P_s(x) = 0\},$$

Let $\pi_n : (\mathbf{S}^{N_{n,d}-1})^s \times \mathbf{S}^n \to (\mathbf{S}^{N_{n,d}-1})^s \hookrightarrow \mathbf{S}^{s(N_{n,d}-1)+s}$ be the projection on the first factor. Then,

(A) *The sequence $(V_n)_{n>0} \in \mathbf{P}_{\mathbb{R}}$.*
(B) *The sequence $\left(R\pi_{n,*}(\mathbb{Q}_{V_n}) \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{s(N_{n,d}-1)+s})\right)_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}$.*

Remark 3.67. Notice that unlike in Example 3.62 above, the dimensions of the fibers of the map π_n grows with n. However, the degrees of the polynomials used in the definition of the sets V_n is restricted to 2.

Proof. The proof of the proposition is somewhat similar to that of Proposition 3.63 and we outline it here omitting details. It follows from the main results in [7] that for each $n > 0$ there exists a family of polynomials $\mathcal{Q}_n$, computable in polynomial time such that the stable homotopy types of the fibers $(V_n)_{\mathbf{x}} = V_n \cap \pi_n^{-1}(\mathbf{x})$ stay invariant over each connected component of the realizations of each realizable sign conditions on $\mathcal{Q}_n$. The degrees and the number of polynomials in $\mathcal{Q}_n$ is bounded polynomially in n (for fixed s). Also, it follows from the main result in [5] (see also [12, 6]), that the Betti numbers of the fibers $(V_n)_{\mathbf{x}}$ are computable in polynomial time (for fixed s). Together they imply the proposition. □

3.5. **Polynomial hierarchy of constructible sheaves.** We will now define the sheaf theoretic version of the polynomial hierarchy. However, before doing so we motivate out definition by recalling the definitions of $\mathbf{NP}^c_{\mathbb{R}}$ and $\mathbf{co\text{-}NP}^c_{\mathbb{R}}$.

If $m(n)$ is any fixed polynomial, then a sequence $\left(X_n \subset \mathbf{S}^{m(n)}\right)_{n>0}$ is in the class $\mathbf{NP}^c_{\mathbb{R}}$, if there exists a polynomial $m_1(n)$ and a sequence $(Y_n \subset \mathbf{S}^{m_1(n)} \times \mathbf{S}^{(m(n)})_{n>0} \in \mathbf{P}_{\mathbb{R}}$ such that for each $n > 0$ the semi-algebraic set X_n is described by the formula $(\exists y \in \mathbf{S}^{m_1(n)})((y, x) \in Y_n)$.

Similarly, $\left(X_n \subset \mathbf{S}^{m(n)}\right)_{n>0}$ is in the class $\mathbf{co\text{-}NP}^c_{\mathbb{R}}$, if there exists a polynomial $m_1(n)$ and a sequence $(Z_n \subset \mathbf{S}^{m_1(n)} \times \mathbf{S}^{(m(n)})_{n>0} \in \mathbf{P}_{\mathbb{R}}$ such that for each $n > 0$ the semi-algebraic set X_n is described by the formula $(\forall z \in \mathbf{S}^{m_1(n)})((z, x) \in Z_n)$.

Notice that in the second case it immediately follows the fact that for all $N > 0$,

$$\begin{aligned} \mathrm{H}^j(\mathbb{Q}_{\mathbf{S}^N}) &= \mathbb{Q}, \text{ for } j = 0, N, \\ &= 0, \text{ otherwise} \end{aligned}$$

that

$$\mathbb{Q}_{X_n} = \tau^{\leq m_1(n)} \tau^{\geq m_1(n)} R\pi_{n,*}(\mathbb{Q}_{Z_n})[m_1(n)],$$

where for each $n > 0$, $\pi_n : \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)} \to \mathbf{S}^{m(n)}$ is the projection map to the second factor.

In the first case, we use a construction used in [13]. Namely, given a sequence $(Y_n \subset \mathbf{S}^{m_1(n)} \times \mathbf{S}^{(m(n)})_{n>0} \in \mathbf{P}_{\mathbb{R}}$, the sequence

$$(J_{\pi_n}(Y_n) \subset [0,1] \times \overline{\mathbf{B}^{m_1(n)}} \times \overline{\mathbf{B}^{m_1(n)}} \times \mathbf{S}^{(m(n)})_{n>0} \in \mathbf{P}_{\mathbb{R}},$$

where $J_{\pi_n}(Y_n) \subset [0,1] \times \overline{\mathbf{B}^{m_1(n)}} \times \overline{\mathbf{B}^{m_1(n)}} \times \mathbf{S}^{m(n)}$ is defined as the union of the following three sets:

$$\begin{gathered}(0,1) \times (Y_n \times_{\pi_n} Y_n),\\ \{(1, y_0, y_1, x) \mid (y_0, x) \in Y_n\},\\ \{(0, y_0, y_1, x) \mid (y_1, x) \in Y_n\}.\end{gathered}$$

Denoting by $J(\pi_n) : J_{\pi_n}(Y_n) \to \mathbf{S}^{m(n)}$ the natural projection map, it is easy to verify that the fibers of the projection map $J(\pi_n) : J_{\pi_n}(Y_n) \to \mathbf{S}^{m(n)}$ are homotopy equivalent to the topological join with itself of the corresponding fibers of the projection $\pi_n : Y_n \to \mathbf{S}^{m(n)}$, and are thus connected if non-empty. It is clear that the sequence $(J_{\pi_n}(Y_n))_{n>0} \in \mathbf{P}_{\mathbb{R}}$. Moreover, $J(\pi_n)(J_{\pi_n}(Y_n)) = X_n$, and for each $x \in X_n$, $(J(\pi_n))^{-1}(x)$ is connected.

Thus, the constructible sheaf $\mathbb{Q}_{X_n}$ can be expressed as

$$\mathbb{Q}_{X_n} = \tau^{\leq 0}\tau^{\geq 0} R(J(\pi_n))_* \mathbb{Q}_{J_{\pi_n}(Y_n)}.$$

Notice that the quantifiers in the usual definition of the classes $\mathbf{NP}_{\mathbb{R}}$ and $\mathbf{co\text{-}NP}_{\mathbb{R}}$ are replaced by the direct image functor and truncation functors.

This motivates the following definition.

Definition 3.68. For each $p \geq 0$ we define a class of sequences $\mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ as follows. Let

$$\mathbf{\Lambda}^{(0)}\boldsymbol{\mathcal{P}}_{\mathbb{R}} = \boldsymbol{\mathcal{P}}_{\mathbb{R}}.$$

For $p > 0$, we define inductively the class $\mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ as the smallest class of sequences of constructible sheaves satisfying the following conditions.

(A) The class $\mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ contains the class of sequences, $(\mathcal{F}_n)_{n>0}$ for which there exists non-negative polynomials $m(n), m_0(n) \in \mathbb{Z}[n]$, and a sequence $(\mathcal{G}_n)_{n>0} \in \mathbf{\Lambda}^{(p-1)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ such that for each $n > 0$,
   (1) $\mathcal{G}_n \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m_0(n)} \times \mathbf{S}^{m(n)}, \mathbb{Q}))$;
   (2) $\mathcal{F}_n = R\pi_{n,*}\mathcal{G}_n$, where $\pi_n : \mathbf{S}^{m_0(n)} \times \mathbf{S}^{m(n)} \to \mathbf{S}^{m(n)}$ is the projection to the second factor.

