VC DENSITY OF DEFINABLE FAMILIES OVER VALUED FIELDS

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Abstract. In this article, we give tight bounds on the Vapnik-Chervonenkis density (VC-density) for definable families over any algebraically closed valued field K (of any characteristic pair) in the language $\mathcal{L}_{\text{div}}$ with signature $(0, 1, +, \times, |)$ (where $x|y$ denotes $|x| \leq |y|$). More precisely, we prove that for any parted formula $\phi(\overline{X}; \overline{Y})$ in the language $\mathcal{L}_{\text{div}}$ with parameters in K, the VC-density of ϕ is bounded by $|\overline{X}|$. This result improves the best known results in this direction proved in [ADH+16], where a bound of $2|\overline{X}|$ is shown to hold on the VC-density in the restricted case where the characteristics of the field K and its residue field are both assumed to be 0. The results in this paper are optimal and without any restriction on the characteristics.

We obtain the aforementioned bound as a consequence of another result on bounding the Betti numbers of semi-algebraic subsets of certain Berkovich analytic spaces, mirroring similar results known already in the case of o-minimal structures and for real closed, as well as, algebraically closed fields. The latter result is the first result in this direction and is possibly of independent interest. Its proof relies heavily on recent results of Hrushovski and Loeser ([HL16]).

Contents

1. Introduction

The independence property of a formula $\phi(\bar{V}; \bar{W})$, in the first-order language of some complete theory, is a very important and widely studied notion in model theory. Let T be a complete theory and $\phi(\bar{V}; \bar{W})$ a formula in the language of T.

S.B. would like to acknowledge support from the National Science Foundation awards and DMS-1620271 and CCF-1618918. D.P. would like to acknowledge support from the National Science Foundation award DMS-1502296.

Then ϕ is said to have the *independence property* if all the sentences in the sequence $(I_n)_{n>0}$ are consequences of T, where I_n is the sentence:

$$(\exists \bar{V}_\emptyset)\cdots(\exists \bar{V}_I)\cdots(\exists \bar{V}_{[1,n]})(\exists \bar{W}_1)\cdots(\exists \bar{W}_i)\cdots(\exists \bar{W}^n)\bigwedge_{i\in I}\phi(\bar{V}_I;\bar{W}_i)\wedge\bigwedge_{i\notin I}\neg\phi(\bar{V}_I;\bar{W}_i)$$

(here $I \in 2^{[1,n]}$). A formula $\phi(\bar{V};\bar{W})$ (with its free variables partitioned into two disjoint sets $\bar{V},\bar{W}$) is said to be NIP if it does *not* have the independence property. A theory is said to be NIP if every formula $\phi(\bar{V};\bar{W})$ in its language is NIP. Finally, a structure is said to be NIP if its theory is NIP.

Examples of NIP theories include the theory of algebraically closed fields of a fixed characteristic (in the language of fields), the theory (RCF) of real closed fields in the language of ordered fields, the theory (ACVF) of any algebraically closed valued field K of some fixed characteristic in the language $\mathcal{L}_{\mathrm{div}}$ (see Section 2 for definition of the language $\mathcal{L}_{\mathrm{div}}$), the theory of the p-adic field $\mathbb{Q}_p$ for any prime p in the language of Macintyre, the theory of any o-minimal structure etc. (see the book [Sim15] for a more exhaustive list).

The theory of integers in the language of rings gives rise to an example of a theory which admits formulas having the independence property. The formula $f(V;W) = (\exists U)W = UV$ (i.e. expressing that V divides W) is easily seen to have the independence property. In particular, the theory of integers is not NIP.

In the following, given a tuple of variables $\bar{W} = (W_1,\ldots,W_k)$ we denote by $|\bar{W}|$ the number k of variables. A fundamental result due to Sauer [Sau72] and Shelah [She72] shows that if $\phi(\bar{V};\bar{W})$ is an NIP formula in the theory of a structure M, then there exists a constant $C_\phi > 0$ such that for all $n > 0$ and all finite sequences $(\bar{w}_1,\ldots,\bar{w}_n) \in (M^{|\bar{W}|})^n$, the order of the set

$$\{I \subset [1,n] | \exists \bar{v}_I \text{ such that } M \models \bigwedge_{i\in I}\phi(\bar{v}_I;\bar{w}_i)\wedge\bigwedge_{i\notin I}\neg\phi(\bar{v}_I;\bar{w}_i)\}$$

is bounded by $C_\phi \cdot n^k$ for some $k \geq 0$. The smallest such k is called the *Vapnik-Chervonenkis density* (VC-density in short) of the formula ϕ (see Definition 1.1).

The notion of VC-density plays an important role in many applications such as learning theory (where the notion originated [VvC68]), discrete geometry and extremal combinatorics [PA95, Mat02]. For the latter, the VC-density of various families plays a crucial role in obtaining tight bounds on various geometrically significant quantities such as incidences and also in designing efficient algorithms. Such bounds are also known to have important applications within model theory itself [JL10, Sim15].

As a result of the aforementioned applications, the Vapnik-Chervonenkis density for formulas in various NIP theories has been an important object of study in model theory ([Sim15]). In order to visualize the notion of VC-density more geometrically, it is useful to reformulate it in terms of sets.

Definition 1.1. (*VC-density*) Let X be a set and $\mathcal{S} \subset 2^X$. The *shatter function* $\pi_{\mathcal{S}} : \mathbb{N} \to \mathbb{N}$ of X is defined by setting

$$\pi_{\mathcal{S}}(n) := \max_{A \subset X, \mathrm{card}(A)=n} \mathrm{card}(\{A \cap Y \mid Y \in \mathcal{S}\}).$$

The *VC-density of* $\mathcal{S}$, denoted by $\mathrm{vcd}_{\mathcal{S}}$, is defined by setting

$$\mathrm{vcd}_{\mathcal{S}} := \limsup_{n\to\infty} \frac{\log(\pi_{\mathcal{S}}(n))}{\log(n)}.$$

Given a definable subset $X \subset V \times W$ in some structure, there is a natural way to associate a VC density to this datum. Let $\pi_V : X \to V$ and $\pi_W : X \to W$ denote the restriction to X of the natural projection maps. For $w \in W$, let $X_w := \pi_V(\pi_W^{-1}(w))$. We will denote by $\mathrm{vcd}(X; V; W)$ the VC-density of the definable family, $\{X_w | w \in W\}$, of subsets of V. More generally, if $\phi(\bar{V}; \bar{W})$ is a first-order formula in the theory of some structure M, we set $\mathrm{vcd}(\phi) := \mathrm{vcd}(X; V; W)$, [1] where $X \subset M^{|\bar{V}|} \times M^{|\bar{W}|}$ is the set defined by ϕ.[2] If M is an NIP structure, then it follows from Sauer and Shelah's result mentioned above that $\mathrm{vcd}(\phi) < \infty$.

The problem of obtaining tight bounds on the VC-density for definable families in an NIP structure has attracted a lot of attention recently. The main result of this article is to obtain such tight bounds for definable families over algebraically closed valued fields in the language $\mathcal{L}_{\mathrm{div}}$. We sketch here a brief history of this problem.

For definable families of hypersurfaces in $\mathbb{F}^k$ of fixed degree over a field $\mathbb{F}$, Babai, Ronyai, and Ganapathy ([RBG01]) gave an elegant argument using linear algebra to show that the VC-density of such families is bounded by k. This bound is easily seen to be optimal. Note that the result in [RBG01] is stated in terms of the number of 0/1 patterns (see Notation 2.2 below for a precise definition) realized by a family of n polynomials in $\mathbb{C}^n$, but the VC-density result can be deduced immediately from this bound. A more refined topological estimate on these realized 0/1 patterns (in terms of the sums of the Betti numbers) is given in [BPR09], where the methods are more in line with the methods in the current paper.

More generally, for definable families of semi-algebraic sets in R^k, where R is an arbitrary real closed field, Basu, Pollack and Roy ([BPR05]) showed that the VC-density is bounded by k. For definable families in M^k, where M is an arbitrary o-minimal structure, the first author [Bas10] adapted the methods in [BPR05] to prove a bound of k for the VC-density for such families. These bounds were again obtained as a consequence of more general results bounding the individual Betti numbers of definable sets defined in terms of the members of the family, and more sophisticated homological techniques (as opposed to just linear algebra) played an important role in obtaining these bounds.

If K is an algebraically closed valued field, then the problem of obtaining tight bounds on the VC-density for definable families in K^k for parted formulas in the one sorted language of valued fields with parameters in K was considered by Aschenbrenner et al. in [ADH+16]. In particular, they prove a bound of $2k$ for the

[1] What we call VC-density of ϕ is called VC-codensity in [ADH+16].

[2] Note that we allow formulas ϕ with parameters.

VC-density in the special case when the characteristic pair of K (i.e. the pair consisting of the characteristic of the field K and that of its residue field) is $(0, 0)$ [ADH+16, Corollary 6.3].

Given that the model-theoretic/algebraic techniques used thus far do not immediately yield the tight upper bound of k on the VC-density for valued fields, it is natural to consider a more topological approach as in [Bas10]. However, for definable families over a (complete) valued field, it is not a priori clear that there exists an appropriate well-behaved cohomology theory (i.e. with the required finiteness/cohomological dimension properties) that makes the approach in [Bas10] feasible in this situation. For example, ordinary sheaf cohomology with respect to the Zariski or Étale site for schemes are clearly not suitable. Fortunately, the recent break-through results of Hrushovski and Loeser [HL16] gives us an opening in this direction. Instead of considering the original definable subset of an affine variety V defined over K, we can consider the corresponding *semi-algebraic* subset of the Berkovich analytification V^{an} of V (see Section 4 below for the definitions). These semi-algebraic subsets have certain key topological tameness properties which are analogous to those used in the case of o-minimal structures, and moreover crucially they are homotopy equivalent to a simplicial complex of dimension at most $\dim(V)$. Therefore, their cohomological dimension is at most $\dim(V)$. In particular, the singular cohomology of the underlying topological spaces satisfies the requisite properties. One can thus replace a finite set of definable subsets of V drawn for a given definable family, by the corresponding semi-algebraic subsets of V^{an}.

Using the results of Hrushovski and Loeser, one can then hope to proceed with the o-minimal case as the guiding principle. While the arguments are somewhat similar in spirit, there are several technical challenges that need to be overcome – for example, an appropriate definition of "tubular neighborhoods" with the required properties (see Section 3.1 below for a more detailed description of these challenges). The bounds on the sum of the Betti numbers of the semi-algebraic subsets of Berkovich spaces that we obtain in this way are exactly analogous to the ones in the algebraic, semi-algebraic as well as o-minimal cases. The fact that the cohomological dimension of the semi-algebraic subsets of V^{an} is bounded by $\dim(V)$ is one key ingredient in obtaining these tight bounds.

Our results on bounding the Betti numbers of semi-algebraic subsets of Berkovich spaces are of independent interest, and the aforementioned results seem to suggest a more general formalism of cohomology associated to NIP structures. For example, one obtains bounds (on the Betti numbers) of the exactly same shape and having the same exponents for definable families in the case of algebraic, semi-algebraic, o-minimal and valued field structures. Moreover, in each of these cases, these bounds are obtained as a consequence of general bounds on the dimension of certain cohomology groups. Therefore, it is perhaps reasonable to hope for some general cohomology theory for NIP structures which would in turn give a uniform method of obtaining tight bounds on VC-density via cohomological methods. More generally, it shows that cohomological methods can play an important role in model theory in general.

