Math. Ann. 266, 507–511 (1984)

Mathematische Annalen

On the Absolute Value of Ramanujan's τ-Function

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In this note we consider Ramanujan's function $\tau(n)$ which is defined by

$$z\prod_{n=1}^{\infty}(1-z^n)^{24}=\sum_{n=1}^{\infty}\tau(n)z^n,\ |z|<1\,.$$

Theorem. *For each real number δ in the interval $0<\delta<1$ define*

$$w(\delta)=\inf_{-1\leqq y\leqq 3} y^{-2}(1+\delta y-(1+y)^{\delta})\,.$$

Then there is a constant c, depending upon δ, so that

$$\sum_{n\leqq x}(|\tau(n)|n^{-11/2})^{2\delta}\leqq cx(\log x)^{-w(\delta)}$$

for all $x\geqq 2$.

For a fixed real $y>-1$ the function $h(z)=1+zy-(1+y)^z$ satisfies $h(0)=h(1)=0$, $h''(z)=-(1+y)^z(\log(1+y))^2<0$, so that $h(z)$ is positive for $0<z<1$ and $y\geqq -1$. Thus $w(\delta)$ is always positive.

The present result shows that the limits A_δ, $0<\delta<2$, which were defined in Elliott [1] are indeed all zero. We refer to that paper for a discussion of other problems concerning the value distribution of $\tau(n)$.

Since $y^{-2}(1+y/2-(1+y)^{1/2})=(1+(1+y)^{1/2})^{-2}/2$ we have $w(1/2)=1/18$ and

$$\sum_{n\leqq x}|\tau(n)|=O(x^{13/2}(\log x)^{-1/18})\,.$$

As we shall indicate at the end of this note, a strong enough form of the Sato-Tate conjecture would lead to an asymptotic estimate

$$\sum_{n\leqq x}|\tau(n)|\sim Bx^{13/2}(\log x)^{(8/3\pi)-1},\ x\to\infty$$

for a non-zero constant B. Note that $1-8/3\pi\simeq 0.15117\simeq 3/20$.

For a proof of the theorem we need the following result taken from the first author's forthcoming book on arithmetic functions.

Lemma. *Let $f(n)$ be a non-negative real-valued function defined on the positive integers, and satisfying $f(ab) \leqq f(a)f(b)$ for every pair of mutually prime a and b. Then for $x \geqq 2$*

$$\sum_{2 \leqq n \leqq x} f(n) \leqq \left(\frac{x}{\log x} + \frac{40x}{(\log x)^2}\right) \Delta \sum_{n \leqq x} \frac{f(n)}{n}$$

with

$$\Delta = \sup_{1 \leqq y \leqq x} y^{-1} \sum_{p^k \leqq y} f(p^k) \log p^k .$$

Moreover

$$\frac{1}{\log x} \sum_{n \leqq x} \frac{f(n)}{n} \leqq c_0 \exp\left(\sum_{p^k \leqq y} \frac{f(p^k) - 1)}{p^k}\right)$$

for an absolute constant c_0.

Here, as usual, p denotes a prime number.

Proof. Consider the sum

$$S(x) = \sum_{n \leqq x} f(n) \log n .$$

Beginning with the representation

$$\log n = \sum_{p^k \| n} \log p^k, \; n > 0 ,$$

we can write

$$S(x) = \sum_{p^k \leqq x} \log p^k \sum_{m \leqq p^{-k}x, (m,p) = 1} f(p^k m) .$$

A typical inner sum does not exceed

$$f(p^k) \sum_{m \leqq p^{-k}x, (m,p) = 1} f(m) \leqq f(p^k) \sum_{m \leqq p^{-k}x} f(m) ,$$

so that

$$\begin{aligned} S(x) &\leqq \sum_{p^k \leqq x} \log p^k f(p^k) \sum_{m \leqq p^{-k}x} f(m) \\ &= \sum_{m \leqq x} f(m) \sum_{p^k \leqq x} f(p^k) \log p^k \leqq x\Delta \sum_{m \leqq x} f(m) m^{-1} . \end{aligned}$$

