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# ON THE TEMPERED SPECTRUM OF QUASI-SPLIT CLASSICAL GROUPS

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**§1. Introduction.** One of the most striking aspects of the Langlands program is the conjectural relation between harmonic analysis and number theory. The proof of a conjecture of Langlands given in [20] shows that determining the poles of certain (conjectural) Langlands $L$-functions is equivalent to determining the nondiscrete tempered spectrum of reductive $p$-adic groups. The theory of endoscopy [16] and twisted endoscopy [13], [14] has proved particularly useful in giving a context within which to explore these problems, at least for classical groups (see the introduction and Section 3 of [21]).

Previously the authors separately determined the nondiscrete tempered spectrum of classical groups supported in their Siegel parabolic subgroups (see [8] and [21]) or, equivalently, computed the symmetric square and the exterior square $L$-functions for $GL_n(F)$ [21], as well as the Asai $L$-functions for $GL_n(E)$, where $E$ is a quadratic extension of $F$ [8]. In fact, these $L$-functions were determined for an arbitrary irreducible admissible representation of $GL_n(F)$ or $GL_n(E)$, accordingly. Finally, in [22], the second author addressed the problem for arbitrary maximal parabolic subgroups of split even special orthogonal groups. In terms of $L$-functions, the work in [22] determines the Rankin-Selberg product $L$-functions attached to irreducible admissible representations of $GL_n(F) \times SO_{2m}(F)$.

The purpose of the present paper is twofold. First we generalize the work in [22] to symplectic and quasi-split special orthogonal groups and thus, using [9], eventually determine the nondiscrete tempered spectrum of these groups. The second purpose is to remove a gap that existed in the proof of Theorems 7.8 and 8.1 of [22]. The final results of the present paper, Theorem 4.8 and Corollary 4.9, while having the similar main (regular) term $R_G$, have a different and more complicated singular contribution than the singular terms given in Theorems 7.8 and 8.1 of [22]. However, in Proposition 5.2 of the present paper, we manage to relate the singular terms from the two different versions to each other (see Remark 4.11). The reader of [22] must therefore consider Theorem 4.8 and Corollary 4.9 of the present paper as correct versions of Theorems 7.8 and 8.1 of [22], and it is most efficient if Sections 4 and 5 of the present paper are sub-

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stituted for Sections 6–9 of [22]. Sections 1–5 of [22], which were the main motivation for this whole project, are precise and correct. We are indebted to Robert Kottwitz for noticing this gap in an earlier version of the present paper, and consequently in [22].

To explain our results, let $\mathbf{G}$ be a symplectic group or a quasi-split special orthogonal group of rank $m$ defined over a $p$-adic field $F$ of characteristic zero. Let $\tau$ and $\tau'$ be discrete series representations of $\mathbf{G}(F)$ and $GL_n(F)$, respectively. We denote by $\tilde{\mathbf{G}}$ either the symplectic or the quasi-split orthogonal group of rank $m+n$ (with the same Witt rank and discriminant as $\mathbf{G}$ in the orthogonal case). Let $I(\tau' \otimes \tau)$ be the representation of $\tilde{\mathbf{G}}(F)$ unitarily induced from the $F$-points of the parabolic subgroup whose Levi component is isomorphic to $GL_n \times \mathbf{G}$. The reducibility of $I(\tau' \otimes \tau)$ is governed by the residue of the standard intertwining operator $A(s, \tau' \otimes \tau, w_0)$ (cf. Section 2), at $s = 0$.