As a consequence of the bound on the Betti numbers (in fact using the bound only on the 0-th Betti number) we prove that VC-density of definable families over an arbitrary algebraically closed valued field is bounded by k. One consequence of our methods (unlike the techniques used in [ADH+16]) is that there are no restrictions on the characteristic pair of the field K.

Finally note that in loc. cit. the authors also obtain a bound of $2k-1$ on the VC-density of definable families over $\mathbb{Q}_p$ in Macintyre's language. However, our methods right now do not yield results in this case.

Outline of the paper: In section 2, we recall our main results and recast them in the setting of Berkovich analytic spaces. In section 3.1, give an outline of the proofs of the main results. In section 3.2 we use recent results of ([HL16]) to prove the our basic homological results on Betti number bounds for certain semi-algebraic subsets of Berkovich analytic spaces. In section 3.3, we use these homological results to deduce our main result on VC-density. Finally, in the Appendix (section 4) we recall the main results of ([HL16]) used in this paper. We also recall some basic definition in the theory of Berkovich analytic spaces and their relation to the spaces in ([HL16]).

2. Main results

In this paper we improve the bounds proved in ([ADH+16, Theorem 1.2]) for definable families over algebraically closed valued fields. In the following, $\mathcal{L}_{\mathrm{div}}$ will denote the language with signature $(0, 1, +, \times, |)$ (where $x|y$ denotes $|x| \leq |y|$). We have the following theorem.

Theorem 2.1 (VC density for definable families over ACVF)**.** *Let K be an algebraically closed valued field with value group $\mathbb{R}_+$. Let $\phi(\overline{X};\overline{Y})$ be a formula with parameters in K in the language $\mathcal{L}_{\mathrm{div}}$. Then,*

$$\mathrm{vcd}(\phi) \leq |\overline{X}|.$$

Theorem 2.1 will be obtained as a consequence of certain geometric results which could be of independent interest. We introduce some notation in order to state these results.

Suppose V and W are sets, and $X \subset V \times W$ is a subset. Let $\pi_V : X \to V, \pi_W : X \to W$ denote the restriction to X of the natural projection maps. Recall, for any $v \in V, w \in W$, we set $X_v = \pi_W(\pi_V^{-1}(v))$, and $X_w = \pi_V(\pi_W^{-1}(w))$.

Notation 2.2. For each $n > 0$, we define a function

$$\chi_{X;V;W;n} : W^n \times V \to \{0,1\}^n$$

as follows. For $\bar{w} := (w_1, \ldots, w_n) \in W^n$ and $v \in V$, we set

$$\begin{aligned} (\chi_{X;V;W,n}(\bar{w}, v))_i &= 0 \text{ if } v \notin X_{w_i} \\ &= 1 \text{ otherwise.} \end{aligned} \tag{2.3}$$

For $\bar{w} \in W^n$, and $\sigma \in \{0,1\}^n$, we will say that σ is *realized by the tuple of sets* $(X_{w_1}, \ldots, X_{w_n})$ *in* V, if there exists $v \in V$ such that $\chi_{X;V;W,n}(\bar{w}, v)) = \sigma$. We will also often refer to elements of $\{0,1\}^n$ colloquially as $0/1$ patterns. In the following, let $\mathrm{vcd}(X;V;W)$ denote the VC-density of the family $\{X_w | w \in W\}$.

The following proposition is well-known, and we give a proof for ease of exposition. The proposition relates $\mathrm{vcd}(X;V;W)$ with the function

$$\chi_{X;V;W} : \mathbb{N} \to \mathbb{N}$$

defined by setting

$$\chi_{X;V;W}(n) := \max_{\bar{w} \in W^n} \mathrm{card}(\chi_{X;V;W,n}(\bar{w}, V)). \tag{2.4}$$

Proposition 2.5. *Suppose that there exists a constant $C > 0$ such that for all $n > 0$, $\chi_{X;V;W}(n) \le Cn^k$. Then,* $\mathrm{vcd}(X;V;W) \le k$.

Proof. Notice that for $v \in V$ and $w \in W$, $w \in X_v \Leftrightarrow v \in X_w$. Let $\mathcal{S} = \{X_v \mid v \in V\}$, and $A = \{w_1, \ldots, w_n\} \subset W$, and $I \subset [1,n]$. For $v \in V$, $w_i \in X_v$ for all $i \in I$, and $w_i \notin X_v$ for all $i \in [1,n] \setminus I$ if and only if $v \in X_{w_i}$ for all $i \in I$, and $v \notin X_{w_i}$ for all $i \in [1,n] \setminus I$. This implies that

$$\mathrm{card}(\{A \cap Y \mid Y \in \mathcal{S}\}) = \chi_{X;V;W;n}(\bar{w}, V) \le Cn^k.$$

The proposition now follows from Definition 1.1. □

An easy observation that nevertheless plays an important role in deducing our bounds for the VC-density is the following.

Observation 2.6. *With the same notation as above, let V', Y' be such that $V \subset V'$, $Y' \subset V' \times W$ and $Y' \cap (V \times W) = Y$. Then, for every $n > 0$,*

$$\chi_{Y;V;W}(n) \le \chi_{Y';V';W}(n).$$

To see this note that a $0/1$ pattern is realized by the tuple $(Y_{w_1}, \ldots, Y_{w_n})$ in V, only if it is realized by the tuple $(Y'_{w_1}, \ldots, Y'_{w_n})$ in V'. On the other hand, the latter follows from the fact that $Y' \cap (V \times W) = Y$, and therefore for all $w \in W$, $Y'_w \cap V = Y_w$.

In the rest of the paper, we will prove upper bounds on the functions $\chi_{X;V;W}$ for certain definable triples (X, V, W) with $X \subset V \times W$ where V and W are definable in some structure, and use Proposition 2.5 to derive upper bounds on $\mathrm{vcd}(X;V;W)$.

In particular, we shall deduce Theorem 2.1 from a topological result on the Betti numbers having formal similarity with a result proved in [Bas10] for o-minimal structures. In order to state the topological result, we shall introduce some more technical machinery, the details of which are postponed for later.

We begin by briefly recalling the notions of semi-algebraic sets in the setting of Berkovich analytic spaces. Let K be a fixed (real valued) complete non-archimedean field. Given an affine variety V over K, we denote by V^{an} the corresponding Berkovich analytic space. We note that there is a natural inclusion $V \subset V^{\mathrm{an}}$. We refer to section 4 for details.

Let $R := K[X_1, \ldots, X_N]$ and $\mathbb{A}^N_K = \mathrm{Spec}(R)$ denote the corresponding affine space. Given an affine variety $V := \mathrm{Spec}(A) \subset \mathbb{A}^N_K$, a subset $X \subset V^{\mathrm{an}}$ is *semi-algebraic* if it is a Boolean combination[3] of sets of the form

$$\{v \in V^{\mathrm{an}} \mid |F(v)| \leq \lambda |G(v)|\} \tag{2.7}$$

where $F, G \in A$ and $\lambda \in \mathbb{R}_+$. Let ϕ denote the corresponding formula in the two sorted language $\mathcal{L}_{\mathrm{div},\leq}$ (see Section 4.2) with atoms of the form $|F| \leq \lambda |G|$. Moreover, we will say that X has a *closed description* if the Boolean combination defining X does not involve taking complements. We note that the formula defining X also gives rise to a definable subset of $V(K) \times W(K)$ which we denote by $X(K)$.

We let $|X|$ denote the topological space underlying X^{an} and $\mathrm{H}^i(|X|, \mathbb{Q})$ denote the corresponding sheaf cohomology of this topological space. If X is a semi-algebraic subset of V^{an} (with V as above), then a recent deep result of Hrushovski and Loeser ([HL16]) shows that $|X|$ is locally contractible with the homotopy type of a finite simplicial complex. In particular, the cohomology groups above are the usual singular (or simplicial) cohomology groups, and these are all finite dimensional.

Suppose now that V and W are affine varieties and X is a semi-algebraic subset of $V^{\mathrm{an}} \times W^{\mathrm{an}}$ having a closed description. For $\bar{w} \in W^n \hookrightarrow (W^{\mathrm{an}})^n$ a K-rational point and $\sigma \in \{0,1\}^n$, we set

$$\mathcal{R}^{\mathrm{an}}(\sigma, \bar{w}) := (\chi_{X; V^{\mathrm{an}}, W^{\mathrm{an}}}(\bar{w}, \cdot))^{-1}(\sigma).$$

The set $\mathcal{R}^{\mathrm{an}}(\sigma, \bar{w})$ is a semi-algebraic subset of V^{an} defined by the formula

$$\bigwedge_{\sigma(i)=1} \phi(\cdot; w_i) \wedge \bigwedge_{\sigma(i)=0} \neg\phi(\cdot; w_i) \wedge \psi(\cdot),$$

where $\phi(\cdot;\cdot)$ is the formula defining X and $\psi(\cdot)$ is the formula defining V. The aforementioned results of Hrushovski and Loeser allow one to consider the corresponding singular cohomology groups. We let $b_i(\mathcal{R}^{\mathrm{an}}(\sigma, \bar{w}))$ denote the corresponding i-th Betti number.

The following theorem, mirroring a similar theorem in the o-minimal case ([Bas10]), is the main technical result of this paper.

Theorem 2.8. *Let $k := \dim(V)$. There exists a constant $C = C_{X;V;W} > 0$, such that for all K-rational points $\bar{w} \in W^n \hookrightarrow (W^{\mathrm{an}})^n$, and $0 \leq i < k$,*

$$\sum_{\sigma \in \{0,1\}^n} b_i(\mathcal{R}^{\mathrm{an}}(\sigma, \bar{w})) \leq C n^{k-i}.$$

The following corollary of Theorem 2.8 is the main ingredient in the proof of Theorem 2.1.

Corollary 2.9. *There exists a constant $C = C_{X;V;W}$, such that for all $n > 0$,*

$$\chi_{X(K);V(K);W(K)}(n) \leq C n^k,$$

where $k = \dim V$.

Proof. We use the special case Theorem 2.8 by setting $i = 0$. In that case, $b_0(\mathcal{R}^{\mathrm{an}}(\sigma, \bar{w}))$ is the number of connected components, which is at least one as soon as $\mathcal{R}^{\mathrm{an}}(\sigma, \bar{w})$ is non-empty. Now use Observation 2.6 with $V' = V^{\mathrm{an}}$, $Y' = X$ and $Y = X(K)$. □

[3] i.e. It is given by finite unions, intersections, or complements of such sets

We shall show in Section 3.2 below how to deduce Theorem 2.1 from the previous corollary.

3. Proofs of the main results

In this section, we prove our main results. Before starting the formal proof we first give a brief outline of our methods.

3.1. **Outline of the methods used to prove the main theorems.** Our main technical result Theorem 2.8 gives a bound for each $i, 0 \leq i \leq k$, and K-rational points $\bar{w} \in W^n$, on the sum over $\sigma \in \{0,1\}^n$ of the i-th Betti numbers of $\mathcal{R}^{\mathrm{an}}(\sigma, \bar{w})$. The technique for achieving this is an adaptation of the topological methods used to prove a similar result in the o-minimal category in [Bas10] (Theorem 2.1). We recall here the main steps of the proof of Theorem 2.1 in [Bas10]. We assume that $V = \mathrm{R}^k, W = \mathrm{R}^\ell$, where R is a real closed field and $X \subset V \times W$ is a closed definable subset in an o-minimal expansion of R.

Step 1. The first step in the proof is constructing definable infinitesimal tubes around the fibers $X_{w_1}, \ldots, X_{w_n}$

Step 2. Let $\sigma \in \{0,1\}^n$, and C be a connected component of

$$\bigcap_{\sigma(i)=1} X_{w_i} \cap \bigcap_{\sigma(i)=0} (V \setminus X_{w_i}).$$

One proves that there exists a unique connected component D of the complement of the boundaries of the tubes constructed in Step 1 such that C is homotopy equivalent to D. The homotopy equivalence is proved using the local conical structure theorem for o-minimal structures.