(B) The class $\mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ is closed under taking direct sums, tensor products, truncations and pull-backs. More precisely,
   (1) If $(\mathcal{F}_n)_{n>0}, (\mathcal{G}_n)_{n>0} \in \mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$, then $(\mathcal{F}_n \oplus \mathcal{G}_n)_{n>0}, (\mathcal{F}_n \otimes \mathcal{G}_n)_{n>0} \in \mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.
   (2) For any polynomial $m(n)$, and a class $(\mathcal{F}_n)_{n>0} \in \mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$, the sequences $(\tau^{\leq m(n)}\mathcal{F}_n)_{n>0}, (\tau^{\geq m(n)}\mathcal{F}_n)_{n>0}$ also belong to the class $\mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.
   (3) For any fixed non-negative polynomial $m_1(n) \in \mathbb{Z}[n]$, let $\pi_n : \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)} \to \mathbf{S}^{m(n)}$ be the projection map on the second factor, and

$$\left(\mathcal{F}_n \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})\right)_{n>0} \in \mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}.$$

   Then, $\left(\pi_n^{-1}(\mathcal{F}_n)\right)_{n>0} \in \mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.

Finally, $\mathbf{\Lambda}^{(\omega)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ is defined as the union

$$\mathbf{\Lambda}^{(\omega)}\boldsymbol{\mathcal{P}}_{\mathbb{R}} = \bigcup_{p\geq 0} \mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$$

The class $\mathbf{\Lambda}^{(\omega)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ is the analogue in the category of constructible sheaves of the class $\mathbf{PH}_{\mathbb{R}}$ and we will denote

$$\boldsymbol{\mathcal{PH}}_{\mathbb{R}} = \mathbf{\Lambda}^{(\omega)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}.$$

The following example is closely related to the language in the B-S-S complexity class $\mathbf{NP}_{\mathbb{R}}$ of all real polynomials in n variables having degree at most 4, whose set of real zeros is non-empty. It is well known [14] that this problem is $\mathbf{NP}_{\mathbb{R}}$-complete.

Example 3.69. For each n let $V_n \subset \mathbf{S}^{n-1} \times \mathbf{S}^{N_{n,4}-1}$ denote the real variety defined by

$$V_n = \{(\mathbf{x}, P) \mid P \in \mathrm{Sym}_{n,4} \cap \mathbf{S}^{N_{n,4}-1}, \mathbf{x} \in \mathbf{S}^{n-1} | P(\mathbf{x}) = 0\}.$$

Let as usual $\pi_n : \mathbf{S}^{n-1} \times \mathbf{S}^{N_{n,4}-1} \to \mathbf{S}^{N_{n,4}-1}$ denote the projection to the second factor.

It is now easy to verify that the sequence $\left(R\pi_{n,*}\mathbb{Q}_{V_n} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{N_{n,4}-1}))\right)_{n>0}$ belongs to the class $\mathbf{\Lambda}^{(1)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$. We conjecture that it does not belong to the class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.

3.6. **Inclusions.** In this section we relate the class of sheaves $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ with the more classical complexity classes $\mathbf{P}^c_{\mathbb{R}}$, and more generally the class $\boldsymbol{\mathcal{PH}}_{\mathbb{R}}$ with the class $\mathbf{PH}^c_{\mathbb{R}}$. This allows us to relate the classical question about separating the class $\mathbf{P}^c_{\mathbb{R}}$ from the class $\mathbf{PH}_{\mathbb{R}}$ with the new question of separating $\boldsymbol{\mathcal{PH}}_{\mathbb{R}}$ from $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.

Proposition 3.70. (A) *A sequence $(S_n \subset \mathbf{S}^{m(n)})_{n>0}$ belongs to the class $\mathbf{P}^c_{\mathbb{R}}$ if and only if $(\mathbb{Q}_{X_n} \in \mathrm{Ob}(\mathbf{D}_{\mathrm{sa}}(\mathbf{S}^{m(n)}))_{n>0}$ belongs to the class $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.*

(B) *If a sequence $(S_n \subset \mathbf{S}^{m(n)})_{n>0}$ belongs to the complexity class $\mathbf{\Sigma}^c_{\mathbb{R},p} \cup \mathbf{\Pi}^c_{\mathbb{R},p}$, then the sequence $(\mathbb{Q}_{S_n} \in \mathrm{Ob}(\mathbf{D}_{\mathrm{sa}}(\mathbf{S}^{m(n)}))_{n>0}$ belongs to the class $\mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.*

In particular, if a sequence $(S_n \subset \mathbf{S}^{m(n)})_{n>0}$ belongs to the class $\mathbf{PH}^c_{\mathbb{R}}$ then $(\mathbb{Q}_{X_n} \in \mathrm{Ob}(\mathbf{D}_{\mathrm{sa}}(\mathbf{S}^{m(n)})_{n>0}$ belongs to the class $\boldsymbol{\mathcal{PH}}_{\mathbb{R}}$.

Proof. The first part has been already proved in Proposition 3.54. Now suppose that $(S_n \subset \mathbf{S}^{m(n)})_{n>0}$ belongs to the complexity class $\mathbf{\Sigma}^c_{\mathbb{R},p}$. We prove that $(\mathbb{Q}_{S_n} \in \mathrm{Ob}(\mathbf{D}_{\mathrm{sa}}(\mathbf{S}^{m(n)}))_{n>0}$ belongs to the class $\mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ using induction on p. Suppose that the claim holds for all smaller values of p. The base case $p = 0$ follows from the first part.

Then, since $(S_n \subset \mathbf{S}^{m(n)})_{n>0}$ belongs to the complexity class $\mathbf{\Sigma}^c_{\mathbb{R},p}$, by definition, there exists polynomials

$$m(n), m_1(n), \ldots, m_p(n)$$

and for for each $n > 0$ the semi-algebraic set S_n is described by a first order formula

$$(Q_1\mathbf{Y}^1 \in \mathbf{S}^{m_1(n)}) \cdots (Q_p\mathbf{Y}^p \in \mathbf{S}^{m_p(n)})\phi_n(X_0, \ldots, X_{m(n)}, \mathbf{Y}^1, \ldots, \mathbf{Y}^p),$$

with ϕ_n a quantifier-free first order formula defining a *closed* semi-algebraic subset of $\mathbf{S}^{m_1(n)} \times \cdots \times \mathbf{S}^{m_p(n)} \times \mathbf{S}^{m(n)}$ and for each $i, 1 \leq i \leq p$ $Q_i \in \{\exists, \forall\}$, with $Q_j \neq Q_{j+1}, 1 \leq j < p$, $Q_1 = \exists$, and the sequence of semi-algebraic sets $(T_n \subset \mathbf{S}^{m_1(n)} \times \cdots \times \mathbf{S}^{m_p(n)} \times \mathbf{S}^{m(n)})_{n>0}$ defined by the formulas $(\phi_n)_{n>0}$ belongs to the class $\mathbf{P}_{\mathbb{R}}$.

The sequence $\left(S'_n \subset \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)}\right)_{n>0}$, and hence the sequence $(J_{\pi_n}(S'_n))_{n>0}$, belongs to the class $\mathbf{\Pi}^c_{\mathbb{R},p-1}$. where each S'_n is defined by the formula

$$(Q_2\mathbf{Y}^2 \in \mathbf{S}^{m_2(n)}) \cdots (Q_p\mathbf{Y}^p \in \mathbf{S}^{m_p(n)})\phi_n(X_0, \ldots, X_{m(n)}, \mathbf{Y}^1, \ldots, \mathbf{Y}^p),$$

and $\pi_n : \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)} \to \mathbf{S}^{m(n)}$ is the projection map on the second factor. Using the inductive hypothesis, the sequence $\left(\mathbb{Q}_{J_{\pi_n}(S'_n)}\right)_{n>0} \in \mathbf{\Lambda}^{(p-1)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$. Then, for each $n > 0$, $\mathbb{Q}_{S_n} = \tau^{\geq 0}\tau^{\leq 0}R\pi_{n,*}\mathbb{Q}_{J_{\pi_n}}(S'_n)$, and it follows from the definition of the class $\mathbf{\Lambda}^{(p-1)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$, that $(\mathbb{Q}_{S_n})_{n>0}$ belongs to the class $\mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.