Suppose $n$ is even and $W_n$ is the standard antidiagonal matrix with respect to which one defines the symplectic or split even orthogonal group of rank $n/2$. We let $\varepsilon : GL_n \to GL_n$ be the automorphism given by $\varepsilon(g) = W_n {}^t g^{-1} W_n^{-1}$. In the symplectic case, $\varepsilon = \theta^*$, and in the orthogonal case, $\varepsilon = \theta$, where these automorphisms are defined in [22]. When one attempts to compute the residue of $A(s, \tau' \otimes \tau, w_0)$, one is led to consider a correspondence between certain $\varepsilon$-conjugacy classes of $GL_n(F)$ and conjugacy classes of $\mathbf{G}(F)$ (see Section 3 and equation (4.3)). We call this the $\varepsilon$-norm correspondence. By Lemma 3.11 and Corollary 3.13, the norm correspondence is surjective whenever $n \geqslant 2m$. If $n < 2m$, then the image of the correspondence contains no regular elliptic conjugacy classes. We show that the norm correspondence that we define agrees, up to a sign, with the norm map defined by Kottwitz and Shelstad on the set of strongly $\varepsilon$-regular conjugacy classes (cf. Lemma 3.18). That is, if $n = 2m$, and $Y$ is a strongly $\varepsilon$-regular element of $GL_n(F)$, then, at least for almost all $Y$, the norm exhausts all those $\mathbf{G}(F)$ conjugacy classes which are $GL_n(F)$ conjugate to $-Y\varepsilon(Y)$. Since the norm correspondence has finite fibers (cf. Lemma 3.11, Corollary 3.13, and Lemma 4.7), we let $\mathscr{A}$ denote its one to finite inverse, extended by the nonsquare central elements (cf. Section 4). It agrees with the image map $\mathscr{A}_{\mathbf{G}/GL_n}$ of [14] from semisimple conjugacy classes in $\mathbf{G}(F)$ to $\varepsilon$-semisimple $\varepsilon$-conjugacy classes in $GL_n(F)$.

We restrict ourselves to the case where both $\tau$ and $\tau'$ are supercuspidal. There are two terms that appear in the residue of $A(s, \tau' \otimes \tau, w_0)$. The first we call the regular or main term, and is that associated to the regular elliptic conjugacy classes in $\mathbf{G}(F)$. If $n < 2m$, then this term does not occur. Suppose that $f_\tau$ is a matrix coefficient of $\tau$ and that $f' \in C_c^\infty(GL_n(F))$ defines a matrix coefficient of $\tau'$ by descent. Then the regular term is given by

$$R_{\mathbf{G}}(f_\tau, f') = \sum_{\{\mathbf{T}_i\}} \mu(T_i) |W(T_i)|^{-1} \int_{T_i} \Phi_\varepsilon(\mathscr{A}(\{\gamma\}), f') \Phi(\{\gamma\}, f_\tau) |D(\gamma)| \, d\gamma,$$

where the sum is over the conjugacy classes of elliptic tori in $\mathbf{G}(\ell)$, with $2\ell =$

$\min(n, 2m)$, $\Phi_\varepsilon$ and $\Phi$ denote $\varepsilon$-twisted and ordinary orbital integrals, respectively, $\mu(T_i)$ is the measure of $T_i = \mathbf{T}_i(F)$, and $D(\gamma)$ is the usual discriminant of Harish-Chandra [10]. In fact, we should be more precise and say that

$$\Phi_\varepsilon(\mathscr{A}(\{\gamma\}), f') = \sum_{\{\gamma'\} \in \mathscr{A}(\{\gamma\})} \Phi_\varepsilon(\{\gamma'\}, f')\Delta(\{\gamma\}, \{\gamma'\}),$$

where $\Delta(\{\gamma\}, \{\gamma'\})$, is a transfer factor given in Section 4.

The singular term is more complicated and given as a limit

$$R_{\text{sing}}(f_\tau, f') = \sum_i |W(T_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname*{Res}_{s=0} \int_{T_i' \setminus \omega_i} \psi_{\mathscr{A}}(s, \gamma)|D(\gamma)|\, d\gamma$$

in which the sum is over conjugacy classes of *all* Cartan subgroups of $\mathbf{G}(\ell)$, and for each $i$, $\omega_i$ is a compact subset of the regular elements $T_i'$ of $T_i = \mathbf{T}_i(F)$. The function $\psi_{\mathscr{A}}$, in which $\mathscr{A}$ is the image correspondence, is defined in Section 4. The fact that all Cartan subgroups must be included is critically important when $n < 2m$. The main result of this paper (Theorem 4.8 and Corollary 4.9) can be formulated as follows.