Step 3. As a consequence of Step 2, in order to bound $\sum_\sigma b_i(\mathrm{R}(\sigma, \bar{w}))$, it suffices (using Alexander duality) to bound the Betti numbers of the union of the boundaries of the tubes constructed in Step 1.

Step 4. Bounding the Betti numbers of the union of the boundaries of the tubes is achieved using certain inequalities which follow from the Mayer-Vietoris exact sequence. In these inequalities only the Betti numbers of at most k-ary intersections of the boundaries play a role.

Step 5. One then uses Hardt's triviality theorem for o-minimal structures to get a uniform bound on each of these Betti numbers that depends only on the definable family under consideration i.e. on X, V, and W. Thus, the only part of the bound that grows with n comes from certain binomial coefficients counting the number of different possible intersections one needs to consider.

The method we use for proving Theorem 2.8 is close in spirit to the proof of Theorem 2.1 in [Bas10] as outlined above but different in many important details. For each of the steps enumerated above we list the corresponding step in the proof of Theorem 2.8.

Step 1′. We construct again certain tubes around the fibers and give explicit descriptions of the tubes in terms of the formula ϕ defining the given semi-algebraic set X^{an}. The definition of these tubes is somewhat more complicated than in the semi-algebraic case (see Notation 3.3 below). The use of two different infinitesimals to define the tube is necessitated by the singular behavior of the semi-algebraic set defined by $|F| \leq \lambda|G|$ near the common zeros of F and G.

Step 2′. The homotopy equivalence property analogous to Step 2 above is proved in Proposition 3.8, and the role of local conical structure theorem in the o-minimal case is now played by a corresponding result of Hrushovski and Loeser (see Theorem 4.3 below).

Step 3′. We avoid the use of Alexander duality by directly using a Mayer-Vietoris type inequality giving a bound on the Betti numbers of intersections of open sets in terms of the Betti numbers of up to k-fold unions (cf. Proposition 3.32).

Step 4′. This step is subsumed by Step 3′.

Step 5′. Finally, instead of using Hardt triviality to obtain a constant bound on the Betti numbers of these "small" unions, we use a theorem of Hrushovski and Loeser (see Theorem 4.4) which states that the number of homotopy types amongst the fibers of any fixed map in the analytic category that we consider is finite (cf. Theorem 4.4 below).

We apply Theorem 2.8 directly to obtain the VC-density bound in the case of the theory of ACVF (using Observation 2.6). One extra subtlety here is in removing the assumption on the formula ϕ (which occurs in the hypothesis of Theorem 2.8), that it defines a closed semi-algebraic set. Actually, in order to prove Theorem 2.1 in general it suffices only to consider ϕ of the special form having just one atom of the form $|F| \leq \lambda|G|$ or $|F| = \lambda|G|$. This reduction from the general case to the special case is encapsulated in a combinatorial result (Proposition 3.36). With the help of Proposition 3.36, Theorem 2.1 becomes a consequence of Theorem 2.8 and Observation 2.6.

We now give the proofs in full detail. The next subsection below gives a proof of Theorem 2.8. In the following subsection, we show how to deduce Theorem 2.1 from Theorem 2.8 and Corollary 2.9.

3.2. **Proof of Theorem 2.8.** In the following, K will be a fixed non-archimedean (complete real-valued) field and V is an affine variety over K. We shall freely use the results of Hrushovski and Loeser ([HL16]) and their relation to Berkovich analytic spaces. For the reader's convenience, an exposition (with references) of the results we require below is provided in the Appendix.

Definition 3.1. (Semi-algebraic sets with closed descriptions) Let $V = \mathrm{Spec}(A)$ be an affine variety and V^{an} the corresponding analytic space. We say that a semi-algebraic subset $X \subset V^{\mathrm{an}}$ has a closed description if X is defined by a negation-free Boolean formula ϕ with atoms $|F_i| \leq \lambda_i|G_i|$, where $F_i, G_i \in A$ and $\lambda \in \mathbb{R}_{\geq 0}$. In this case, X is a closed semi-algebraic subset of V^{an}.

Notation 3.2. (closed cube) Suppose $\mathbb{A}^\ell = \mathrm{Spec}(K[X_1, \ldots, X_\ell])$. For a real number $R > 0$, we denote by $B_\ell(0, R)$ the closed semi-algebraic subset of $\mathbb{A}_K^{\ell,\mathrm{an}}$ defined by the formula

$$\bigwedge_{1 \leq i \leq \ell} |X_i| \leq R.$$

Notice that $B_\ell(0, R)$ is a compact topological space. If $V = \mathrm{Spec}(A) \subset \mathbb{A}^\ell$ is an affine scheme, then we set $B_V(0, R) := B_\ell(0, R) \cap V^{\mathrm{an}}$. Note that this a closed semi-algebraic subset of V^{an}.

Notation 3.3. (Open, closed $(\varepsilon,\varepsilon')$-tubes) Suppose that $Z \subset V^{\mathrm{an}} \subset \mathbb{A}^{\ell,\mathrm{an}}$ is a semi-algebraic subset of V^{an} ($V = \mathrm{Spec}(A)$) having a closed description. Let Z be defined by a negation-free Boolean formula ϕ with atoms $|F_i| \le \lambda_i |G_i|$, where $F_i, G_i \in A$ and $\lambda \in \mathbb{R}_{\ge 0}$. For $\varepsilon > 1, \varepsilon' > 0$, we will denote by $Z^+_{\phi,\varepsilon,\varepsilon'}$ the semi-algebraic subset of V^{an} defined by the Boolean formula $\phi^+_{\varepsilon,\varepsilon'}$ obtained from ϕ, by replacing each atom $|F_i| \le \lambda_i|G_i|$ by the formula $(|F_i| < (\lambda_i\varepsilon)|G_i|) \vee ((|F_i| < \varepsilon') \wedge (|G_i| < \varepsilon'))$. For each $R > 0$, we set $Z^+_{\phi,\varepsilon,\varepsilon',R} := B_V(0,R) \cap Z^+_{\phi,\varepsilon,\varepsilon'}$.

We will denote by $\overline{Z^+_{\phi,\varepsilon,\varepsilon'}}$ the semi-algebraic subset of V^{an} defined by the Boolean formula $\overline{\phi^+_{\varepsilon,\varepsilon'}}$ obtained from ϕ, by replacing each atom $|F_i| \le \lambda_i|G_i|$ by the formula $(|F_i| \le (\lambda_i\varepsilon)|G_i|) \vee ((|F_i| \le \varepsilon') \wedge (|G_i| \le \varepsilon'))$. For each $R > 0$, we set $\overline{Z^+_{\phi,\varepsilon,\varepsilon',R}} := B_V(0,R) \cap \overline{Z^+_{\phi,\varepsilon,\varepsilon'}}$.

We denote by $Z^-_{\phi,\varepsilon,\varepsilon',R}$ the complement in $B_V(0,R)$ of $Z^+_{\phi,\varepsilon,\varepsilon',R}$ Notice that by definition, $Z^+_{\phi,\varepsilon,\varepsilon',R}$ (resp. $Z^-_{\phi,\varepsilon,\varepsilon',R}$) is an open (resp. closed) subset of $B_V(0,R)$. Moreover, both of these are semi-algebraic as subsets of V^{an}. Finally, we will denote by $Z^0_{\phi,\varepsilon,\varepsilon',R} = \overline{Z^+_{\phi,\varepsilon,\varepsilon',R}} \cap Z^-_{\phi,\varepsilon,\varepsilon',R}$.

The next proposition is the key ingredient for the proof of Theorem 2.8. Let $X \subset V^{\mathrm{an}} \times W^{\mathrm{an}} \subset \mathbb{A}^{\ell,\mathrm{an}}$ be a semi-algebraic set with a closed description. Recall, for a K-rational point $\bar{w} \in W^n \hookrightarrow (W^{\mathrm{an}})^n$ and $\sigma \in \{0,1\}^n$, we defined

$$\mathcal{R}^{\mathrm{an}}(\sigma,\bar{w}) := \chi_n^{\mathrm{an}}(-,\bar{w})^{-1}(\sigma),$$

where

$$\chi_n : V^{\mathrm{an}} \times (W^{\mathrm{an}})^n \to \{0,1\}^n,$$

was defined by setting $(\chi_n(v,\bar{w}))_i = 0$ if $v \notin X_{w_i}$ and 1 otherwise. Note that $\mathcal{R}^{\mathrm{an}}(\sigma,\bar{w}) \subset V^{\mathrm{an}}$ is a semi-algebraic subset.

Remark 3.4. In the following we will often use the phrase "for all sufficiently small $\varepsilon' > 0$, $\varepsilon > 1$, $\delta' > 0$, $\delta > 1$". What we mean is that "there exists an $\varepsilon'_0 > 0$ such for all $\varepsilon' \in (0,\varepsilon'_0]$, there exists an $\varepsilon_0 > 1$, such that for all $\varepsilon \in (1,\varepsilon_0]$, and so on ...". We will use the phrase "for all sufficiently small $\varepsilon' > 0$, $\varepsilon > 1$, $\delta' > 0$, $\delta > 1$", and similar ones without remark from now on.

Remark 3.5. Below, given a topological space P, $\mathrm{H}^i(P)$ will denote the sheaf cohomology with rational coefficients. We only deal with spaces P that deformation retract onto a finite simplicial complex. These are then the usual singular or simplicial cohomology groups of the underlying topological space. Moreover, they will be finite dimensional $\mathbb{Q}$-vector spaces.

Remark 3.6. In ([HL16]), Hrushovski and Loeser work with the two sorted structure $(K,\mathbb{R}_+)$, and associate to any quasi-projective variety V the set $B_{\mathbf{F}}(V)$ of $\mathbf{F}$-definable types $B_{\mathbf{F}}(V)$ nearly orthogonal to $\mathbb{R}_+$. These have a natural topology, and are canonically homeomorphic to V^{an}. We refer to the Appendix for details.

Remark 3.7. More generally, to any $\mathbf{F}$-definable subset $X \subset V$, they associate the set $B_{\mathbf{F}}(X)$ of $\mathbf{F}$-definable types on X which are nearly orthogonal to $\mathbb{R}_+$. These are defined to be the *semi-algebraic* subsets of $B_{\mathbf{F}}(V)$. These are exactly the semi-algebraic subsets as defined in this section ([HL16], Chapter 14.1, pg. 187). We refer to the Appendix for details.

Proposition 3.8. *For each K-rational point $\bar{w} \in W^n \hookrightarrow (W^{\mathrm{an}})^n$, $\sigma \in \{0,1\}^n$, for all sufficiently large $R > 0$ and all sufficiently small $\varepsilon' > 0$, $\varepsilon > 1$, $\delta' > 0$, $\delta > 1$, one has an isomorphism*

$$\mathrm{H}^*(\mathcal{R}^{\mathrm{an}}(\sigma, \bar{w})) \cong \mathrm{H}^*(S_{\sigma,\delta,\delta',\varepsilon,\varepsilon',R}),$$

where $S_{\sigma,\delta,\delta'\varepsilon,\varepsilon',R}$ is defined by

$$S_{\sigma,\delta,\delta',\varepsilon,\varepsilon',R} := \bigcap_{i,\sigma(i)=1} ((X^+_{\phi,\delta,\delta',R})_{w_i}) \cap \bigcap_{i,\sigma(i)=0} ((X^-_{\phi,\varepsilon,\varepsilon',R})_{w_i}).$$

Remark 3.9. Note that $(X^+_{\phi,\delta,\delta',R})_{w_i}, (X^-_{\phi,\varepsilon,\varepsilon',R})_{w_i} \subset V^{\mathrm{an}}$ are semi-algebraic subsets.