Integrating by parts

$$\begin{aligned} \sum_{2 \leqq n \leqq x} f(n) &= \int_{2-}^{x} (\log y)^{-1} dS(y) \\ &= S(x)(\log x)^{-1} - f(2) + \int_{2}^{x} S(y) y^{-1} (\log y)^{-2} dy . \end{aligned}$$

We estimate this integral by applying the upper bound

$$S(y) \leqq y\Delta \sum_{m \leqq x} f(m) m^{-1}, \; 2 \leqq y \leqq x ,$$

noting that for $x \geqq 2$

$$\int_2^x (\log y)^{-2} dy \leqq 40x(\log x)^{-2}.$$

This justifies the first assertion of the lemma.

As for the second assertion we combine the inequalities

$$\sum_{n \leqq x} \frac{f(n)}{n} \leqq \prod_{p \leqq x} \left(1 + \sum_{k \leqq \frac{\log x}{\log p}} \frac{f(p^k)}{p^k}\right)$$

$$\leqq \exp\left(\sum_{p^k \leqq x} \frac{f(p^k)}{p^k}\right)$$

with the elementary estimate

$$\sum_{p^k \leqq x} \frac{1}{p^k} = \log\log x + \text{constant} + O((\log x)^{-1}).$$

See, for example, Hardy and Wright [4], Chap. XXII.

Proof of the Theorem

Let $g(n) = (|\tau(n)| n^{-11/2})^2$. It follows from Deligne's estimate $|\tau(p)| < 2p^{11/2}$ that $g(p^k) \leqq (k+1)^2$. See, for example, Hardy [3], Chap. X.

As a function of the complex variable s the sum

$$\sum_{n=1}^{\infty} g(n) n^{-s} = \sum_{n=1}^{\infty} \tau(n)^2 n^{-11-s}$$

was considered by Rankin [6]. Of his results we need only that this function may be analytically continued into a neighborhood of $s=1$ except that it has a simple pole at that point. In terms of the Riemann zeta function $\zeta(s)$ a non-zero limit

$$\lim_{s \to 1} \zeta(s)^{-1} \sum_{n=1}^{\infty} g(n) n^{-s}$$

exists. Expressing these Dirichlet series in terms of their Euler products and taking logarithms we deduce the finite existence of

$$\lim_{s \to 1} \sum_p p^{-s}(g(p) - 1).$$

If we define

$$\alpha(u) = \sum_{p \leqq e^u} p^{-1}(g(p) - 1),$$

then this last relation may be given the form

$$\lim_{y \to 0} \int_0^{\infty} e^{-yu} d\alpha(u) = c_1$$

for some constant c_1. Moreover, for $1 \leqq u \leqq x$

$$|\alpha(x) - \alpha(u)| \leqq \sum_{e^u < p \leqq e^x} \frac{3}{p} = 3\log\frac{x}{u} + O(u^{-1})$$

so that

$$\lim_{\substack{u,x\to\infty\\ u/x\to 1}} |\alpha(x)-\alpha(u)|=0;$$

the function $\alpha(u)$ is slowly oscillating. We may therefore apply the Hardy-Littlewood tauberian theorem (see Elliott [2], Lemma 2.18) to obtain

$$\lim_{x\to\infty} \sum_{p\leqq x} p^{-1}(g(p)-1)=c_1 .$$

In a similar way we treat the function

$$\sum_{n=1}^{\infty} g(n)^2 n^{-s} = \sum_{n=1}^{\infty} \tau(n)^4 n^{-22-s},$$

but with the results of Rankin replaced by those of the second and third authors, Moreno and Shahidi [5]. Now there is a double pole giving the existence of

$$\lim_{s\to 1} \zeta(s)^{-2} \sum_{n=1}^{\infty} g(n)^2 n^{-s},$$

and after another application of the Hardy-Littlewood tauberian theorem,

$$\lim_{x\to\infty} \sum_{p\leqq x} p^{-1}(g(p)^2-2)=c_2$$

for some constant c_2.