THEOREM. *The intertwining operator $A(s, \tau' \otimes \tau, w_0)$ has a pole at $s = 0$ if and only if $cR_{\mathbf{G}} + R_{\text{sing}} \neq 0$ for some data, where $c = (2n \log q)^{-1}$. Consequently, suppose that $\tau' \simeq \tilde{\tau}'$.*

(a) *The induced representation $I(\tau' \otimes \tau)$ is irreducible if and only if $cR_{\mathbf{G}} + R_{\text{sing}} \neq 0$.*

(b) *Assume $\tau$ is generic, that is, has a Whittaker model. If $I(\tau' \otimes \tau)$ is irreducible, then $I(s, \tau' \otimes \tau)$, $s \in \mathbb{R}$ is reducible exactly at $s = \pm 1/2$ or $s = \pm 1$, and at only one of these pairs.*

The term $R_{\mathbf{G}}$ can be easily expressed as a pairing between the character of $\tau$ and the $\varepsilon$-twisted character of $\tau'$ (Section 5). Its nonvanishing will then be the reason to call $\tau'$ the "$\varepsilon$-twisted endoscopic transfer" of $\tau$ (Definition 5.1). When $n \geqslant 2m$, the nonvanishing of $R_{\mathbf{G}}$ must be equivalent to the existence of a pole at $s = 0$ for the Rankin-Selberg product $L$-function $L(s, \tau' \times \tau)$ defined in [20].

On the other hand, if $\tau'$ comes from $SO_{n+1}(F)$ by twisted endoscopy as defined in [21], and $R_{\mathbf{G}} \equiv 0$, which we expect to be the case for any $\tau'$ as such, then $R_{\text{sing}} \not\equiv 0$ (Proposition 5.2). Thus, $R_{\text{sing}} \equiv 0$ must imply that $\tau'$ comes from $SO_n^*(F)$ (cf. [21]). Consequently, the two terms $R_{\mathbf{G}}$ and $R_{\text{sing}}$ pretty much separate the poles of the two $L$-functions $L(s, \tau' \times \tau)$ and $L(s, \tau', \wedge^2 \rho_n)$, respectively.

When $n < 2m$, $R_{\mathbf{G}} \equiv 0$, and therefore $R_{\text{sing}}$ controls the poles of both $L$-functions, and distinguishing them requires further analysis. But in all cases we have the following weaker result (Proposition 5.3) in which we may and do assume $\tau' \simeq \tilde{\tau}'$:

PROPOSITION (a) *Suppose $\tau'$ comes from $SO_{n+1}(F)$ (nonvanishing condition (5.2) of [21]), or, equivalently, $L(s, \tau', \wedge^2 \rho_n)$ has a pole at $s = 0$. Then $L(s, \tau' \times \tau)$ is holomorphic at $s = 0$.*

(b) *Otherwise, that is, if $\tau'$ does not come from $SO_{n+1}(F)$, then $L(s, \tau' \times \tau)$ has a pole at $s = 0$ if and only if $cR_G + R_{\text{sing}} \neq 0$ for some data.*

Finally, we would like to thank Laurent Clozel and Robert Kottwitz for their suggestions, comments, and interest.