Now suppose that $(S_n \subset \mathbf{S}^{m(n)})_{n>0}$ belongs to the complexity class $\mathbf{\Pi}^c_{\mathbb{R},p}$.

Then, there exists polynomials

$$m(n), m_1(n), \ldots, m_p(n)$$

and for for each $n > 0$ the semi-algebraic set S_n is described by a first order formula

$$(Q_1\mathbf{Y}^1 \in \mathbf{S}^{m_1(n)}) \cdots (Q_p\mathbf{Y}^p \in \mathbf{S}^{m_p(n)})\phi_n(X_0, \ldots, X_{m(n)}, \mathbf{Y}^1, \ldots, \mathbf{Y}^p),$$

with ϕ_n a quantifier-free first order formula defining a *closed* semi-algebraic subset of $\mathbf{S}^{m_1(n)} \times \cdots \times \mathbf{S}^{m_p(n)} \times \mathbf{S}^{m(n)}$ and for each $i, 1 \leq i \leq p$ $Q_i \in \{\exists, \forall\}$, with $Q_j \neq Q_{j+1}, 1 \leq j < p$, $Q_1 = \forall$, and the sequence of semi-algebraic sets $(T_n \subset \mathbf{S}^{m_1(n)} \times \cdots \times \mathbf{S}^{m_p(n)} \times \mathbf{S}^{m(n)})_{n>0}$ defined by the formulas $(\phi_n)_{n>0}$ belongs to the class $\mathbf{P}_{\mathbb{R}}$.

The sequence $\left(S'_n \subset \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)}\right)_{n>0}$ belongs to the class $\mathbf{\Sigma}^c_{\mathbb{R},p-1}$ where each S'_n is defined by the formula

$$(Q_2\mathbf{Y}^2 \in \mathbf{S}^{m_2(n)}) \cdots (Q_p\mathbf{Y}^p \in \mathbf{S}^{m_p(n)})\phi_n(X_0, \ldots, X_{m(n)}, \mathbf{Y}^1, \ldots, \mathbf{Y}^p).$$

Using the inductive hypothesis, the sequence $\left(\mathbb{Q}_{S'_n}\right)_{n>0} \in \mathbf{\Lambda}^{(p-1)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$. Then, for each $n > 0$, $\mathbb{Q}_{S_n} = \tau^{\geq 0}\tau^{\leq 0}R\pi_{n,*}\mathbb{Q}_{J_{\pi_n}}(S'_n)[m_1(n)]$, and it follows from the definition of the class $\mathbf{\Lambda}^{(p-1)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$, that $(\mathbb{Q}_{S_n})_{n>0}$ belongs to the class $\mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$. □

It is an immediate consequence of Proposition 3.70 that:

Theorem 3.71. *For each $p \geq 0$, the equality $\mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}} = \boldsymbol{\mathcal{P}}_{\mathbb{R}}$ implies $\mathbf{\Sigma}^c_{\mathbb{R},p} \cup \mathbf{\Pi}^c_{\mathbb{R},p} = \mathbf{P}^c_{\mathbb{R}}$.*

In view of Theorem 3.71 we conjecture that:

Conjecture 3.72. *For each $p > 0$,*

$$\mathbf{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}} \neq \boldsymbol{\mathcal{P}}_{\mathbb{R}}.$$

Remark 3.73. Notice that Conjecture 3.72 is a priori weaker than the more standard conjecture that $\mathbf{\Sigma}^c_{\mathbb{R},p} \cup \mathbf{\Pi}^c_{\mathbb{R},p} \neq \mathbf{P}^c_{\mathbb{R}}$.

Remark 3.74. The reverse implication in Theorem 3.71 is probably not true, even though we do not have a counter-example at this point. In fact, sequences $\left(S_n \subset \mathbf{S}^{m(n)}\right)_{n>0}$ for which $\left(\mathbb{Q}_{S_n}\right)_{n>0} \in \mathbf{\Lambda}^{(1)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ "interpolates" between the classes $\mathbf{NP}^c_{\mathbb{R}}$ and $\mathbf{co\text{-}NP}^c_{\mathbb{R}}$. For example, consider a sequence $\left(S_n \subset \mathbf{S}^{m(n)}\right)_{n>0}$ for which there exists a sequence $\left(T_n \subset \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)}\right)_{n>0}$, such that $S_n = \{x \in \mathbf{S}^{m(n)} \mid \mathrm{H}^{m_2(n)}(T_{n,x}, \mathbb{Q}) \cong \mathbb{Q}\}$, where $m_1(n), m_2(n)$ are non-negative polynomials, and $T_{n,x} = \pi_n^{-1}(x) \cap T_n$

where π_n is the projection map $\pi_n : \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)} \to \mathbf{S}^{m(n)}$ to the second factor. This sequence is unlikely to be in either $\mathbf{NP}^c_{\mathbb{R}}$ and $\mathbf{co\text{-}NP}^c_{\mathbb{R}}$, even though clearly $(\mathbb{Q}_{S_n})_{n>0} \in \mathbf{\Lambda}^{(1)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$.

4. Topological complexity of constructible sheaves

A standard way in semi-algebraic geometry to measure the topological complexity of a semi-algebraic set is by the sum of their Betti numbers. The topological complexity of a set S, measured by the sum of its Betti numbers, has been shown to be related to hardness of testing membership in S [48, 36]. Also, the topological complexity is a rough guide to the best possible complexity one can hope for of algorithms computing topological invariants of a set S. For example, if the topological complexity is bounded by a polynomial in the input parameters, then one often gets algorithms also with polynomial complexity (see for example [2, 5, 4, 12] in the case of semi-algebraic sets defined by few quadratic polynomials for an instance of this phenomena).

For a sequence of semi-algebraic sets $(S_n \subset \mathbb{R}^n)$ one can study the dependence of the Betti numbers of S_n as a function of n, and in particular estimate the growth of this sequence. It follows from complexity estimates on effective quantifier elimination (see for example [9]), as well as estimates on the Betti numbers of semi-algebraic sets in terms of the number and degrees of the polynomials occurring in their definition [29], that the Betti numbers of the sets S_n, for any sequence $\left(S_n \subset \mathbb{R}^{m(n)}\right)_{n>0} \in \mathbf{PH}_{\mathbb{R}}$, are bounded singly exponentially as a function of n. This is gives an upper bound on the growth of topological complexity of the sequence $\left(S_n \subset \mathbb{R}^{m(n)}\right)_{n>0}$. We now state this fact more formally.

Theorem 4.1. *Let $\left(S_n \subset \mathbb{R}^{m(n)}\right)_{n>0}$ belong to the class $\mathbf{PH}_{\mathbb{R}}$. Then, there exists a polynomial $q(n)$ (depending on the sequence $(S_n)_{n>0}$), such that*

$$b(S_n) \leq 2^{q(n)}.$$

In other words, $b(S_n)$ is bounded singly exponentially as a function of n. A similar statement holds for the compact class $\mathbf{PH}^c_{\mathbb{R}}$ as well.

Proof. Recall that, if $\left(S_n \subset \mathbb{R}^{m(n)}\right)_{n>0}$ belong to the class $\mathbf{PH}_{\mathbb{R}}$, then for each $n > 0$ the semi-algebraic set S_n is described by a first order formula

$$(Q_1\mathbf{Y}^1) \cdots (Q_p\mathbf{Y}^p)\phi_n(X_1, \ldots, X_{k(n)}, \mathbf{Y}^1, \ldots, \mathbf{Y}^p), \tag{4.1}$$

with ϕ_n a quantifier free formula in the first order theory of the reals, and for each $i, 1 \leq i \leq p$, $\mathbf{Y}^i = (Y^i_1, \ldots, Y^i_{k_i(n)})$ is a block of $k_i(n)$ variables, $Q_i \in \{\exists, \forall\}$, with $Q_j \neq Q_{j+1}, 1 \leq j < p$, $Q_1 = \exists$, and the sequence of semi-algebraic sets $(T_n \subset \mathbb{R}^{k(n)+k_1(n)+\cdots+k_p(n)})_{n>0}$ defined by the quantifier-free formulas $(\phi_n)_{n>0}$ belongs to the class $\mathbf{P}_{\mathbb{R}}$.