Lemma 3.10. *With notation as in Proposition 3.8:*

1. *There exists a $\delta_0 > 0$ such that for all $1 < t_1 \leq t_2 \leq \delta_0$, the inclusion map $S_{\sigma,t_1,\delta',\varepsilon,\varepsilon',R} \hookrightarrow S_{\sigma,t_2,\delta',\varepsilon,\varepsilon',R}$ is a homotopy equivalence. Here δ_0 depends on $\delta', \varepsilon, \varepsilon', R$.*
2. *There exists $\delta'_0 > 0$ such that for all $0 < t'_1 \leq t'_2 \leq \delta'_0$, the inclusion map $\bigcap_{t>1} S_{\sigma,t,t'_1,\varepsilon,\varepsilon',R} \hookrightarrow \bigcap_{t>1} S_{\sigma,t,t'_2,\varepsilon,\varepsilon',R}$ is a homotopy equivalence. Here δ'_0 depends on $\varepsilon, \varepsilon', R$.*
3. *Let $S'_{\sigma,\varepsilon,\varepsilon',R} := \bigcap_{t>1} \bigcap_{t'>0} S_{\sigma,t,t',\varepsilon,\varepsilon',R}$. There exists $\varepsilon_0 > 0$ such that for all $1 < s_1 \leq s_2 \leq \varepsilon_0$, then natural inclusion $S'_{\sigma,s_2,\varepsilon',R} \hookrightarrow S'_{\sigma,s_1,\varepsilon',R}$ is a homotopy equivalence. Here ε_0 depends on ε', R.*
4. *There exists $\varepsilon'_0 > 0$ such that for all $0 < s'_1 \leq s'_2 \leq \varepsilon'_0$, then natural inclusion $\bigcup_{s>1} S'_{\sigma,s,s'_2,R} \hookrightarrow \bigcup_{s>1} S'_{\sigma,s,s'_1,R}$ is a homotopy equivalence. Here ε'_0 depends on R.*
5. *One has*
$$\mathcal{R}^{\mathrm{an}}(\sigma, \bar{w}) \cap B_V(0, R) = \bigcup_{s>1,s'>0} S'_{\sigma,s,s',R},$$
6. *There exists $R_0 > 0$, such that for all $R > R_0$, the natural inclusion*
$$\mathcal{R}^{\mathrm{an}}(\sigma, \bar{w}) \cap B_V(0, R) \hookrightarrow \mathcal{R}^{\mathrm{an}}(\sigma, \bar{w})$$
is a homotopy equivalence.

Proof. We prove each part separately below.

Proof of Part (1). Let $\mathbb{X} \subset V^{\mathrm{an}}$ be the union of the sets $S_{\sigma,t,\delta'\varepsilon,\varepsilon',R}$ for $t \geq 1$. The subset $\mathbb{X}$ is easily seen to be semi-algebraic since taking the union over t is the same as eliminating an existential quantifier (of the value sort). Consider now the definable map $G : \mathbb{X} \to \mathbb{R}_+$ given by setting $G(x) := \inf\{t | x \in S_{\sigma,t,\delta',\varepsilon,\varepsilon',R}\}$. Since $S_{\sigma,t,\delta',\varepsilon,\varepsilon',R}$ is a non-decreasing sequence in the parameter t, $G^{-1}([t, \infty)) = S_{\sigma,t,\delta',\varepsilon,\varepsilon',R}$ and an application of Theorem 4.3 gives the desired result. □

Proof of Part (2). A similar argument as in the proof of Part 1 proves this part, noting that $\bigcap_{t>1} S_{\sigma,t,t',\varepsilon,\varepsilon',R}$ is a definable family and non-decreasing in t'. □

Proof of Part (3). The proof is similar to that of Part 1. First, note that $S''_{\sigma,\varepsilon,\varepsilon',R}$ is semi-algebraic. In particular, the union over $s > 1$ of $S''_{\sigma,s,\varepsilon',R} \subset V^{\mathrm{an}}$ is a semi-algebraic subset of V^{an}. Let $\mathbb{X}$ denote this union. Let $G : \mathbb{X} \to \mathbb{R}_+$ be given by $G(x) = \sup\{t | x \in S''_{\sigma,s,\varepsilon',R}\}$. One can now apply Theorem 4.3 as before. □

Proof of Part (4). The proof is similar to that of Part 3. □

Proof of Part (5). One can reduce to the case $n = 1$. If $\sigma = 0$, then both sides are empty. If $\sigma = 1$, then the left side is $X_w \cap B(0, R)$. One can check that this is exactly the right hand side in this case. $\square$

Proof of Part (6). One can prove this using a similar argument as in Part 3. $\square$

$\square$

We also recall some continuity properties of Čech cohomology which will be used in the proof of Proposition 3.8 below.

Lemma 3.11. *Let X be a paracompact Hausdorff space, having the homotopy type of a finite simplicial complex. Below, we let H^i denote the Čech Cohomology (with rational coefficients) of the space X.*

1. *Let $\{U_i\}_{i>0}$ be a decreasing sequence of open subspaces of X, and $S := \bigcap_i U_i$. Suppose that the family U_i is cofinal in the family of open neighborhoods of S in X. Then the natural map*
$$\mathrm{H}^k(S) \to \varinjlim \mathrm{H}^k(U_i)$$
is an isomorphism.
2. *Let $\{C_i\}$ be a decreasing sequence of compact subspaces of X, and $S := \bigcap_i C_i$. Then the natural map*
$$\mathrm{H}^k(S) \to \varinjlim \mathrm{H}^k(C_i)$$
is an isomorphism.
3. *Let $\{C_i\}_{i>0}$ be an increasing sequence of compact subspaces of S, and $S := \bigcup_i C_i$. Suppose that the family C_i is cofinal in the family of compact subspaces of S. Then the natural map*
$$\mathrm{H}^k(S) \to \varprojlim \mathrm{H}^k(C_i)$$
is an isomorphism.

Proof of Part (1). This is Theorem 5 in [LR68]. $\square$

Proof of Part (2). Follows from [Ive86, page 195, Lemma 9.3]. $\square$

Proof of Part (3). First observe that in our situation, singular cohomology is isomorphic to Čech cohomology. The statement now follows from the fact that singular homology of any space is isomorphic to the direct limit of the singular homology of its compact subspaces [Spa66, Theorem 4.4.6], and the fact that the singular cohomology group $\mathrm{H}^*(S, \mathbb{Q})$ is canonically isomorphic to $\mathrm{Hom}(\mathrm{H}_*(S, \mathbb{Q}), \mathbb{Q})$, since $\mathbb{Q}$ is a field. $\square$

Proof of Proposition 3.8. The proof is somewhat long and proceeds through several intermediate claims.

Claim 1. *The natural inclusions*
$$S'_{\sigma,\varepsilon,\varepsilon',R} = \bigcap_{t'>0} \bigcap_{t>1} S_{\sigma,t,t',\varepsilon,\varepsilon',R} \hookrightarrow \bigcap_{t>1} S_{\sigma,t,t',\varepsilon,\varepsilon',R} \hookrightarrow S_{\sigma,t,t',\varepsilon,\varepsilon',R}$$
induce an isomorphism
$$\mathrm{H}^*(S'_{\sigma,\varepsilon,\varepsilon',R}) \cong \varinjlim_{t'\to 0} \varinjlim_{t\to 1} \mathrm{H}^*(S_{\sigma,t,t',\varepsilon,\varepsilon',R}). \tag{3.12}$$

Proof of Claim 1. We will prove it in two steps. First consider the inclusion

$$\bigcap_{t>1} S_{\sigma,t,t',\varepsilon,\varepsilon',R} \hookrightarrow S_{\sigma,t,t',\varepsilon,\varepsilon',R}.$$

We prove that this inclusion induces an isomorphism

$$\mathrm{H}^*(\bigcap_{t>1} S_{\sigma,t,t',\varepsilon,\varepsilon',R}) \cong \varinjlim_{t\to 1} \mathrm{H}^*(S_{\sigma,t,t',\varepsilon,\varepsilon',R}). \tag{3.13}$$

Let $T = \bigcap_{i,\sigma(i)=0}((X^-_{\phi,R,\varepsilon,\varepsilon'})_{w_i})$. Then T is a closed subspace of a paracompact Hausdorff space and hence also paracompact and Hausdorff. In fact, since each $(X^-_{\phi,R,\varepsilon,\varepsilon'})_{w_i}$ is compact, T is a compact Hausdorff space. Notice that for each $t > 1, t' > 0$, $S_{\sigma,t,t',\varepsilon,\varepsilon',R} \subset T$, and hence $\bigcap_{t>1} S_{\sigma,t,t',\varepsilon,\varepsilon',R} \subset T$ as well. We will now show that the sequence $S_{\sigma,t,t',\varepsilon,\varepsilon',R}$ is a cofinal system of open neighborhoods of $\bigcap_{t>1} S'_{\sigma,t,t',\varepsilon,\varepsilon',R}$ in T. Assuming this fact, the result follows from Part (1) of Lemma 3.11.

We first assume that the formula ϕ is a conjunction of weak inequalities $|F_j| \leq \lambda_j|G_j|, j \in J$, where J is a finite set. For each i, we set $F_{ij} := F_j(\cdot, w_i)$ and $G_{ij} := G_j(\cdot, w_i)$. Let U be an open neighborhood of $\bigcap_{t>1} S_{\sigma,t,t',\varepsilon,\varepsilon',R}$ in T, and let U^c denote $T \setminus U$. Since T is compact, U^c is compact. We will show that there is a t_0 such that $S_{\sigma,R,t_0,t',\varepsilon,\varepsilon'} \subset U$.
For each i with $\sigma(i) = 1$, $j \in J$, let U^c_{ij} denote the intersection of U^c with the semi-algebraic set defined by $(|F_{ij}| \geq \lambda t|G_{ij}|) \wedge ((|F_{ij}| \geq t') \vee (|G_{ij}| \geq t'))$. Then, $U^c = \bigcup_{i,\sigma(i)=1,j\in J} U^c_{ij}$, and each U^c_{ij} is compact.

For each i with $\sigma(i) = 1$, $j \in J$ consider the continuous function $\theta_{ij} : U^c_{ij} \to \mathbb{R}_+ \cup \{\infty\}$ defined by

$$\theta_{ij}(x) \quad = \quad \frac{|F_{ij}(x)|}{\lambda_j|G_{ij}(x)|}. \tag{3.14}$$

Notice that the numerator and denominator of the right hand side of (3.14) cannot vanish simultaneously for $x \in U^c_{ij}$. Also, notice that $\theta_{ij}(x) > 0$ for all $x \in U^c_{ij}$, since $\theta_{ij}(x) = 0$ implies that $|F_{ij}(x)| \leq \lambda_j|G_{ij}(x)|$. The function θ_{ij} is continuous on the compact set U^c_{ij} (since $|\cdot|$ is continuous), and thus attains its minimum (say θ^0_{ij}). Note that $\theta^0_{ij} > 0$. Let $t_0 = \frac{1}{2}\min_{i,\sigma(i)=1,j\in J} \theta^0_{ij}$. We claim that $S_{\sigma,t_0,t',\varepsilon,\varepsilon',R} \subset U$. This is equivalent to showing that U^c is contained in $S^c_{\sigma,t_0,t',\varepsilon,\varepsilon',R} = T \setminus S_{\sigma,t_0,t',\varepsilon,\varepsilon',R}$. Let $x \in U^c_{ij}$. Then, x belongs to the semi-algebraic set defined by $((|F_{ij}| \geq t') \vee (|G_{ij}| \geq t'))$. Then, since $\theta_{ij}(x) \geq t_0$, $|F_{ij}(x)|\lambda_j^{-1}|G_{ij}(x)|^{-1} \geq \theta^0_{ij} > t_0$, which implies that $|F_{ij}(x)| > (\lambda_j t_0)|G_{ij}(x)|$. But then x does not belong to the semi-algebraic set defined by $(|F_{ij}(x)| < (\lambda_i t_0)|G_{ij}(x)|) \vee ((|F_{ij}| \leq t') \wedge (|G_{ij}| \leq t'))$, and so $x \notin S_{\sigma,R,t_0,t',\varepsilon,\varepsilon'}$. This proves the cofinality of the sequence $S_{\sigma,t,t'\varepsilon,\varepsilon',R}$ in the special case when ϕ is a conjunction of weak inequalities.