**§2. Preliminaries.** Let $F$ be a $p$-adic field of characteristic zero. For an integer $l > 0$, let

$$w_l = \begin{pmatrix} & & & & 1 \\ & & & \cdot & \\ & & \cdot & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \in M_l(F),$$

and

$$u_l = \begin{pmatrix} & & & & -1 \\ & & & \cdot & \\ & & \cdot & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix} \in M_l(F).$$

Let $J_{2r}$ be a quasi-split symmetric or symplectic form of dimension $2r$, defined over $F$. If $J_{2r}$ is symmetric and $V \simeq F^k$ is the maximal anisotropic subspace of $F^{2r}$ under $J_{2r}$, then we assume that

$$J_{2r} = \begin{pmatrix} & & w_l \\ & \Lambda_k & \\ w_l & & \end{pmatrix},$$

with $2r = 2l + k$, and $\Lambda_k$ an anisotropic form of dimension $k$. In the symplectic case we take $J_{2r} = u_{2r}$. For $X \in M_n(F)$, we let $\tilde{\theta}(X) = w_n {}^tX w_n^{-1}$. If $Y \in GL_n(F)$, then we set $\theta(Y) = \tilde{\theta}(Y^{-1})$. Similarly, we let $\tilde{\theta}^*(X) = u_n {}^tX u_n^{-1}$, and $\theta^*(Y) = \tilde{\theta}^*(Y^{-1})$. We set

$$\tilde{\varepsilon}(X) = \begin{cases} \tilde{\theta}^*(X), & \text{if } J_{2r} \text{ is symplectic,} \\ \tilde{\theta}(X), & \text{otherwise,} \end{cases}$$

and write $\varepsilon(Y)$ for $\tilde{\varepsilon}(Y^{-1})$.

Let $\tilde{\mathbf{G}}$ be the special orthogonal, or symplectic group defined with respect to $J_{2r}$. Thus, $\tilde{\mathbf{G}} = \{g \in GL_{2r} | {}^tgJ_{2r}g = J_{2r}\}^\circ$, with the superscript indicating the con-

nected component. In the symmetric case, we assume that $k > 0$, since $k = 0$ is dealt with in [22]. (Note that this implies that $k = 2$, and $\tilde{\mathbf{G}}$ is then determined by the discriminant of the associated symmetric form (see [18]).) If $\mathbf{G}$ is orthogonal, then we let $E$ be the quadratic extension of $F$ over which $\mathbf{G}$ splits.

If $J_{2r}$ is symplectic, let $\mathbf{T}$ be the maximal split torus of diagonal elements in $\tilde{\mathbf{G}}$. If $J_{2r}$ is symmetric, then we take

$$\mathbf{T} = \left\{ \begin{pmatrix} x_1 & & & & & & & & \\ & x_2 & & & & & & & \\ & & \ddots & & & & & & \\ & & & x_l & & & & & \\ & & & & X & & & & \\ & & & & & x_l^{-1} & & & \\ & & & & & & \ddots & & \\ & & & & & & & x_2^{-1} & \\ & & & & & & & & x_1^{-1} \end{pmatrix} | x_i \in \mathbf{G}_m, X \in SO(\Lambda_2) \right\}.$$

Let $\mathbf{T}_d$ be the maximal split subtorus. Thus, in the symmetric case, $\mathbf{T}_d$ is the collection of elements of $\mathbf{T}$ with $X = I_2$. Let $\mathbf{B} = \mathbf{TU}$ be the Borel subgroup, with $\mathbf{U}$ the collection of upper triangular unipotent matrices in $\tilde{\mathbf{G}}$. Let $\Delta$ be the set of simple roots of $\mathbf{T}$ in $\mathbf{U}$, and suppose that $\Theta = \Delta \backslash \{e_n - e_{n+1}\}$. Let

$$\mathbf{A} = \mathbf{A}_\Theta = \left( \bigcap_{\alpha \in \Theta} \ker \chi_\alpha \right)^\circ,$$

where $\chi_\alpha$ is the root character attached to $\alpha$. Then

$$\mathbf{A} = \left\{ \begin{pmatrix} xI_n & & \\ & I_{2m} & \\ & & x^{-1}I_n \end{pmatrix} | x \in \mathbf{G}_m \right\}.$$