It follows from the definition of the class $\mathbf{P}_{\mathbb{R}}$ that the sets T_n has a description by a quantifier-free first order formula with atoms of the form $P\{>,=,<\}0, P \in \mathbb{R}[X_1, \ldots, X_n]$, and the number and degrees of the polynomials appearing in the formula are bounded singly exponentially in n. It then follows from the complexity estimates on effective quantifier elimination in the theory of the reals (see for example [32, 40],[9, Theorem 14.16]), that the same is true for the sets S_n as well. It now follows from [29, Theorem 6.8] (see also [9, Theorem 7.50]) that $b(S_n)$ is bounded singly exponentially in n. □

Remark 4.2. Note that for the compact class $\mathbf{PH}^c_{\mathbb{R}}$, the statement of Theorem 4.1 also follows directly from Theorem 4.6 and Proposition 3.70 below.

In the sheaf context, in order to measure the complexity of a given constructible sheaf $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(X))$ for any semi-algebraic set X, we we will use the sum of the dimensions of the hypercohomology groups, $\mathbb{H}^i(X, \mathcal{F})$ (Definition 3.25).

Notation 4.3. For a constructible sheaf $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(X))$, we will denote

$$\begin{aligned} b_i(\mathcal{F}) &= \dim_{\mathbb{Q}}(\mathbb{H}^i(X, \mathcal{F}), \\ b(\mathcal{F}) &= \sum_i b_i(\mathcal{F}). \end{aligned}$$

Definition 4.4. We call $b(\mathcal{F})$ the topological complexity of $\mathcal{F}$.

It is well known (see for example [33, Chapter III]) that:

Proposition 4.5. *If X is a locally closed semi-algebraic set, then*

$$b_i(\mathbb{Q}_X) = b_i^{BM}(X).$$

Theorem 4.6. *Let $\left(\mathcal{F}_n \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})\right)_{n>0}$ be a sequence of constructible sheaves belonging to the class $\mathbf{\Lambda}^{(\omega)}(\boldsymbol{\mathcal{P}}_{\mathbb{R}})$. Then, there exists a polynomial $q(n)$ (depending on the sequence $(\mathcal{F}_n)_{n>0}$), such that*

$$b(\mathcal{F}_n) \leq 2^{q(n)}.$$

In other words, $b(\mathcal{F}_n)$ is bounded singly exponentially as a function of n.

Before proving Theorem 4.6 we need a few preliminaries.

Proposition 4.7. *Let $\mathbf{K}$ be a compact semi-algebraic set, and $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{K}))$, and suppose that $C \subset \mathbf{S}^n$ is a locally closed and contractible subset of $\mathbf{S}^n$. Suppose that $\mathrm{H}^*(\mathcal{F}_{\mathbf{x}})$ is locally constant on C. Then, $\mathcal{F}|_{C'}$ is isomorphic to a constant sheaf in $\mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(C'))$ for any locally closed subset $C' \subset C$.*

Proof. The proposition follows from the fact that any locally constant sheaf over a contractible set is isomorphic to a constant sheaf (Proposition 3.13), and the fact that the restriction of a constant sheaf to a subspace again yields a constant sheaf. The same applies in the derived category as well. □

Proposition 4.8. *(Covering by contractibles) Let $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_{n+1}]$ be a finite family of polynomials, with $\mathrm{card}(\mathcal{P}) = s$, and $d = \max_{P \in \mathcal{P}} \deg(P)$. Then, there exists a family of polynomials $\mathcal{Q} \subset \mathbb{R}[X_1, \ldots, X_{n+1}]$ having the following properties.*

(A) *The degrees of the polynomials in $\mathcal{Q}$, as well as $\mathrm{card}(\mathcal{Q})$ are bounded by $(sd)^{n^{O(1)}}$.*
(B) *The signs of the polynomials in $\mathcal{P}$ are invariant at all points of any semi-algebraically connected component D of the realization $\mathcal{R}(\rho, \mathbf{S}^n)$ of any realizable sign condition $\rho \in \{0, +1, -1\}^{\mathcal{Q}}$.*
(C) *For each realizable sign condition $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$, the realization $S = \mathcal{R}(\sigma, \mathbf{S}^n)$ is a disjoint union of locally closed semi-algebraic sets D. Each such D is a semi-algebraically connected component of some realizable sign condition on $\mathcal{Q}$, and is contained in a contractible semi-algebraic subset $D' \subset S$ which is closed in S.*

Proof. The proposition follows by applying Algorithm 16.14 (Covering by contractible sets) in [9] taking the family of polynomials $\mathcal{P}$ as input. We omit the details. □

Following the same notation as in Proposition 4.8.

Corollary 4.9. *Suppose $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^n))$ such that $\mathcal{F}_{\mathbf{x}}$ is locally constant along each connected component C of the realization of each realizable sign condition $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$. Then, $\mathcal{F}|_D$ is a constant sheaf for each D, for D a connected component of the realization of any realizable sign condition of $\mathcal{Q}$.*

Proof. This follows immediately from Propositions 4.8 and 4.7. □

We will also need some properties regarding the invariance of sheaf cohomology under "infinitesimal" "thickenings" and "shrinkings" of the underlying semi-algebraic sets. In the case of ordinary cohomology (or in sheaf theoretic language cohomology with values in a constant sheaf) these facts are standard and follows from homotopy invariance of the cohomology groups and the local conic structure of semi-algebraic sets [16, Theorem 9.3.6]. However, in general sheaf cohomology is not homotopy invariant (see for example [28, Example 3.1.6]), and so more care needs to be taken. The following proposition follows directly from [34, Proposition 2.7.1].

Proposition 4.10. *(Shrinking) Let $\mathbf{K}$ be a compact semi-algebraic set and $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{K}))$. Let X be a locally closed semi-algebraic subset of $\mathbf{K}$ and suppose that $(X_n \subset X)_{n\in\mathbb{N}}$ is an increasing family of compact semi-algebraic subsets of X such that, $X_n \subset Int(X_{n+1})$ for each n, and $X = \cup_n X_n$. Then, the natural map induced by restriction*

$$\phi_j : \mathbb{H}^j(X, \mathcal{F}|_X) \to \varprojlim_n \mathbb{H}^j(X_n, \mathcal{F}|_{X_n})$$

is an isomorphism for all $j \in \mathbb{Z}$. In particular, since all the hypercohomology groups, $\mathbb{H}^j(X_n, \mathcal{F}|_{X_n}$ are finite dimensional, and vanish for $|j| > N$ for some N, we have that for all n large enough,

$$\mathbb{H}^j(X, \mathcal{F}|_X) \cong \varprojlim_n \mathbb{H}^j(X_n, \mathcal{F}|_{X_n}).$$

We also have the following.

Proposition 4.11. *(Thickening) Let $\mathbf{K}$ be a compact semi-algebraic set and $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{K}))$. Let Z be a closed semi-algebraic subset of $\mathbf{K}$ and suppose that $(Z_n \subset X)_{n\in\mathbb{N}}$ is a decreasing family of compact semi-algebraic subsets of $\mathbf{K}$ such that $Z = \cap_n Z_n$. Then, the natural map induced by restriction*

$$\phi_j : \varinjlim_n \mathbb{H}^j(Z_n, \mathcal{F}|_{Z_n}) \to \mathbb{H}^j(Z, \mathcal{F}|_Z)$$

is an isomorphism for all $j \in \mathbb{Z}$. In particular, we have that for all n large enough,

$$\mathbb{H}^j(Z, \mathcal{F}|_Z) \cong \mathbb{H}^j(Z_n, \mathcal{F}|_{Z_n}).$$

Proof. See [34, Remark 2.6.9]. □

We need some notation.