In general ϕ will be a disjunction of conjunctions. Let $\phi = \bigvee_{h\in H} \phi^{(h)}$, where each $\phi^{(h)}$ is a conjunction of weak inequalities, $|F_{jh}| \leq \lambda_{jh}|G_{jh}|$, $j \in J_h$, and H, J_h are finite sets. Let $I_\sigma = \{i \in [1, n] \mid \sigma_i = 1\}$. Then,

$$S_{\sigma,t,t',\varepsilon,\varepsilon',R} = \bigcup_{\psi\in H^{I_\sigma}} S^{(\psi)}_{\sigma,t,t',\varepsilon,\varepsilon',R},$$

where

$$S^{(\psi)}_{\sigma,t,t',\varepsilon,\varepsilon',R} = \bigcap_{i,\sigma_i=1} ((X^{+}_{\phi(\psi(i)),t,t',R})_{w_i}) \cap T.$$

An open neighborhood U of $\bigcap_{t>1} S_{\sigma,t,t'\varepsilon,\varepsilon',R}$ in T is clearly also an open neighborhood of $\bigcap_{t>1} S^{(\psi)}_{\sigma,t,t'\varepsilon,\varepsilon',R}$ for each $\psi \in H^{I_\sigma}$. Using what has already been proved for conjunctive formulas, for each $\psi \in H^{I_\sigma}$, there exists $t_0^{(\psi)}$, such that

$$S^{(\psi)}_{\sigma,t_0^{(\psi)},t',\varepsilon,\varepsilon',R} \subset U.$$

Now take $t_0 = \min_{\psi\in H^{I_\sigma}} t_0^{(\psi)}$. Then, $S_{\sigma,t_0,t',\varepsilon,\varepsilon',R} = \bigcup_{\psi\in H^{I_\sigma}} S^{(\psi)}_{\sigma,t_0,t',\varepsilon,\varepsilon',R} \subset U$. This proves (3.13).

Notice that $(\bigcap_{t>1} S_{\sigma,t,t',\varepsilon,\varepsilon',R})_{t'>0}$ is a decreasing sequence of compact semi-algebraic sets. The proof of the fact that

$$\mathrm{H}^*(\bigcap_{t'>0}\bigcap_{t>1} S_{\sigma,t,t',\varepsilon,\varepsilon',R}) \cong \varinjlim_{t'\to 0} \mathrm{H}^*(\bigcap_{t>1} S_{\sigma,t,t',\varepsilon,\varepsilon',R}) \tag{3.15}$$

is now very similar to that of (3.13) and omitted.
Eqns. (3.13) and (3.15) together imply (3.12). This finishes the proof of Claim 1. □

Claim 2. *For every fixed $\varepsilon > 1, \varepsilon' > 0$, there exists $\delta' > 1$ sufficiently small, and for each δ', there exists $\delta > 0$ sufficiently small, such that*

$$\mathrm{H}^*(S'_{\sigma,\varepsilon,\varepsilon',R}) \cong \mathrm{H}^*(S_{\sigma,\delta,\delta',\varepsilon,\varepsilon',R}). \tag{3.16}$$

Proof of Claim 2. It follows from Lemma 3.10 (Part (1)) that there exists t_0' such that for all $0 < t_2' \le t_1' \le t_0'$, the inclusion map $\bigcap_{t>1} S_{\sigma,t,t_2',\varepsilon,\varepsilon',R} \hookrightarrow \bigcap_{t>1} S_{\sigma,t,t_1',\varepsilon,\varepsilon',R}$ induces an isomorphism

$$\mathrm{H}^*(\bigcap_{t>1} S_{\sigma,t,t_2',\varepsilon,\varepsilon',R}) \to \mathrm{H}^*(\bigcap_{t>1} S_{\sigma,t,t_1',\varepsilon,\varepsilon',R}),$$

which implies that

$$\varinjlim_{t'\to 0} \mathrm{H}^*(\bigcap_{t>1} S_{\sigma,t,t'\varepsilon,\varepsilon',R}) \cong \mathrm{H}^*(\bigcap_{t>1} S_{\sigma,t,t_0',\varepsilon,\varepsilon',R}) \cong \varinjlim_{t\to 1} \mathrm{H}^*(S_{t,t_0',\varepsilon,\varepsilon',R}), \tag{3.17}$$

where the first isomorphism is a consequence of Part (2) of Lemma 3.11, and the last isomorphism follows from (3.12).

It again follows from Lemma 3.10 (Part (2)) that for each fixed t', and hence for $t' = t_0'$, there exists t_0 such that for all $1 < t_2 \le t_1 \le t_0$ (which depends on t_0'), the inclusion map $S_{\sigma,t_2,t_0',\varepsilon,\varepsilon',R} \hookrightarrow S_{\sigma,t_1,t_0',\varepsilon,\varepsilon',R}$ induces an isomorphism

$$\mathrm{H}^*(S_{\sigma,t_2,t_0',\varepsilon,\varepsilon',R}) \to \mathrm{H}^*(S_{\sigma,t_1,t_1',\varepsilon,\varepsilon',R}),$$

which implies that

$$\varinjlim_{t\to 1} \mathrm{H}^*(S_{\sigma,t,t_0'\varepsilon,\varepsilon',R}) \cong \mathrm{H}^*(S_{\sigma,t_0,t_0',\varepsilon,\varepsilon',R}). \tag{3.18}$$

Eqns. (3.12), (3.17) and (3.18) together now imply that

$$\mathrm{H}^*(S'_{\sigma,\varepsilon,\varepsilon',R}) \cong \mathrm{H}^*(S_{\sigma,t_0,t_0',\varepsilon,\varepsilon',R}). \tag{3.19}$$

This proves Claim 2 after taking $\delta' = t_0'$ and $\delta = t_0$. □

Claim 3. *One has a natural isomorphism*

$$\mathrm{H}^*(\mathcal{R}^{\mathrm{an}}(\sigma,\bar{w}))\cap B_V(0,R))\cong \varprojlim_{s'\to 0}\varprojlim_{s\to 1}\mathrm{H}^*(S'_{\sigma,s,s',R}). \tag{3.20}$$

Proof of Claim 3. As in the proof of Claim 1 we will prove this claim in two steps. We will first prove that for every fixed $s'>0$, there exists a natural isomorphism,

$$\mathrm{H}^*(\bigcup_{s>1} S'_{\sigma,s,s',R})\cong \varprojlim_{s\to 1}\mathrm{H}^*(S'_{\sigma,s,s',R}). \tag{3.21}$$

First, note that each $S'_{\sigma,s,s',R}$ is a compact subset of $B_V(0,R)$. Then 3.21 will follow from from Part (3) of Lemma 3.11, and Part (4) of Lemma 3.10, once we know that the family $(S'_{\sigma,s,s',R})_{s>1}$ is cofinal in the family of compact subspaces of $\bigcup_{s>1} S'_{\sigma,s,s',R}$. We now prove that the sequence $(S'_{\sigma,s,s',R})_{s>1}$ is cofinal in the family of compact subspaces of $\bigcup_{s>1} S'_{\sigma,s,s',R}$.

We first prove this in the case ϕ is a disjunction of weak inequalities $|F_j|\leq \lambda_j|G_j|, j\in J$. Using the same notation as in Claim 1 we will denote for each i, $F_j(\cdot,w_i)$ by F_{ij} and $G_j(\cdot,w_i)$ by G_{ij}.

It follows from the definitions that $S'_{\sigma,s,s',R}$ is equal to the intersection of the subsets of T defined by $(|F_{ij}|\geq s\lambda_j|G_{ij}|)\wedge((|F_{ij}|\geq s')\vee(|G_{ij}|\geq s')), 1\leq i\leq n, j\in J$, where $T=\bigcap_{t'>0}\bigcap_{t>1}(\bigcap_{i,\sigma(i)=1}((X^+_{\phi,t,t',R})_{w_i})$.

Let C be a compact subset of $\bigcup_{s>1} S'_{\sigma,s,s',R}$. Consider the function $\theta: C\to \mathbb{R}_+\cup\{\infty\}$ defined by setting

$$\theta(x):=\min_{i,\sigma(i)=0,j\in J}\frac{|F_{ij}(x)|}{\lambda_j|G_{ij}(x)|}.$$

Then θ is continuous (since $|\cdot|$ is continuous and $|F_{ij}|$ and $|G_{ij}|$ never vanish simultaneously) and $\theta(x)>1$ for all $x\in C$. Since C is compact, θ achieves its minimum say s_0. Then, $C\subset S'_{\sigma,s_0,s',R}$, proving cofinality of the family $(S'_{\sigma,s,s',R})_{s>1}$ in the family of compact subspaces of $\bigcup_{s>1} S'_{\sigma,s,s',R}$.

The case for a general ϕ can be deduced from the above case in a similar fashion as in Claim 1. This finishes the proof of (3.21).

We now prove that

$$\varprojlim_{s'\to 0}\mathrm{H}^*(\bigcup_{s>1} S'_{\sigma,s,s',R})\cong \mathrm{H}^*(\bigcup_{s'>0}\bigcup_{s>1} S'_{\sigma,s,s',R}). \tag{3.22}$$

The isomorphism in (3.22) would follow from Part (3) of Lemma 3.11 if we prove the cofinality of the sequence $(\bigcup_{s>1} S'_{\sigma,s,s',R})_{s'>0}$ in the family of compact subsets of $\bigcup_{s'>0}\bigcup_{s>1} S'_{\sigma,s,s',R}$.

We first prove this again in the case ϕ is a disjunction of weak inequalities $|F_j|\leq \lambda_j|G_j|, j\in J$. Let C be a compact subset of $\bigcup_{s'>0}\bigcup_{s>1} S'_{\sigma,s,s',R}$. Consider the function $\theta: C\to\mathbb{R}$ defined by setting

$$\theta(x):=\min_{i,\sigma_i=0,j\in J}\max(|F_{ij}(x)|,|G_{ij}(x)|).$$

Then, θ is continuous (since $|\cdot|$ is continuous) and $\theta(x) > 0$ for all $x \in C$. Since C is compact, θ achieves its minimum say s'_0. Then, $C \subset \bigcup_{s>1} S'_{\sigma,s,s'_0,R}$, proving cofinality of the family $(\bigcup_{s>1} S'_{\sigma,s,s',R})_{s'>0}$ in the family of compact subspaces of $\bigcup_{s'>0} \bigcup_{s>1} S'_{\sigma,s,s',R}$.

The case for a general ϕ can be deduced from the above case in a similar fashion as in Claim 1. This finishes the proof of (3.22).