We assume that $n$ is even. If $\mathbf{M} = Z_{\mathbf{G}}(\mathbf{A})$, then $\mathbf{M} \simeq GL_n \times \mathbf{G}(m)$, where

$$\mathbf{G}(m) = \begin{cases} Sp_{2m}, & \text{if } \tilde{\mathbf{G}} = Sp_{2r}, \\ SO_{J_{2m}}, & \text{if } \tilde{\mathbf{G}} = SO_{J_{2r}}. \end{cases}$$

In fact,

$$\mathbf{M} = \left\{ \begin{pmatrix} g & & \\ & h & \\ & & \varepsilon(g) \end{pmatrix} | g \in GL_n, h \in \mathbf{G}(m) \right\}.$$

Let $\mathbf{N} = \prod_{\alpha \in \Phi^+ \setminus \Sigma^+(\Theta)} \mathbf{N}_\alpha$, where

$$\Sigma^+(\Theta) = \left\{ \beta \in \Phi^+ | \beta = \sum_{\Theta} c_\gamma \gamma, c_\gamma \geqslant 0 \right\}.$$

We set $\mathbf{P} = \mathbf{MN}$. Let $P = \mathbf{P}(F), M = \mathbf{M}(F)$, and $N = \mathbf{N}(F)$. We are interested in determining when $\mathrm{Ind}_P^{\tilde{G}}(\pi)$ is reducible, if $\pi$ is a discrete series representation of $M$. In order to do this, we study the poles of the standard intertwining operators that arise, using the theory developed by Harish-Chandra [24]. We begin by looking at the structure of $N$ to determine the types of integrals that need to be evaluated. The proof of the following lemma follows from a straightforward matrix computation.

LEMMA 2.1. *Suppose that* $n \in \mathbf{N}$, *and* $n = \begin{pmatrix} I_n & X & Y \\ 0 & I_{2m} & X' \\ 0 & 0 & I_n \end{pmatrix}$. *Then we have*

$$X' = \begin{cases} u_{2m}{}^t X u_n, & \text{if } \tilde{\mathbf{G}} = Sp_{2r}; \\ -J_{2m}{}^t X w_n, & \text{if } \tilde{\mathbf{G}} = SO_{J_{2r}}, \end{cases}$$

*and*

$$Y + \tilde{\varepsilon}(Y) = XX'. \quad \square \tag{2.1}$$

*Remark.* If $n \in \mathbf{N}$ is as above, then we may denote $n$ by $n(X, Y)$.

Let $\mathbf{G}' = GL_n$. For $\delta \in G'$, the set

$$\{g^{-1}\delta\varepsilon(g) | g \in G')\}$$

is called the $\varepsilon$-conjugacy class of $\delta$ in $GL_n(F)$. Furthermore, the group

$$\mathbf{G}'_{\varepsilon,\delta} = \{g \in \mathbf{G}' | g^{-1}\delta\varepsilon(g) = \delta\}$$

is called the $\varepsilon$-twisted centralizer of $\delta$ in $\mathbf{G}'$. We let $\mathscr{C}'$ be the set of $\varepsilon$-conjugacy classes in $G'$, and denote the conjugacy classes in $G(m)$ by $\mathscr{C}$.

Let

$$g_\theta = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix} \in GL_n(F).$$

Note that $u_n = g_\theta w_n$. If $n = 2m$, we rewrite (2.1) as

$$-Yg_\theta + \tilde{\theta}(Yg_\theta) = Xg_\theta\tilde{\theta}(X) \tag{2.2}$$

if $\tilde{\mathbf{G}} = Sp_{2r}$, and

$$Y + \tilde{\theta}(Y) = X(-J_n w_n)\tilde{\theta}(X) \tag{2.3}$$

if $\tilde{\mathbf{G}} = SO_{J_{2r}}$.