Notation 4.12. Let $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_n]$ be a family of polynomials with $\mathrm{card}(\mathcal{P}) = s$. Let $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2s})$.

For two sign conditions $\sigma_1, \sigma_2 \in \{0, 1, -1\}^{\mathcal{P}}$, we denote $\sigma_1 \prec \sigma_2$ if and only if $\sigma_2(P) = 0 \Rightarrow \sigma_1(P) = 0$ for all $P \in \mathcal{P}$.

For a sign condition $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$, let

$$\mathrm{level}(\sigma) = \mathrm{card}(\{P \in \mathcal{P} \mid \sigma(P) = 0\}),$$

and let $\sigma_{\bar{\varepsilon}}$ be the formula

$$\bigwedge_{P \in \mathcal{P}, \sigma(P) = 0} (-\varepsilon_{2\ell-1} \leq P \leq \varepsilon_{2\ell-1}) \wedge \bigwedge_{P \in \mathcal{P}, \sigma(P) > 0} (P \geq \varepsilon_{2\ell}) \wedge \bigwedge_{P \in \mathcal{P}, \sigma(P) < 0} (P \leq -\varepsilon_{2\ell}),$$

where $\ell = \mathrm{level}(\sigma)$.

By the phrase "$0 < \bar{\varepsilon} \ll 1$" we will mean "$0 < \varepsilon_1 \ll \varepsilon_2 \ll \cdots \ll \varepsilon_{2s-1} \ll \varepsilon_{2s} \ll 1$" (i.e. "for all sufficiently small positive ε_{2s}, and then for all sufficiently small positive ε_{2s-1}, etc.").

Proposition 4.13. *Let $\mathbf{K}$ be a compact semi-algebraic subset of $\mathbb{R}^n$. Suppose $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{K}))$ such that $\mathrm{H}^*(\mathcal{F}_{\mathbf{x}})$ is constant along each connected component C of the realization of each realizable sign condition $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$ on $\mathbf{K}$. Then for all $0 < \bar{\varepsilon} \ll 1$ (see Notation 4.12 above):*

(A) *If C is a connected component of the realization of the realizable sign condition $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$ on $\mathbf{K}$, then there exists a unique connected component, $C_{\bar{\varepsilon}}$, of $\mathcal{R}(\sigma_{\bar{\varepsilon}}, \mathbf{K})$, such that $C \cap C_{\bar{\varepsilon}} \neq \emptyset$.*

(B) *The semi-algebraic set $C_{\bar{\varepsilon}}$ is closed in $\mathbf{K}$ and homotopy equivalent to C.*

(C) *For ever $q \in \mathbb{Z}$, and $\mathbf{x} \in C$,*

$$\mathbb{H}^q(C, \mathcal{F}|_C) \cong \mathbb{H}^q(C_{\bar{\varepsilon}}, \mathcal{F}|_{C_{\bar{\varepsilon}}}) \cong \bigoplus_{i+j=q} \mathrm{H}^i(C, \mathbb{Q}) \otimes \mathrm{H}^j(\mathcal{F}_{\mathbf{x}}).$$

(D) *More generally, suppose that $C_0, \ldots, C_p$ are connected components of the realizations of $\sigma_0, \ldots, \sigma_p \in \{0, 1, -1\}^{\mathcal{P}}$ on $\mathbf{K}$, and that $C' = \mathbb{C}_{0,\bar{\varepsilon}} \cap \cdots \cap C_{p,\bar{\varepsilon}} \neq \emptyset$. Then, there exists a permutation π of $\{0, \ldots, p\}$ such that*

$$\sigma_{\pi(0)} \prec \cdots \prec \sigma_{\pi(p)},$$
$$C' \subset C_{\sigma_{\pi(p)}}.$$

Moreover, C' is closed in $\mathbf{K}$, and homotopy equivalent to $C'' = C_{\sigma_{\pi(p)}} \cap C'$, and for every $q \in \mathbb{Z}$ and $\mathbf{x} \in C'$

$$\mathbb{H}^q(C', \mathcal{F}|_{C'}) \cong \mathbb{H}^q(C'', \mathcal{F}|_{C''}) \cong \bigoplus_{i+j=q} \mathbb{H}^i(C', \mathbb{Q}) \otimes \mathrm{H}^j(\mathcal{F}_{\mathbf{x}}).$$

Proof. We prove the first part of the proposition. The second part follows by an induction on p.

For $\gamma > 0$ let σ_γ be the formula

$$\bigwedge_{P \in \mathcal{P}, \sigma(P) > 0} (P \geq \gamma) \wedge \bigwedge_{P \in \mathcal{P}, \sigma(P) < 0} (P \leq -\gamma),$$

and $\sigma_{\gamma,\delta}$ be the formula defined by

$$\bigwedge_{P \in \mathcal{P}, \sigma(P) = 0} (-\delta \leq P \leq \delta) \wedge \bigwedge_{P \in \mathcal{P}, \sigma(P) > 0} (P \geq \gamma) \wedge \bigwedge_{P \in \mathcal{P}, \sigma(P) < 0} (P \leq -\gamma).$$

It is easy to show that (see for example, [9, Lemma 16.17]) that for $0 < \delta \ll \gamma \ll 1$, and for each connected component C of $\mathcal{R}(\sigma, \mathbf{K})$ there exists a unique connected component $C_{\gamma,\delta}$ of $\sigma_{\gamma,\delta}$, and that $C_{\gamma,\delta}$ is homotopy equivalent to C.

Now choose a sequence $\gamma_i \searrow 0$ and for each γ_j choose $\delta_{i,j} \searrow 0$. Then for all i large enough, and having chosen i, for all j large enough C_{γ_i,δ_j} is well defined, and it is clear that each C_{γ_i,δ_j} is compact and $\cap_j C_{\gamma_i,\delta_i} = C_{\gamma_i}$. Using Proposition 4.11 we obtain that For every $q \in \mathbb{Z}$, and for all j large enough,

$$\mathbb{H}^q(C_{\gamma_i}, \mathcal{F}|_{C_{\gamma_i}}) \quad \cong \quad \mathbb{H}^q(C_{\gamma_i,\delta_j}, \mathcal{F}|_{C_{\gamma_i,\delta_j}}).$$

Now observe that $\cup_i C_{\gamma_i} = C$, and using Proposition 4.10 we have for every $q \in \mathbb{Z}$,

$$\mathbb{H}^q(C, \mathcal{F}|_C) \quad \cong \quad \mathbb{H}^q(C_{\varepsilon_i}, \mathcal{F}|_{C_{\gamma_i}}),$$

for all i large enough. Moreover, noticing that $\mathrm{H}^*(\mathcal{F}_{\mathbf{x}})$ is constant over C_{γ_i} (using the fact that $\mathrm{H}(\mathcal{F}_{\mathbf{x}})$ is constant over C and Proposition 4.7), we have that for every $q \in \mathbb{Z}$ and $\mathbf{x} \in C_{\gamma_i}$

$$\mathbb{H}^q(C_{\gamma_i}, \mathcal{F}|_{C_{\gamma_i}}) \quad \cong \quad \bigoplus_{i+j=q} \mathrm{H}^i(C_{\gamma_i}, \mathbb{Q}) \otimes \mathrm{H}^j(\mathcal{F}_{\mathbf{x}}).$$

Now using the fact that C_{γ_i} is homotopy equivalent to C for i large enough, we have for every $q \in \mathbb{Z}$,

$$\mathrm{H}^q(C_{\gamma_i}, \mathbb{Q}) \quad \cong \quad \mathrm{H}^q(C, \mathbb{Q}).$$

Together Eqns. (4.2), (4.2), (4.2) and (4.2) imply that For ever $q \in \mathbb{Z}$, and $\mathbf{x} \in C$,

$$\mathbb{H}^q(C, \mathcal{F}|_C) \cong \mathbb{H}^q(C_{\gamma_i,\delta_j}, \mathcal{F}|_{C_{\gamma_i,\delta_j}}) \cong \bigoplus_{k+m=q} \mathrm{H}^k(C, \mathbb{Q}) \otimes \mathrm{H}^m(\mathcal{F}_{\mathbf{x}}).$$

for all i large enough, and after having chosen i, for all j large enough. The first part of the proposition now follows by replacing δ_i by $\varepsilon_{2\mathrm{level}(\sigma)-1}$ and γ_i by $\varepsilon_{2\mathrm{level}(\sigma)}$.