It now follows from (3.21) and (3.22) that

$$\mathrm{H}^*(\bigcup_{s'>0} \bigcup_{s>1} S'_{\sigma,s,s',R}) \cong \varprojlim_{s'\to 0} \varprojlim_{s\to 1} \mathrm{H}^*(S'_{\sigma,s,s',R}). \tag{3.23}$$

Moreover, by Lemma 3.10 (Part (5)), $\bigcup_{s'>0} \bigcup_{s>1} S'_{\sigma,s,s',R} = \mathcal{R}^{\mathrm{an}}(\sigma,\bar{w}) \cap B_V(0,R)$. This finishes the proof of Claim 3. $\square$

Claim 4. *There exists $\varepsilon' > 0$ sufficiently small, and for each such $\varepsilon' > 0$, there exists $\varepsilon > 1$ sufficiently small, such that*

$$\mathrm{H}^*(\mathcal{R}^{\mathrm{an}}(\sigma,\bar{w}) \cap B_V(0,R)) \cong \mathrm{H}^*(S'_{\sigma,\varepsilon,\varepsilon',R}). \tag{3.24}$$

Proof of Claim 4. It follows from Lemma 3.10 (Part (3)) that there exists s'_0 such that for all $0 < s'_2 \le s'_1 \le s'_0$, the inclusion map $\bigcup_{s>1} S'_{\sigma,s,s'_1,R} \hookrightarrow \bigcup_{s>1} S'_{\sigma,s,s'_2,R}$ induces an isomorphism

$$\mathrm{H}^*(\bigcup_{s>1} S'_{\sigma,s,s'_1,R}) \to \mathrm{H}^*(\bigcup_{s>1} S'_{\sigma,s,s'_2,R}).$$

It follows that

$$\varprojlim_{s'\to 0} \mathrm{H}^*(\bigcup_{s>1} S'_{\sigma,s,s',R}) \cong \mathrm{H}^*(\bigcup_{s>1} S'_{\sigma,s,s'_0,R}). \tag{3.25}$$

It follows from Lemma 3.10 (Part (4)) that for each fixed s', and hence for $s' = s'_0$, there exists s_0 such that for all $1 < s_2 \le s_1 \le s_0$, the inclusion map $S'_{\sigma,s_1,s'_0,R} \hookrightarrow S'_{\sigma,s_2,s'_0,R}$ induces an isomorphism

$$\mathrm{H}^*(S'_{\sigma,s_1,s'_0,R}) \to \mathrm{H}^*(S'_{\sigma,s_2,s'_0,R}),$$

which implies that

$$\varprojlim_{s\to 1} \mathrm{H}^*(S'_{\sigma,s,s'_0,R}) \cong \mathrm{H}^*(S'_{\sigma,s_0,s'_0,R}). \tag{3.26}$$

Eqns. (3.20), (3.21) and (3.22) together now imply that

$$\mathrm{H}^*(\mathcal{R}^{\mathrm{an}}(\sigma,\bar{w}) \cap B_V(0,R)) \cong \mathrm{H}^*(S'_{\sigma,s_0,s'_0,R}). \tag{3.27}$$

Claim 4 follows by choosing $\varepsilon = s_0$ and $\varepsilon' = s'_0$. This ends the proof of Claim 4. $\square$

Finally, by Lemma 3.10 (Part (6)), for R large enough one has

$$\mathrm{H}^*(\mathcal{R}^{\mathrm{an}}(\sigma,\bar{w}) \cap B_V(0,R)) \cong \mathrm{H}^*(\mathcal{R}^{\mathrm{an}}(\sigma,\bar{w})). \tag{3.28}$$

The proposition now follows from the isomorphisms (3.16), (3.24) and (3.28) and Lemma 3.10 (Part (5)). $\square$

Proposition 3.29. *For $R > 0$ and each K-rational point $\bar{w} \in W^n \hookrightarrow (W^{\mathrm{an}})^n$, $\sigma \in \{0,1\}^n$, there exists $\delta > 1, \delta' > 0, \varepsilon > 1, \varepsilon' > 1$, such that for each connected component C of $S_{\sigma,\delta,\delta',\varepsilon,\varepsilon',R}$, there exists a unique connected component D of the semi-algebraic set*

$$U_{\phi,\delta,\delta',\varepsilon,\varepsilon',R} = \bigcap_{1 \le i \le n} (U_{i,\varepsilon,\varepsilon',R} \cap U_{i,\delta,\delta',R}), \tag{3.30}$$

where $U_{i,t,t',R} = B_V(0,R) \setminus (X^0_{\phi,t,t',R})_{w_i}$ for $1 \le i \le n$, with $\mathrm{H}^*(D) \cong \mathrm{H}^*(C)$.

Proof. In order to simplify notation, we will assume that ϕ is a conjunction of weak inequalities $|F_j| \le \lambda_j |G_j|, j \in J$, where J is a finite set. As before for each i we let $F_{ij} := F_j(\cdot, w_i)$ $G_{ij} := G_j(\cdot, w_i)$.

We first observe that $S_{\sigma,\delta,\delta',\varepsilon,\varepsilon',R} \subset U_{\phi,\delta,\delta',\varepsilon,\varepsilon',R}$. To see this let for $t' > 0, t > 1$, and $i \in [1,n]$, $\theta_{i,t,t'} : V^{\mathrm{an}} \to \mathbb{R}$ be the continuous function defined by

$$\theta_{i,t,t'}(x) = \min_{j \in J}\{\max(\lambda_j t |G_{ij}(x)| - |F_{ij}(x)|, \min(t' - |F_{ij}(x)|, t' - |G_{ij}(x)|))\}.$$

It is easy to check that for $x \in (X^+_{\phi,\delta,\delta'})_{w_i}$, $\theta_{i,\delta,\delta'}(x) > 0$, and for $x \in (X^0_{\phi,\delta,\delta'})_{w_i}$, $\theta_{i,\delta,\delta'}(x) = 0$.

Now let $x \in S_{\sigma,\delta,\delta',\varepsilon,\varepsilon',R}$. Then, for each i with $\sigma(i) = 1$, $x \in (X^+_{\phi,\delta,\delta',R})_{w_i}$ and hence $x \notin (X^0_{\phi,\delta,\delta',R})_{w_i}$.

It is also easy to check, using the fact that $\delta' < \varepsilon'$ and $\delta < \varepsilon$, that $\theta_{i,\delta,\delta'}(x) > 0$ implies that $\theta_{i,\varepsilon,\varepsilon'}(x) > 0$ as well. This in turn implies that if $x \in (X^+_{\phi,\delta,\delta',R})_{w_i}$, then $x \notin (X^0_{\phi,\varepsilon,\varepsilon',R})_{w_i}$. Hence, we have that $x \notin (X^0_{\phi,\delta,\delta',R})_{w_i} \cup (X^0_{\phi,\varepsilon,\varepsilon',R})_{w_i}$ for all i with $\sigma(i) = 1$. A similar argument which we omit shows that the same is true for i with $\sigma(i) = 0$.

Now let C be a connected component of $S_{\sigma,\delta,\delta',\varepsilon,\varepsilon',R}$, and D be the connected component of $U_{\phi,\delta,\delta',\varepsilon,\varepsilon',R}$ containing C. We claim that $D = C$. Let $x \in D$, and let y be any point of C. Then, since $y \in D$ and D is path connected, there exists a path $\gamma : [0,1] \to D$, with $\gamma(0) = y$ and $\gamma(1) = x$, and $\gamma([0,1]) \subset D$. We claim that $\gamma([0,1]) \subset S_{\sigma,\delta,\delta',\varepsilon,\varepsilon',R}$, which immediately implies that $D = C$.

We first show that for each i with $\sigma(i) = 1$, $\gamma([0,1]) \subset (X^+_{\phi,\delta,\delta',R})_{w_i}$. Consider for each i with $\sigma(i) = 1$, the continuous function $\theta_i : [0,1] \to \mathbb{R}$ defined by

$$\theta_i(t) = \theta_{i,\delta,\delta'}(\gamma(t)).$$

Notice that $\theta_i(t) = 0$ implies that $\gamma(t) \in (X^0_{\phi,\delta,\delta',R})_{w_i}$, and hence since $\gamma([0,1]) \subset B_V(0,R) \setminus (X^0_{\phi,\delta,\delta',R})_{w_i}$ for each i, θ_i cannot vanish anywhere on $[0,1]$. Moreover, also notice that $\theta_i(t) > 0$ if and only if $\gamma(t) \in (X^+_{\phi,\delta,\delta',R})_{w_i}$. Since, $\gamma(0) = y \in S_{\sigma,\delta,\delta',\varepsilon,\varepsilon',R}$, this implies that $\theta_i(0) > 0$, and hence $\theta_i(t) > 0$, for each $t \in [0,1]$, and hence

$$\gamma([0,1]) \subset \bigcap_{i, \sigma(i)=1} (X^+_{\phi,\delta,\delta',R})_{w_i}.$$

A similar argument (which we omit) shows that

$$\gamma([0,1]) \subset \bigcap_{i,\sigma(i)=0} (X^{-}_{\phi,\varepsilon,\varepsilon',R})_{w_i}.$$

This proves that $D = C$.
The case for the general ϕ is similar and omitted. □

In particular, the above Proposition and Proposition 3.8 implies:

Proposition 3.31. *For each K-rational point $\bar{w} \in W^n \hookrightarrow (W^{\mathrm{an}})^n$, there exists $\delta > 1, \delta' > 0, \varepsilon > 1, \varepsilon' > 0$, and $R > 0$ such that for each $\sigma \in \{0,1\}^n$ and $0 \le i < k$, one has*

$$\sum_{\sigma \in \{0,1\}^n} b_i(\mathcal{R}^{\mathrm{an}}(\sigma, \bar{w})) \le b_i(U_{\phi,\delta,\delta',\varepsilon,\varepsilon',R})$$

Let V be an affine variety of dimension k, and $U_1, \ldots, U_n$ open semi-algebraic subsets of V^{an}. For $I \subset [1,n]$, we denote by $U^I := \bigcup_{i\in I} U_i$ and $U_I := \bigcap_{i \in I} U_i$. We have the following proposition, which is very similar to [BPRon, Proposition 7.33, Part (ii)].

Proposition 3.32. *With notation as above, for each m, $0 \le m \le k$,*

$$b_m(U_{[1,n]}) \le \sum_{j=1}^{k-m} \sum_{J\subset[1,n],\mathrm{card}(J)=j} b_{m+j-1}(U^J) + \binom{n}{k-m} b_k(V^{\mathrm{an}}).$$

In particular, setting $m = 0$,

$$\begin{aligned} b_0(U_{[1,n]}) &\le \sum_{j=1}^{k} \sum_{J\subset[1,n],\mathrm{card}(J)=j} b_{j-1}(U^J) + \binom{n}{k} b_k(V^{\mathrm{an}}) \\ &\le \left(\sum_{j=1}^{k-1} \binom{n}{j} + 2\binom{n}{k} \right) C, \end{aligned}$$

where $C = \max(\max_{J\subset[1,n],\mathrm{card}(J)\le k} b(U^J), b_k(V^{\mathrm{an}}))$. Here $b(U^J) := \Sigma_{i<k} b_i(U^J)$.

Proof. We first prove the claim when $n = 1$. If $0 \le m \le k-1$, the claim is

$$b_m(U_1) \le b_m(U_1) + b_k(V^{\mathrm{an}}),$$

which is clear. If $m = k$, the claim is $b_k(U_1) \le b_k(V^{\mathrm{an}})$, which is true using Part (d) of Corollary 4.6.