We apply the lemma to the function $g(n)^\delta$. In the present circumstances

$$\Delta \leqq \sup_{y\leqq x} y^{-1} \sum_{p^k\leqq y} (k+1)^{2\delta} \log p^k$$

which by an old estimate of Tchebycheff is bounded uniformly for all $x\geqq 2$. Thus for some constant c_3

$$x^{-1} \sum_{2\leq n\leq x} g(n)^\delta \leq c_3 \exp\left(\sum_{p^k\leq x} p^{-k}(g(p^k)^\delta -1)\right).$$

The contribution to this last sum which arises from the prime-powers p^k with $k\geqq 2$ is bounded uniformly for all $x\geqq 2$, whilst from the definition of $w(\delta)$ the terms over the primes do not contribute more than

$$\delta \sum_{p\leqq x} p^{-1}(g(p)-1) - w(\delta) \sum_{p\leqq x} p^{-1}(g(p)-1)^2 .$$

Here we have made essential use of Deligne's bound $|\tau(p)|<2p^{11/2}$ since that secures $-1\leqq g(p)-1\leqq 3$ for every p.

The coefficient of δ in this expression is bounded. The coefficient of $w(\delta)$ may be estimated to be

$$\sum_{p\leqq x} p^{-1} + \sum_{p\leqq x} p^{-1}(g(p)^2-2) - 2\sum_{p\leqq x} p^{-1}(g(p)-1)$$
$$= \log\log x + \text{constant} + c_2 - 2c_1 + O(1)$$

as $x\to\infty$, and the theorem is clear.

Concluding Remarks

Let $\tau(p)p^{-11/2}=2\cos\theta_p$, so that after Deligne θ_p is real. Let us assume the Sato-Tate conjecture in the form

$$\sum_{\substack{p\leq x\\ \theta_p\leq\alpha}} 1=\frac{2}{\pi}\int_0^\alpha \sin^2\theta d\theta\cdot\frac{x}{\log x}+O\left(\frac{x}{(\log x)^2}\right),$$

to hold uniformly in $0\leqq\alpha\leqq\pi$, as $x\to\infty$.

Integrating by parts, with $s=\sigma+i\tau$, $\sigma=\mathrm{Re}(s)$, it is not difficult to deduce that

$$\sum_p |\tau(p)|p^{-11/2-s}=-(8/3\pi)\log(s-1)+c_3+O(1)$$

as $\sigma\to 1+$, uniformly in each fixed Stolz angle $|\tau|\leqq K(\sigma-1)$. This in turn leads to

$$\sum_{n=1}^{\infty} |\tau(n)|n^{-11/2-s}=D(s-1)^{-(8/3\pi)}+O((\sigma-1)^{-(8/3\pi)})$$

for a certain non-zero constant D, valid in the same angle(s).

This, together with the general distributional theory of arithmetic functions (see, for example, Elliott [2] Vol. 2) enables one to prove that

$$\sum_{n\leq x} |\tau(n)|n^{-11/2}\sim Fx(\log x)^{(8/3\pi)-1}$$

as $x\to\infty$, for some non-zero constant F. An asymptotic estimate for the average of $|\tau(n)|$ is now obtained using an integration by parts.

A similar argument, depending upon the above form of the Sato-Tate conjecture, would lead to an asymptotic estimate for the average of $|\tau(n)|^\delta$, for each fixed $\delta>0$.

References

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Received July 22, 1983

Note added in proof. M. Ram Murty has also proved in (Math. Ann. 262, 431–446 (1983), Theorem 3) a non-vanishing result for L-functions similar to that used in the proof of the main theorem.