The second part of the proposition follows from an induction on p using the first part as the base case. But since the steps are very similar to the one above we omit this part of the proof. □

Proof. (Proof of Theorem 4.6.) Let $\left(\mathcal{F}_n \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^{m(n)})\right)_{n>0}$ be a sequence of constructible sheaves belonging to the class $\mathbf{\Lambda}^{(p)}(\boldsymbol{\mathcal{P}}_{\mathbb{R}})$ for some $p \geq 0$.

We first prove that there exists a polynomial $q_1(n)$ (depending on the sequence $(\mathcal{F}_n)_{n>0}$) and for each $n > 0$ a family of polynomials $\mathcal{P}_n \subset \mathbb{R}[X_1, \ldots, X_{m(n)+1}]$, such that:

(A) both the card($\mathcal{P}_n$) and the degrees of the polynomials in $\mathcal{P}_n$ are bounded by $2^{q_1(n)}$;

(B) the semi-algebraic partition $\Pi(\mathcal{P}_n, \mathbf{S}^{m(n)})$ (cf. Notation 3.52) is subordinate to $\mathcal{F}_n$;

(C) moreover, for $\mathbf{x} \in C$,

$$\sum_i \dim_{\mathbb{Q}}(\mathrm{H}^i((\mathcal{F}_n)_{\mathbf{x}})) \leq 2^{q_1(n)},$$

and

(D)

$$\mathrm{H}^i((\mathcal{F}_n)_{\mathbf{x}}) = 0,$$

for $|i| \geq q_1(n)$.

The proof of the above claim is by induction on p. If $p = 0$, then the claim follows directly from the definition of $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ and Proposition 3.53. Now suppose that the claim is true for all smaller values of p.

Now a sequence $(\mathcal{F}_n)_{n>0}$ belongs to $\boldsymbol{\Lambda}^{(p)}(\boldsymbol{\mathcal{P}}_{\mathbb{R}})$, by definition either:

(A) for each $n > 0$

$$\mathcal{F}_n = R\pi_{n,*}(\mathcal{G}_n),$$

and the sequence

$$(\mathcal{G}_n \in \mathrm{Ob}(\mathbf{D}_{\mathrm{sa}}(\mathbf{K}_n))_{n>0} \in \boldsymbol{\Lambda}^{(p-1)}(\boldsymbol{\mathcal{P}}_{\mathbb{R}}),$$

where $m_1(n) \in \mathbb{Z}[n]$ is a non-negative polynomial, $\mathbf{K}_n = \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)}$. and $\pi_n : \mathbf{S}^{m_1(n)} \times \mathbf{S}^{m(n)} \to \mathbf{S}^{m(n)}$ is the projection to the second factor. Or:

(B)

$$\mathcal{F}_n = R\pi_{n,*}(\mathcal{G}_n),$$

can be obtained from a finite number of sequences of the first kind, by taking tensor products, direct sums, truncations and pull-backs.

However, the claim to be proved is easily shown to be preserved under operations of tensor products, direct sums, truncations and pull-backs (by the same method of proof as in the proof of Proposition 3.55. Thus, it suffices to consider only the first case.

By the induction hypothesis there exists polynomial $q_2(n)$ (depending on the sequence $(\mathcal{G}_n)_{n>0}$) and for each $n > 0$ a family of polynomials

$$\mathcal{Q}_n \subset \mathbb{R}[Y_0, \ldots, Y_{m_1(n)}, X_0, \ldots, X_{m(n)}],$$

such that:

(A) both $s_n = \mathrm{card}(\mathcal{Q}_n)$ and the degrees of the polynomials in $\mathcal{Q}_n$ are both bounded by $2^{q_2(n)}$;

(B) the semi-algebraic partition $\Pi(\mathcal{Q}_n, \mathbf{K}_n)$ is subordinate to $\mathcal{G}_n$;

(C) moreover, for each $\mathbf{z} \in D$,

$$\sum_i \dim_{\mathbb{Q}} \mathrm{H}^i((\mathcal{G}_n)_{\mathbf{z}} \leq 2^{q_2(n)},$$

and

(D)

$$\mathrm{H}^i((\mathcal{G}_n)_{\mathbf{z}}) = 0,$$

for $|i| \geq q_2(n)$.

Now, let $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2s_n})$. Considering the variables $\bar{\varepsilon}$ as variables, let

$$\bar{\mathcal{P}}'_n \subset \mathbb{R}[\bar{\varepsilon}, Y_0, \ldots, Y_{m_1(n)}, X_0, \ldots, X_{m(n)}]$$

be defined as a set of polynomials such that for all $0 < \bar{\varepsilon} \ll 1$,

$$\bigcup_{P \in \bar{\mathcal{P}}'_n} \mathrm{Z}(P(\bar{\varepsilon}, \cdot), \mathbf{S}^{m(n)})$$

contains all images of the critical points of the projection map π_n restricted to all varieties $\mathrm{Z}(\mathcal{Q}'_{n,\bar{\varepsilon}}(\bar{\varepsilon}, \cdot), \mathbf{K}_n)$, for all subsets $\mathcal{Q}'_{n,\bar{\varepsilon}} \subset \mathcal{Q}_{n,\bar{\varepsilon}}$ with $\mathrm{card}(\mathcal{Q}'_{n,\bar{\varepsilon}}) \leq m_1(n) + m(n)$. Now writing the polynomials in $\bar{\mathcal{P}}'_n$ as polynomials in the variables $\bar{\varepsilon}$, let $\mathcal{P}'_n$ denote the union of all the coefficients of all the polynomials in $\bar{\mathcal{P}}'_n$.

It follows from arguments similar to those in the proof of the main theorem in [10], that the degrees of the polynomials in $\bar{\mathcal{P}}'_n$ are bounded singly exponentially in n, and noting further that each one of them depends on at most $m_1(n)+m(n)$ of the ε_i's, it follows that the number and the degrees of the polynomials in $\mathcal{P}'_n$ are also bounded singly exponentially in n. Moreover, for $0<\bar{\varepsilon}\ll 1$, and every connected component C' of the realization $\mathcal{R}(\sigma',\mathbf{S}^{m(n)})$ of a sign condition $\sigma'\in\{0,1,-1\}^{\mathcal{P}'_n}$ and $\mathbf{x}\in C'$, there is a homeomorphism, $\phi_{\mathbf{x}}: C'\times\pi_n^{-1}(\mathbf{x})\to\pi^{-1}(C')$, such that the following diagram commutes (i.e. the homeomorphism $\phi_{\mathbf{x}}$ preserves the fibers of π_n)

$$\begin{array}{ccc} C'\times\pi_n^{-1}(\mathbf{x}) & \xrightarrow{\phi_{\mathbf{x}}} & \pi_n^{-1}(C') \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_n} \\ C' & \xrightarrow{Id_{C'}} & C' \end{array}$$