The claim is now proved by induction on n. Assume that the induction hypothesis holds for all $n-1$ open semi-algebraic subsets of V^{an}, and for all $0 \le m \le k$.
It follows from the standard Mayer-Vietoris sequence that

$$b_m(U_{[1,n]}) \le b_m(U_{1,n-1]}) + b_m(U_n) + b_{m+1}(U_{[1,n-1]} \cup U_n). \tag{3.33}$$

Applying the induction hypothesis to the set $U_{[1,n-1]}$, we deduce that

$$\begin{aligned} b_m(U_{[1,n-1]}) \le & \sum_{j=1}^{k-m} \sum_{J\subset[1,n-1],\mathrm{card}(J)=j} b_{m+j-1}(U^J) \\ & + \binom{n-1}{k-m} b_k(V^{\mathrm{an}}). \end{aligned} \tag{3.34}$$

Next, applying the induction hypothesis to the set,

$$U_{[1,n-1]} \cup U_n = \bigcap_{1\le j\le n-1} (U_j \cup U_n),$$

we get that

$$\begin{aligned} b_{m+1}(U_{[1,n-1]} \cup U_n) &\le \sum_{j=1}^{k-m-1} \sum_{J\subset[1,n-1],\mathrm{card}(J)=j} b_{m+j}(U^{J\cup\{n\}}) \\ &+ \binom{n-1}{k-m-1} b_k(V^{\mathrm{an}}). \end{aligned} \tag{3.35}$$

We obtain from inequalities (3.33), (3.34), and (3.35) that

$$b_m(U_{[1,n]}) \le \sum_{j=1}^{k-m} \sum_{J\subset[1,n],\mathrm{card}(J)=j} b_{m+j-1}(U^J) + \binom{n}{k-m} b_k(V^{\mathrm{an}}),$$

which finishes the induction. □

Proof of Theorem 2.8. Using Proposition 3.31 we obtain that for each $i, 0 \le i \le k$,

$$\sum_{\sigma\in\{0,1\}^n} b_i(\mathcal{R}^{\mathrm{an}}(\sigma,\bar{w})) \le b_i(U_{\phi,\delta,\delta',\varepsilon,\varepsilon',R}).$$

From the definition of $U_{\phi,\delta,\delta',\varepsilon,\varepsilon',R}$ in (3.30), we have that $U_{\phi,\delta,\delta',\varepsilon,\varepsilon',R}$ is an intersection of the sets $B_V(0,R) \setminus (X^0_{\phi,\varepsilon,\varepsilon',R})_{w_i}, B_V(0,R) \setminus (X^0_{\phi,\delta,\delta',R})_{w_i}, 1 \le i \le n$. The theorem follows using Proposition 3.32, and from the fact that all j-ary unions of the semi-algebraic sets $B_V(0,R) \setminus (X^0_{\phi,\varepsilon,\varepsilon',R})_{w_i}), B_V(0,R) \setminus ((X^0_{\phi,\delta,\delta',R})_{w_i}), 1 \le i \le n, 1 \le j \le k-i$ are fibers of a fixed projection which depends solely on V and the formula ϕ defining X. In particular, it follows from [HL16, Theorem 14.3.1] that the number of homotopy types of such fibers is finite and hence the Betti numbers of these unions are bounded by some constant that depends solely on V. □

3.3. **Proof of Theorems 2.1.** In the proofs of Theorem 2.1 we will need the following technical generalization of Proposition 2.5. We begin by introducing some notation.

Let I be a finite set and for each $\alpha \in I$, let X_α be a definable subset of $V \times W$ in some structure M. Let $i_\alpha : X_\alpha \hookrightarrow V \times W$ denote the inclusion map. Suppose that X is a definable subset of $V \times W$ obtained as a Boolean combination of the X_α's. Let $W' := \coprod_{\alpha\in I} W$, and for $\alpha \in I$ we $j_\alpha : W \hookrightarrow W'$ denote the canonical inclusion. Let $X' = \bigcup_{\alpha\in I} \mathrm{Im}((j_\alpha \times 1_V) \circ i_\alpha) \subset W' \times V$. Note that X' is a definable subset of $W' \times V$.

Proposition 3.36. *Suppose that there exists a constant $C > 0$ such that for all $n > 0$,*

$$\chi_{X';V;W'}(n) \le Cn^k.$$

Then $\mathrm{vcd}(X; V; W) \le k$.

Notation 3.37. In the proof below, given a subset $S < S'$, we let $\Delta_S : S' \to \{0,1\}$ denote the characteristic function of S.

Proof. For $v \in V$, and $S \subset W$ (resp. $S' \subset W'$) we set $S_v := S \cap X_v$ (resp. $S'_v := S' \cap X'_v$). We prove that for all $\bar{w} \in W^n$, $\mathrm{card}(\chi_{X;V,W;n}(\bar{w}, V)) \leq Cn^k$. Let $\bar{w} \in W^n$. We claim that for $v, v' \in V$,

$$\chi_{X;V;W;n}(\bar{w}, v) \neq \chi_{X;V;W;n}(\bar{w}, v') \implies \chi_{X';V;W';n}(j_n(\bar{w}), v) \neq \chi_{X';V;W';n}(j_n(\bar{w}), v'),$$

where $j_n : W^{[1,n]} \to W'^{I \times [1,n]}$ is defined by

$$j(w_1, \ldots, w_n)_{(\alpha,i)} = j_\alpha(w_i).$$

To prove the claim first observe that since $\chi_{X;V;W;n}(\bar{w}, v) \neq \chi_{X;V;W;n}(\bar{w}, v')$, there exists $i \in [1, n]$ such that $\Delta_{X_{w_i}}(v) \neq \Delta_{X_{w_i}}(v')$.

Since X is a Boolean combination of the $X_\alpha, \alpha \in I$, there must exist $\alpha \in I$ such that $\Delta_{(X_\alpha)_{w_i}}(v) \neq \Delta_{(X_\alpha)_{w_i}}(v')$. It now follows from the definition of X', W' that $\chi_{X';V;W';n}(j_n(\bar{w}), v) \neq \chi_{X';V;W';n}(j_n(\bar{w}), v')$. This implies that

$$\mathrm{card}(\chi_{X;V;W;n}(\bar{w}, V)) \leq \mathrm{card}(\chi_{X';V;W';n}(j_n(\bar{w}), V)).$$

The hypothesis of the proposition now implies that

$$\mathrm{card}(\chi_{X;V;W,n}(\bar{w}, V)) \leq \mathrm{card}(\chi_{X';V;W';n}(j_n(\bar{w}), V)) \leq Cn^k.$$

The proposition now follows from Proposition 2.5. □

Proof of Theorem 2.1. Let $k = |\overline{X}|$ and $\ell = |\overline{Y}|$. Now observe that we can assume that the atoms of ϕ are of the form $|F| \leq \lambda|G|$, with $\lambda \in \mathbb{R}_+$, and $F, G, \in K[\overline{X}, \overline{Y}]$. Let $(\phi_\alpha)_{\alpha \in I}$ be the finite tuple of (closed) atomic formulas appearing in ϕ.
Let

$$\phi'' = \left(\bigvee_{\alpha \in I} \left(\phi_\alpha(\overline{X}, \overline{Y}^{(\alpha)}) \wedge (Z_\alpha = 1) \right) \right) \wedge \bigvee_{\alpha \in I} \theta_\alpha((Z_\alpha)_{\alpha \in I}),$$

where $\theta_\alpha((Z_\alpha)_{\alpha \in I})$ is the closed formula

$$(Z_\alpha = 1) \wedge \bigwedge_{\beta \neq \alpha} (Z_\beta = 0).$$

Notice that (using the same notation as in Proposition 3.36) the diagram

$$\begin{tikzcd} & \mathcal{R}(\phi'') \arrow[dl, "\pi''_1"'] \arrow[dr, "\pi''_2"] & \\ K^k & & \mathrm{Im}(\pi''_2) \hookrightarrow K^{|I|\ell+|I|} \end{tikzcd}$$

is isomorphic to the diagram

$$\begin{tikzcd} & (\mathcal{R}(\phi))' \arrow[dl, "\pi'_1"'] \arrow[dr, "\pi'_2"] & \\ K^k & & \mathrm{Im}(\pi'_2) \hookrightarrow (K^\ell)' = \coprod_\alpha K^\ell. \end{tikzcd}$$

By isomorphism, we mean that there are natural bijections $\mathcal{R}(\phi'') \to (\mathcal{R}(\phi)'$ and $\mathrm{Im}(\pi''_2) \to \mathrm{Im}(\pi'_2)$ making the resulting morphism of diagrams above commute (with identity as the map on K^k). If K is complete, the it follows from Corollary 2.9 that there exists a constant $C_{\phi''} > 0$ such that for all $\bar{w}' \in (K^{|I|\ell+|I|})^n$,

$$\mathrm{card}(\chi_{\mathcal{R}(\phi''), K^k, K^{|I|\ell+|I|}, n}(\bar{w}, \mathcal{R}(\phi'')) \leq C_{\phi''} n^k.$$

It now follows from Proposition 3.36 that $\mathrm{vcd}(\phi) \leq k$ in the case when K is complete. The general case now follows passing to the completion and using Observation 2.6. □

4. Appendix

4.1. **Berkovich analytic spaces.** In this section, we give a brief exposition of some basic results from the theory of Berkovich analytic spaces. We refer to the papers ([Ber90], [Ber93]) for more details. As before, K is a complete non-archimedean field

Berkovich has defined a good notion of analytic spaces over any such field. We briefly recall the construction. Given positive real numbers $r_1, \cdots, r_n > 0$, let

$$K\langle r_1^{-1}, \cdots, r_n^{-1} X_n\rangle = \{\Sigma a_J X^J \in K[[X_1, \cdots, X_n]] | |a_J| r^J \to 0\}$$

denote the usual Tate algebra of convergent power series on the polydisk of radius $r = (r_1, \cdots, r_n)$. Note that this is a K-Banach algebra. A K-affinoid algebra is a K-Banach algebra with a continuous surjection from some Tate algebra such that the residue norm induced on A (as a quotient) is equivalent to the given norm on A. If such a surjection can be found with $r_i = 1$, then we say that A is a strictly K-affinoid domain.

Given a (commutative) K-Banach algebra A, its spectrum, denoted $\mathcal{M}(A)$, is the set of bounded multiplicative seminorms on A. The topology on $\mathcal{M}(A)$ is the weakest topology for which the functions $\mathcal{M}(A) \to \mathbb{R}$ defined by $|\cdot| \to |F|$, for $F \in A$, are all continuous. For a point $|\cdot|_0 \in \mathcal{M}(A)$, a basis of open neighborhoods is given by the sets $U(F_1, \cdots, F_n; \epsilon_1, \cdots, \epsilon_n) := \{|\cdot| \in \mathcal{M}(A) || |F_j|_0 - |F_j|| < \epsilon_j\}$. We note that $\mathcal{M}(A)$ is a compact and Hausdorff topological space. Moreover, given any $F, G \in \mathcal{A}$, the subset $\{x \in \mathcal{M}(A) || F(x)| \leq \lambda |G(x)|\}$ is a closed subset of $\mathcal{M}(A)$. Finally, we note that the construction is (contravariantly) functorial in bounded K-linear maps $A' \to A$ between commutative K-Banach algebras.

A closed subset $U \subset \mathcal{M}(A)$ is an affinoid subdomain if there is a bounded map $A \to A'$ of K-affinoid algebras such that the corresponding image in $\mathcal{M}(A)$ is contained in U, and it is universal in the following sense: For any bounded map $A \to B$ of K-affinoid algebras such that the image of $\mathcal{M}(B) \subset U$, one has a factorization $A \to B$ factors through the $A \to A'$.

