Solutions to Hwk4

Section 3.4

6. \( F(x) = x + f(x)g(x) \), \( f(\xi) = 0, f'(\xi) \neq 0 \). The theory developed in this section shows that the sequence defined by \( x_{n+1} = F(x_n) \) will converge cubically to \( \xi \) if \( F'(\xi) = 0, F''(\xi) \neq 0 \). (We want at least cubically convergent sequence, so we do not insist on \( F'''(\xi) \neq 0 \).) \( F'(x) = 1 + f'(x)g(x) + f(x)g'(x) \)
\[ \Rightarrow F'(\xi) = 1 + f'(\xi)g(\xi) + f(\xi)g'(\xi) = 1 + f'(\xi)g(\xi) \]
We want this to be zero. So \( g(\xi) = -1/f'(\xi) \). (Condition 1) \( F''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x) \)
\[ \Rightarrow F''(\xi) = f''(\xi)g(\xi) + 2f'(\xi)g'(\xi) + f(\xi)g''(\xi) = f''(\xi)[-1/f'(\xi)] + 2f'(\xi)g'(\xi) \]
For this to be zero we require \( g'(\xi) = f''(\xi)/[2(f'(\xi))^2] \). (Condition 2) \textbf{Note:} If \( g(x) = -[f'(x)]^{-1} \) then \( g'(x) = [f'(x)]^{-2}[f''(x)] \), so \( g'(\xi) \) is off by a factor of 2.

7. Eventually 0.9998477 appears. Let \( f(x) = \cos x \). Then \[ |\cos x - \cos y| = |\sin \xi| \cdot |x - y| \] for \( x < \xi < y \) since \( |\sin \xi| < 1 \), \( F(x) \) is a contraction and thus has a fixed point.

12. \( x = \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}} \). Let \( x_1 = \sqrt{p}, x_2 = \sqrt{p + \sqrt{p}}, x_3 = \sqrt{p + \sqrt{p + \sqrt{p}}} \), and so on. Observe that \( x_3 = \sqrt{p + x_1}, x_3 = \sqrt{p + x_2}, \) and so on. In general \( x_{n+1} = \sqrt{p + x_n} \) (I).

Let \( f(x) = \sqrt{p + x} \). Equation (I) is the result of using functional iteration on \( f \). If \( \lim x_n \) exists, denote it by \( x \). Take limits in Equation (I) to get \( x = \sqrt{p + x} \). Hence \( x^2 = p + x, x^2 - x - p = 0, x = (1 + \sqrt{1 + 4p})/2 \). This is the limit of the sequence. For example if \( p = 2, x = 2 \). Try it on your pocket calculator.

13. Use the ideas of Problem 3.4.12. Let \( x_1 = 1/p, x_2 = 1/(p + (1/p)), x_3 = 1/(p + (1/p + (1/p))) \) etc. So \( x_2 = 1/(p + x_1), x_3 = 1/(p + x_2) \), and so on. Hence \( x_{n+1} = 1/(p + x_n) \). If \( \lim_{n \to \infty} x_n = x \) then \( x = 1/(p + x) \). Hence \( x(p + x) = 1, x^2 + px - 1 = 0, x = (-p + \sqrt{p^2 + 4})/2 \). This illustrates functional iteration with \( f(x) = 1/(p + x) \). If \( p > 0, f \) is a contraction. Use Mean Value Theorem:
\[ |f(x) - f(y)| = |f'(\xi)| \cdot |x - y| = |1/(p + \xi)| \cdot |x - y| \]
Since \( p > 1, \) all \( x_n \)'s will be \( \geq 0 \), and \( 1/(p + x)^2 \leq 1/p^2 < 1 \). So \( f \) is a contraction on \([0, \infty]\). \( f \) actually maps \([0, 1]\) into \([0, 1]\), so has a fixed point in \([0, 1]\).

40. \( F(x) = x(x^2 + 3R)/(3x^2 + r), F(\sqrt{R}) = \sqrt{R}, F'(\sqrt{R}) = 0, F''(\sqrt{R}) = 0, F'''(\sqrt{R}) \neq 0 \).