(here π denotes the projection map to the first factor), and moreover for each $\mathbf{x}'\in C'$, and for each tuple $(\sigma_0,\ldots,\sigma_p)$, where for each i, $\sigma_i\in\{0,1,-1\}^{\mathcal{Q}_n}$, and connected components D_{σ_i} of the realization $\mathcal{R}(\sigma_i,\mathbf{K}_n)$ for each $i, 0\le i\le p$, $\phi_{\mathbf{x}}$ restricts to a homeomorphism between $(D_{\sigma_0,\bar{\varepsilon}}\cap\cdots\cap D_{\sigma_p,\bar{\varepsilon}})_{\mathbf{x}}, (D_{\sigma_0,\bar{\varepsilon}}\cap\cdots\cap D_{\sigma_p,\bar{\varepsilon}})_{\mathbf{x}'}$, (following the same notation as in Proposition 4.13). Note that, for each $\mathbf{x}\in C'$, there is a spectral sequence, $E_r^{p,q}(\mathbf{x})$ with

(4.2)
$$E_2^{p,q}(\mathbf{x}) = \bigoplus_{\sigma_0,\prec\cdots\prec\sigma_p,\sigma_j\in\{0,1,-1\}^{\mathcal{Q}_n}} \mathbb{H}^q((D_{\sigma_0,\bar{\varepsilon}}\cap\cdots\cap D_{\sigma_p,\bar{\varepsilon}})_{\mathbf{x}}, \mathcal{G}_n|_{(D_{\sigma_0,\bar{\varepsilon}}\cap\cdots\cap D_{\sigma_p,\bar{\varepsilon}})_{\mathbf{x}}})$$

that abuts to $\mathbb{H}^*(\pi_n^{-1}(\mathbf{x}),\mathcal{G}_n|_{\pi_n^{-1}(\mathbf{x})})$. The homeomorphism $\phi_{\mathbf{x}}$ thus induces an isomorphism between the spectral sequences $E_r^{p,q}(\mathbf{x})$ and $E_r^{p,q}(\mathbf{x}')$ and hence between the groups $\mathbb{H}^*(\pi_n^{-1}(\mathbf{x}),\mathcal{G}_n|_{\pi_n^{-1}(\mathbf{x})})$ and $\mathbb{H}^*(\pi_n^{-1}(\mathbf{x}'),\mathcal{G}_n|_{\pi_n^{-1}(\mathbf{x}')})$. In particular, this implies that

$$\mathrm{H}^q(C',\mathrm{H}^p\mathcal{G}_n|_{C'})\cong\mathrm{H}^q(C',\mathbb{Q})\otimes\mathbb{H}^p(\pi_n^{-1}(\mathbf{x}),\mathcal{G}_n|_{\pi_n^{-1}(\mathbf{x})}).$$

Using Proposition 4.8, there exists a family of polynomials

$$\mathcal{P}_n\subset\mathbb{R}[X_0,\ldots,X_{m(n)}]$$

such that the card$(\mathcal{P}_n)$, as well as the degrees of the polynomials in $\mathcal{P}_n$ are bounded by $(\mathrm{card}(\mathcal{P}'_n)\max_{P\in\mathcal{P}'_n}\deg(P))^{m_1(n)^{O(1)}}$, and such that over each connected component C of the realization, $\mathcal{R}(\sigma,\mathbf{S}^{m(n)})$, of any realizable sign condition $\sigma'\in\{0,1,-1\}^{\mathcal{P}_n}$, $\mathbb{H}^i(\pi_n^{-1}(x),\mathcal{G}_n|_{\pi^{-1}(\mathbf{x})})$ is constant (not just locally constant) for $\mathbf{x}\in C$.

We now bound the dimensions $\dim_{\mathbb{Q}}\mathbb{H}^i(\pi_n^{-1}(\mathbf{x}),\mathcal{G}_n|_{\pi^{-1}(\mathbf{x})})$.

By induction hypothesis, we have that $\mathrm{H}^q((\mathcal{G}_n)_{(\mathbf{y},\mathbf{x})})=0$ for all q with $|q|\ge q_2(n)$. It follows that for any $\mathbf{x}\in\mathbf{S}^{m(n)}$, $\mathbb{H}^i(\pi_n^{-1}(\mathbf{x}),\mathcal{G}_n|_{\pi^{-1}(\mathbf{x})})=0$ for all i with $|i|>N(n)=q_2(n)+m_1(n)$.

It then follows from the E_2-term of the spectral sequence abutting to

$$\mathbb{H}^i(\pi_n^{-1}(\mathbf{x}),\mathcal{G}_n|_{\pi^{-1}(\mathbf{x})})$$

(Eqn. 4.2), that

$$\dim_{\mathbb{Q}}\mathbb{H}^i(\pi_n^{-1}(\mathbf{x}),\mathcal{G}_n|_{\pi^{-1}(\mathbf{x})})\le\sum_{p+q=i,|q|\le N(n)}\sum_{\bar{\sigma}}\dim_{\mathbb{Q}}\mathbb{H}^q((D'_{\bar{\sigma},\bar{\varepsilon}})_{\mathbf{x}},\mathcal{G}_n|_{(D'_{\bar{\sigma},\bar{\varepsilon}})_{\mathbf{x}}}),$$

where

$$\begin{aligned}\bar{\sigma} &= \sigma_0 \prec \cdots \prec \sigma_p, \sigma_j \in \{0,1,-1\}^{\mathcal{Q}_n},\\ D' &= (D_{\sigma_0,\bar{\varepsilon}} \cap \cdots \cap D_{\sigma_p,\bar{\varepsilon}}).\end{aligned}$$

It follows from Part (D) of Proposition 4.13 that,

$$\begin{aligned}\mathbb{H}^q((D'_{\bar{\sigma},\bar{\varepsilon}})_{\mathbf{x}}, \mathcal{G}_n|_{(D'_{\bar{\sigma},\bar{\varepsilon}})_{\mathbf{x}}}) &\cong \mathbb{H}^q((D_{\sigma_p} \cap D'_{\bar{\sigma},\bar{\varepsilon}})_{\mathbf{x}}, (\mathcal{G}_n)_{\mathbf{z}})\\ &\cong \bigoplus_{i+j=q} \mathbb{H}^i(D_{\sigma_p} \cap D'_{\bar{\sigma},\bar{\varepsilon}}, \mathbb{Q}) \otimes \mathrm{H}^j(\mathcal{G}_n)_{(\mathbf{y},\mathbf{x})})).\end{aligned}$$

for any $\mathbf{z} = (\mathbf{y}, \mathbf{x}) \in (D_{\sigma_p} \cap D'_{\bar{\sigma},\bar{\varepsilon}})_{\mathbf{x}}$. It follows that $\sum_i \dim_{\mathbb{Q}} \mathbb{H}^i(\pi_n^{-1}(\mathbf{x}), \mathcal{G}_n|_{\pi^{-1}(\mathbf{x})})$ is bounded singly exponentially. This finishes the induction.

Using Proposition 4.8 one more time, we can pass from the partition of $\mathbf{S}^n$, into connected locally closed semi-algebraic subsets C where each C is a connected component of $\mathcal{R}(\sigma, \mathbf{S}^{m(n)})$ for some $\sigma \in \{0,1,-1\}^{\mathcal{P}_n}$, to a cover by closed subsets whose elements are connected components $C_{\sigma,\bar{\varepsilon}}$, of $\mathcal{R}(\sigma_{\bar{\varepsilon}}, \mathbf{S}^{m(n)})$, $\sigma \in \{0,1,-1\}^{\mathcal{P}_n}$ and $\sigma_{\bar{\varepsilon}}$ as in Notation 4.12.

The singly exponential upper bound on the $\dim_{\mathbb{Q}} \mathbb{H}^k(\mathbf{S}^{m(n)}, \mathcal{F}_n)$ now follows by bounding the E_2-term of hypercohomology spectral sequence corresponding to this cover using Proposition 3.31. The theorem follows. □

4.1. **Complexity of generalized quantifier elimination.** The following result which follows directly from the proof of Theorem 4.6 above but which does not refer to any complexity classes could be of independent interest. It is the sheaf theoretic analogue of an effective singly exponential complexity bound for eliminating one block of quantifiers in the first order theory of the reals [40, 9]. In particular, the implied algorithm in the following theorem could be viewed as the sheaf-theoretic analogue of Algorithm 14.1 (Block Elimination) in [9] restricted to the compact situation. We omit the proof of this theorem which is embedded in the proof of the intermediate claim inside the the proof of Theorem 4.6 above.