Let $\mathcal{N}$ be the collection of $\varepsilon$-conjugacy classes in $GL_n(F)$ for which (2.1) has $F$-rational solutions. Then (2.2) shows that, if $\tilde{\mathbf{G}} = Sp_{2r}$, then

$$\mathcal{N} = \{\{Y\} \mid -Yg_\theta + \tilde{\theta}(Yg_\theta) \sim g_\theta\},$$

where the equivalence is that of $\tilde{\theta}$-skew symmetric forms. Since $-Yg_\theta + \tilde{\theta}(Yg_\theta)$ is singular only on a set of measure zero in $GL_n$, and there is a unique equivalence class of nondegenerate $\tilde{\theta}$-skew symmetric forms, we see that $\mathcal{N}$ is almost all of $\mathcal{C}'$. Similarly, (2.3) shows that if $\tilde{\mathbf{G}} = SO_{J_{2r}}$ then, up to a set of measure zero, $\mathcal{N}$ is the set of $\theta$-conjugacy classes for which $Y + \tilde{\theta}(Y)$ is equivalent to $-J_n w_n$ as $\tilde{\theta}$-symmetric forms.

Note that if $\{Y\} \in \mathcal{N}$, and $(X, Y)$ is a solution to (2.1), then

$$Y^{-1} + \tilde{\varepsilon}(Y^{-1}) = Y^{-1}XX'\tilde{\varepsilon}(Y)^{-1} = (Y^{-1}X)(Y^{-1}X)'.$$

Therefore, $\mathcal{N}$ is closed under $\{Y\} \mapsto \{Y^{-1}\}$.

Let $\tilde{w}_0$ be the nontrivial class in the Weyl group $W(\tilde{\mathbf{G}}, \mathbf{A})$. Then $\tilde{w}_0$ is represented by

$$w_0 = \begin{pmatrix} & & I_n \\ & I_{2m} & \\ I_n & & \end{pmatrix}.$$

The actions of $w_0$ on $\mathbf{M}$ and $\mathbf{A}$ are $(g, h) \mapsto (\varepsilon(g), h)$, and $x \mapsto x^{-1}$, respectively.

As in the work of Harish-Chandra, we have the following notation. For a connected reductive $p$-adic group $G$, we let $\mathcal{E}_c(G)$ be the collection of equivalence classes of irreducible admissible representations of $G$. We denote by $\mathcal{E}(G)$ those classes in $\mathcal{E}_c(G)$ which are unitarizable. We let $\mathcal{E}_2(G)$ be the discrete series classes. We use ${}^\circ\mathcal{E}_c(G)$ to denote the supercuspidal classes, and set ${}^\circ\mathcal{E}(G) = \mathcal{E}(G) \cap {}^\circ\mathcal{E}_c(G)$.

Let $(\tau', V') \in {}^\circ\mathcal{E}(G')$ and $(\tau, V) \in {}^\circ\mathcal{E}(G(m))$. Then $\tau' \otimes \tau$ is a unitary supercuspidal representation of $M$. Let

$$I(s, \tau' \otimes \tau) = \mathrm{Ind}_{MN}^{\tilde{G}}((\tau' \otimes |\det(\ )|^s) \otimes \tau \otimes \mathbf{1}_N).$$

We denote the space of $I(s, \tau' \otimes \tau)$ by $V(s, \tau' \otimes \tau)$. To understand the reducibility of $I(\tau' \otimes \tau) = I(0, \tau' \otimes \tau)$, one must determine the poles of the standard intertwining operator

$$A(s, \tau' \otimes \tau, w_0)f(g) = \int_N f(w_0^{-1}ng)\, dn$$

associated to $\tau' \otimes \tau$ (cf. [24]). Here $f \in V(s, \tau' \otimes \tau)$. By Bruhat's theorem (see [11]) we may assume that $w_0(\tau' \otimes \tau) \simeq \tau' \otimes \tau$. This is equivalent to assuming $\tau' \simeq \tilde{\tau}'$ [4, §7].