A K-analytic space is a locally topological Hausdorff space which has an atlas consisting of K-affinoids. We do not recall the definition here. But, we recall the definition of the analytic affine space $\mathbb{A}_K^{\mathrm{an},n}$. The underlying topological space is the set of multiplicative semi-norms on the ring of polynomials $K[x_1, \ldots, x_n]$ with the topology given as before. Namely, it is the weakest topology for which all the maps $x \to |f(x)|$, for all $F \in K[x_1, \ldots, x_n]$, are continuous.

Let $An(K)$ denote the category of K-analytic spaces. There is a natural analytification functor $Sch(K) \to An(K)$. Given a scheme X over K, let X^{an} denote the corresponding analytic space. In the following, we shall assume all our analytic spaces are separated. Note, if X is separated then X^{an} is also separated. If X

is a separated analytic space, then the underlying topological space, denoted $|X|$, is a locally compact Hausdorff topological space. There is a natural morphism ringed spaces $X^{\mathrm{an}} \to X$. Moreover, there is a natural (functorial) injective map $X(K) \to |X|$ which is dense if K is algebraically closed. Here $|X|$ denoted the topological space underlying X^{an}. It follows from the results of Berkovich (in the smooth case), and Hrushovski-Loeser (for arbitrary quasi-projective varieties) that the topological spaces $|X|$ enjoy many nice properties. We recall these in the next subsection.

4.2. **Model theory basics.** In this section, we recall some results from the theory of non-archimedean tame topology due to Hrushovski and Loeser ([HL16]). In particular, we will deal with the model theory of valued fields. In this section, we shall denote by K a valued field with values in the ordered multiplicative group Γ of the positive real numbers, and we will denote by $\mathbb{R}_+ = \Gamma \cup \{0\}$.

We consider a two sorted language with the two sorts corresponding to valued fields and the value group. The signature of this two sorted language will be

$$(0, 1, +_K, \times_K, |\cdot| : K \to \mathbb{R}_+, \leq_{\mathbb{R}_+}, \times_{\mathbb{R}}),$$

where the subscript K denotes constants, functions, relations etc., of the field sort and the subscript $\mathbb{R}_+$ denotes the same for the value group sort. When the context is clear we will drop the subscripts.

We denote by $|\cdot|$ the valuation written multiplicatively. The valuation $|\cdot|$ satisfies:

$$\begin{aligned} |x+y| &\geq \min\{|x|, |y|\}, \\ |x \cdot y| &= |x||y|, \\ |0| &= 0. \end{aligned}$$

Remark 4.1. Note that we follow Berkovich's convention and write our valuations multiplicatively. However, in ([HL16]), all valuations are written additively.

Following [HL16], we will denote by $\mathbf{F}$ the two sorted structure $(K, \mathbb{R}_+)$. Given a quasi-projective variety V defined over K and an $\mathbf{F}$-definable subset X of $V \times \mathbb{R}_+$, Huroshovski and Loeser associate to X (functorially) a topological space $B_{\mathbf{F}}(X)$. By defintion, this is the set of $\mathbf{F}$-definable types on X which are nearly orthogonal to the definable set $\mathbb{R}_+$. We say that $B_{\mathbf{F}}(X)$ a *semi-algebraic* subset of $B_{\mathbf{F}}(V)$. Note that X itself can be identified in $B_{\mathbf{F}}(X)$ as the set of realized types, and hence there is a canonically defined injection $X \hookrightarrow B_{\mathbf{F}}(X)$.

For any $\mathbf{F}$-definable map $f : V \to \mathbb{R}_+$, any type $p \in B_{\mathbf{F}}(X)$ and $a \models p$, $|f(a)| \in \mathbb{R}_+$ (since p is weakly orthogonal to the value group $\mathbb{R}_+$). It follows that $f(p) := f(a)$ does not depend on the choice of a.

If $g \in K[V]$, the mapping $|g| : V \to \mathbb{R}_+, x \mapsto |g(x)|$ is $\mathbf{F}$-definable. One equips the set $B_{\mathbf{F}}(X)$ with a topology generated by the basic open sets which are finite intersections of the form

$$\{p \in B_{\mathbf{F}}(X \cap U) \mid |g|(p) \in W\}$$

where $g \in K[V]$, U a Zariski open subset of V, and W is an open subset of $\mathbb{R}_+$. In fact, this gives a basis of semi-algebraic subsets.

Remark 4.2. If K is complete, then $B(V)$ is canonically homeomorphic to the associated Berkovich analytic space V^{an}, and a subset of $B(V)$ is semi-algebraic if and only if it is in the sense of section 3.1.

Given an $\mathbf{F}$-definable map $f : X \to \mathbb{R}_+$, we will denote by $B_{\mathbf{F}}(f) : B_{\mathbf{F}}(X) \to \mathbb{R}_+$ the map whose graph is $B_{\mathbf{F}}(\mathrm{graph}(f))$. We will say that $B_{\mathbf{F}}(f)$ is a *semi-algebraic* map. Note that $B_{\mathbf{F}}(|\cdot|) : B_{\mathbf{F}}(V) \to \mathbb{R}_+$ is continuous with respect to the topology defined on $B_{\mathbf{F}}(V)$.

The following Theorems which are easily deduced from the main theorems in [HL16, Chapter 14] will play a key role in the results of this paper. We will use the same notation as above.

Theorem 4.3. [HL16, Theorem 14.4.4] *Let $f : X \to \mathbb{R}_+$ be an $\mathbf{F}$-definable map. For $t \in \mathbb{R}_+$, let $B_{\mathbf{F}}(X)_{\geq t}$ denote the semi-algebraic subset $B_{\mathbf{F}}(X \cap (f \geq t)) = B_{\mathbf{F}}(X) \cap (B_{\mathbf{F}}(f) \geq t)$ of $B_{\mathbf{F}}(V)$. Then, there exists $\varepsilon_0 > 0$, such that for all $\varepsilon, \varepsilon'$, $0 < \varepsilon \leq \varepsilon' < \varepsilon_0$, the inclusion $B_{\mathbf{F}}(X)_{\geq \varepsilon'} \hookrightarrow B_{\mathbf{F}}(X)_{\geq \varepsilon}$ is a homotopy equivalence.*

Theorem 4.4. [HL16, Theorem 14.3.1, Part (1)] *Let $X \subset Y \times \mathbb{P}^m$ be an $\mathbf{F}$-definable set and let $\pi : X \to Y$ be the projection map to the first factor. Then there are finitely many homotopy types amongst the fibers $(B_{\mathbf{F}}(\pi^{-1}(y)))_{y \in Y}$.*

Theorem 4.5. [HL16, Theorem 14.2.4] *Let V be a quasi-projective variety defined over K, and X an $\mathbf{F}$-definable subset of V. Then there exists a sequence of finite simplicial complexes $(X_i)_{i \in \mathbb{N}}$ embedded in $B_{\mathbf{F}}(X)$ of dimension $\leq \dim(V)$, deformation retractions $\pi_{i,j} : X_i \to X_j, j < i$, and deformation retractions $\pi_i : B_{\mathbf{F}}(X) \to X_i$, such that $\pi_{i,j} \circ \pi_i = \pi_j$ and the canonical map $B_{\mathbf{F}}(X) \to \varprojlim_i X_i$ is a homeomorphism.*

As an immediate consequence of Proposition 4.5 we have using the same notation:

Corollary 4.6. *Let V be an affine variety, and let $B_{\mathbf{F}}(X)$ be a semi-algebraic subset of V.*

(a) Every connected component of $B_{\mathbf{F}}(X)$ is path connected.
(b) For any field of coefficients $\mathbb{F}$, $\mathrm{H}^n(B_{\mathbf{F}}(X), \mathbb{F}) = 0$ for $n > \dim(V)$.
(c) $\dim_{\mathbb{F}} \mathrm{H}^*(B_{\mathbf{F}}(X), \mathbb{F}) < \infty$.
(d) The restriction homomorphism $\mathrm{H}^{\dim(V)}(B_{\mathbf{F}}(V), \mathbb{F}) \to \mathrm{H}^{\dim(V)}(B_{\mathbf{F}}(X), \mathbb{F})$ is surjective.

Note that Parts (a), (b) and (c) follow directly from the previous propositions.

Proof of Part (d). We suppose that $V \subset \mathbb{A}^n$, and let $B(0, R)$ denote the ball of radius R in $B_{\mathbf{F}}(\mathbb{A}^n)$. Note that the latter is a semi-algebraic subset of $B_{\mathbf{F}}(\mathbb{A}^n)$, and we let $B_V(0, R) := B(0, R) \cap B_{\mathbf{F}}(V)$ denote the resulting semi-algebraic subset of $B_{\mathbf{F}}(V)$. Note that this is a compact topological space. Similar remarks apply to $B_X(0, R) := B(0, R) \cap X$. Finally, note that, arguing as in Lemma 6, for sufficiently larger R the natural inclusions $B_X(0, R) \hookrightarrow B(X)$ and $B_V(0, R) \hookrightarrow B(V)$ induce homotopy equivalences.

It follows from the previous remarks that it is sufficient to prove that for all sufficiently large $R > 0$ the natural induced morphism

$$\mathrm{H}^{\dim(V)}(B_V(0,R), F) \to \mathrm{H}^{\dim(V)}(B_X(0,R), F)$$

is surjective for any (abelian) sheaf F on $B_V(0,R)$. It follows from Theorem 4.5 that $B_{\mathbf{F}}(V)$ and hence $B_V(0,R)$ (for all sufficiently large $R > 0$) has the homotopy type of a finite simplicial polyhedron of dimension at most $\dim(V)$. Since $B_V(0,R)$ is compact, it follows that $\mathrm{H}^{\dim(V)+1}(B_V(0,R), F) = \mathrm{H}_c^{\dim(V)+1}(B_V(0,R), F) = 0$ for all sheaves F. In particular, the cohomological dimension (in the sense of [Ive86, page 196, Definition 9.4]) of $B_V(0,R)$ is $\leq \dim(V)$

It follows again from Theorem 4.5 that there exists a compact polyhedron $Z \subset B_X(0,R)$ (for sufficiently large R) such that Z is a deformation retract of $B_{\mathbf{F}}(X)$. Let $i : Z \hookrightarrow B_X(0,R)$ be the inclusion map. Note that i induces isomorphisms in cohomology. Since the inclusion of Z in $B_V(0,R)$ factors through i, and i induces isomorphisms in cohomology, it follows (using the long exact sequence cohomology for pairs of pairs) that

$$\mathrm{H}^*(B_V(0,R), B_X(0,R); F) \cong \mathrm{H}^*(B_V(0,R), Z; F)$$

for any sheaf F on $B_V(0,R)$.

Let $N = \dim(V) + 1$. We now prove that

$$\mathrm{H}^N(B_V(0,R), B_X(0,R); F) \cong \mathrm{H}^N(B_V(0,R), Z; F) = 0.$$

This gives the desired result by an application of the long exact sequence in cohomology associated to the pair $(B_V(0,R), B_X(0,R))$.

Recall that $B_V(0,R)$ is a compact space, and Z is a closed subspace of $B_V(0,R)$. It follows now [Ive86, page 198, Proposition 9.7] that the cohomological dimension of $U \triangleq (B_{\mathbf{F}}(V) \cap B_V(0,R)) \setminus Z$ is also $\leq \dim(V)$. This implies that $\mathrm{H}_c^{\dim(V)+1}(U, F) \cong \mathrm{H}^{\dim(V)+1}(B_V(0,R), Z; F) = 0$ which finishes the proof. $\square$

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