Theorem 4.14. *(Complexity of generalized quantifier elimination) Let*

$$\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^m \times \mathbf{S}^n)),$$

and let $\mathcal{P} \subset \mathbb{R}[Y_0, \ldots, Y_m, X_0, \ldots, X_n]$ be a finite set of polynomials, such that the semi-algebraic partition $\Pi(\mathcal{P}, \mathbf{S}^m \times \mathbf{S}^n)$ is subordinate to $\mathcal{F}$. Moreover, suppose that $\mathrm{card}(\mathcal{P}) = s$, and also that the degrees of the polynomials in $\mathcal{P}$ are all bounded by d. Let $\pi : \mathbf{S}^m \times \mathbf{S}^n$ denote the projection map to the second factor. Then, there exists a family of polynomials $\mathcal{Q} \subset \mathbb{R}[X_0, \ldots, X_n]$, with $\mathrm{card}(\mathcal{Q})$, as well as the degrees of the polynomials in $\mathcal{Q}$ bounded by $(sd)^{(m+n)^{O(1)}}$, such that the semi-algebraic partition $\Pi(\mathcal{Q}, \mathbf{S}^n)$ is subordinate to the constructible sheaf $R\pi_*\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{S}^n))$. Moreover, there exists an algorithm for computing semi-algebraic description of the partition $\Pi(\mathcal{Q}, \mathbf{S}^n)$, given $\mathcal{P}$ as input, with complexity bounded by $(sd)^{(m+n)^{O(1)}}$.

Proof. See above. □

Remark 4.15. The restriction to spheres in Theorem 4.14 is to ensure properness of the projection map π. It is possible that with more work it would be possible to extend the theorem to the non-compact case and consider projections $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, and consider not just the functor $R\pi_*$ but the derived image functor with proper support, $R\pi_!$ as well. We do not undertake this task in the current paper.

5. Constructible functions and sheaves: Toda's theorem

In this section, we discuss the connections between the complexities of constructible functions and constructible sheaves. We formulate a new conjecture that that could be seen as analogous to Toda's theorem in discrete complexity theory [42]. Toda's theorem gives an inclusion of the polynomial hierarchy **PH** in the class $\mathbf{P}^{\#\mathbf{P}}$ where the right hand side is the set of languages accepted by a Turing machine in polynomial times but with access to an oracle computing functions in $\#\mathbf{P}$. The class $\#\mathbf{P}$ consists of sequences of functions, $(f_n : \{0,1\}^n \to \mathbb{N})_{n>0}$ which counts the number of points in the fibers of a linear projection of a language in **NP**. We refer the reader to [13] where this geometric definition of the class $\#\mathbf{P}$ is elaborated.

In [13] (respectively, [11]) a geometric definition was given of a class, $\#\mathbf{P}^{\dagger}_{\mathbb{R}}$ (respectively, $\#\mathbf{P}^{\dagger}_{\mathbb{C}}$) of sequences of functions $(f_n : \mathbf{S}^n \to \mathbb{Z}[T])_{n>0}$ (respectively, $(f_n : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{Z}[T])_{n>0}$) where the functions f_n took values in the Poincaré polynomials of the fibers of a projection of a language (in the B-S-S) sense in $\mathbf{NP}^c_{\mathbb{R}}$ (respectively, $\mathbf{NP}^c_{\mathbb{C}}$). We omit the precise definitions of these classes, but point out that these functions are in fact constructible functions (or more precisely each component of these functions corresponding to the different coefficients of the image polynomial is a constructible function). An analogue of Toda's theorem was proved in [13].

Theorem 5.1. [13]
$$\mathbf{PH}^c_{\mathbb{R}} \subset \mathbf{P}_{\mathbb{R}}^{\#\mathbf{P}^{\dagger}_{\mathbb{R}}}.$$

A similar result was proved in the complex case in [11].

The relationship implicit in Toda's theorem (and its real and complex analogue) raises the interesting question of whether such a relationship is also true in the sheaf theoretic case. In particular, the following classical proposition (see [34]) is very suggestive.

We first need a new notation.

Notation 5.2. For X a semi-algebraic set we denote by $\mathrm{CF}(X)$ the set of constructible functions on X.

Notation 5.3. If X is a semi-algebraic set, and $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(X))$, then we will denote by $\mathrm{Eu}(\mathcal{F})$ the constructible function on X defined by
$$\mathrm{Eu}(\mathcal{F}) = \sum_j (-1)^j \dim_{\mathbb{Q}} \mathrm{H}^j(\mathcal{F}_{\mathbf{x}}).$$

Proposition 5.4. [34] *Let X, Y be compact semi-algebraic sets, and $f : Y \to X$ a semi-algebraic continuous map. Then, we have the following commutative diagram:*
$$\begin{array}{ccc} \mathrm{Ob}(\mathbf{D}_{\mathrm{sa}}(Y)) & \xrightarrow{Rf_*} & \mathrm{Ob}(\mathbf{D}_{\mathrm{sa}}(X)) \\ \downarrow \mathrm{Eu} & & \downarrow \mathrm{Eu} \\ \mathrm{CF}(Y) & \xrightarrow{f_* = \int \cdot \, d\,\chi} & \mathrm{CF}(X). \end{array}$$

Proof. See [34]. □

We now define the sheaf theoretic analogue of the class $\#\mathbf{P}$ and its generalizations.

Definition 5.5. Let $m(n) \in \mathbb{Z}[n]$ be a non-negative polynomial. We say that a sequence of constructible functions $\left(f_n : \mathbf{S}^{m(n)} \to \mathbb{Z}\right)_{n>0}$ is in the class $\#\boldsymbol{\mathcal{P}}_{\mathbb{R}}$, if there exists a sequence of constructible sheaves $(\mathcal{F}_n)_{n>0} \in \boldsymbol{\Lambda}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ such that for each $n > 0$, $f_n = \mathrm{Eu}(\mathcal{F}_n)$. More generally, we will say that $\left(f_n : \mathbf{S}^{m(n)} \to \mathbb{Z}\right)_{n>0}$ is in the class $\mathbf{Eu}(\boldsymbol{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}})$, if there exists a a sequence of constructible sheaves $(\mathcal{F}_n)_{n>0} \in \boldsymbol{\Lambda}^{(p)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ such that for each $n > 0$, $f_n = \mathrm{Eu}(\mathcal{F}_n)$. Note that $\#\boldsymbol{\mathcal{P}}_{\mathbb{R}} = \mathrm{Eu}(\boldsymbol{\Lambda}^{(1)}\boldsymbol{\mathcal{P}}_{\mathbb{R}})$.

We have the following conjecture which can be seen as a reformulation of Toda's fundamental theorem [42] in sheaf theoretic terms.

Conjecture 5.6.

$$\mathbf{Eu}(\boldsymbol{\Lambda}^{(\omega)}\boldsymbol{\mathcal{P}}_{\mathbb{R}}) = \#\boldsymbol{\mathcal{P}}_{\mathbb{R}}.$$

6. Conclusions

In this paper we have begun the study of a complexity theory of constructible functions and sheaves patterned along the line of the Blum-Shub-Smale theory for constructible/semi-algebraic sets. We have formulated versions of the **P** vs **NP** questions for classes of constructible functions as well as sheaves. An immediate goal would be to develop an analogue of "completeness" results in classical complexity theory and identify certain functions and sheaves to be complete in their class.

As mentioned in the introduction, aside from in semi-algebraic geometry, constructible functions and sheaves appear in many areas of mathematics, in particular in the theory of linear partial differential equations and micro-local analysis as developed by Kashiwara and Schapira [34], motivic integration [25], and also in a more applied setting of signal processing [3]. A more distant goal would be to study these applications from the complexity viewpoint.

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