Suppose that ${}^LM$ is the $L$-group of $\mathbf{M}$ and ${}^L\mathfrak{n}$ is the Lie algebra of ${}^LN$ (see [5]). If $r$ is the adjoint representation of ${}^LM$ on ${}^L\mathfrak{n}$, then $r = r_1 \oplus r_2$, where $r_2 = \wedge^2\rho_n$, with $\rho_n$ the standard representation of $GL_n(\mathbb{C})$, and $r_1$ is the tensor product of $\rho_n$ with the standard representation of the $L$-group of $\mathbf{G}(m)$. Moreover, the labeling of these representations as $r_1$ and $r_2$ is consistent with the ordering given in [19]. Consequently, the results of [20] imply that if $\tau$ is generic, then the poles of $A(s, \tau' \otimes \tau, w_0)$ are the same as the poles of

$$L(s, \tau' \times \tau)L(2s, \tau', \wedge^2\rho_n),$$

where $L(s, \tau' \times \tau) = L(s, \tau' \otimes \tau, r_1)$. Furthermore, at most one of the two $L$-functions in the product can have a pole at $s = 0$ (cf. [20, Thm. 7.6]). The poles of $L(s, \tau', \wedge^2\rho_n)$ were determined by Shahidi in [21]. Thus, our work here determines which representations $\tau'$ give rise to a pole of $L(s, \tau' \times \tau)$ at $s = 0$ for some $\tau$. As explained in [20], this completely determines $L(s, \tau' \times \tau)$ if $\tau$ and $\tau'$ are supercuspidal. The method of [20], as used in [21] and [8], can then be used to determine $L(s, \tau' \times \tau)$ for any pair of irreducible admissible representations $\tau$ and $\tau'$.

Denote by $\overline{\mathbf{N}}$ the unipotent radical opposed to $\mathbf{N}$. We let

$$V(s, \tau' \otimes \tau)_0 = \{h \in V(s, \tau' \otimes \tau) | \text{supp } h \subset \bar{N} \text{ modulo } P\}.$$

By a lemma of Rallis [21], it is enough to compute the poles that arise when $A(s, \tau' \otimes \tau, w_0)$ is applied to functions in $V(s, \tau' \otimes \tau)_0$ and evaluated at the identity. Thus, we note the following result.

LEMMA 2.2. *Let $\overline{\mathbf{N}}$ be the unipotent radical opposed to $\mathbf{N}$. If $n = n(X, Y) \in N$, then $w_0^{-1}n \in P\bar{N}$ if and only if $Y \in GL_n(F)$, in which case*

$$w_0^{-1}n = \begin{pmatrix} \varepsilon(Y) & -Y^{-1}X & I_n \\ 0 & I_{2m} - X'Y^{-1}X & X' \\ 0 & 0 & Y \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 \\ (Y^{-1}X)' & I_{2m} & 0 \\ Y^{-1} & Y^{-1}X & I_n \end{pmatrix}. \quad \square$$

We note, for future reference, that $(Y^{-1}X)' = X'\varepsilon(Y)$.

The proof of the following result is computational and is based on equation (2.1).

LEMMA 2.3. *Suppose that* $(X, Y)$ *is an F-rational solution to (2.1). Then* $I_{2m} - X'Y^{-1}X \in G(m)$. □

Let $h \in V(s, \tau' \otimes \tau)_0$. Fix open compact subsets $L \subset M_n(F)$ and $L' \subset M_{n\times 2m}(F)$. We assume that, for some $v' \in V'$, $v \in V$, we have

$$h\left(\begin{pmatrix} I & 0 & 0 \\ X'\varepsilon(Y) & I & 0 \\ Y^{-1} & Y^{-1}X & I \end{pmatrix}\right) = \xi_L(Y^{-1})\xi_{L'}(Y^{-1}X)(v' \otimes v),$$

where $\xi_L$ and $\xi_{L'}$ are the characteristic functions of $L$ and $L'$, respectively. Choose $\tilde{v}' \in \tilde{V}'$ and $\tilde{v} \in \tilde{V}$. Let $\psi_{\tau'}$ and $f_\tau$ be the matrix coefficients of $\tau'$ and $\tau$ given by the pairs $(v', \tilde{v}')$ and $(v, \tilde{v})$, respectively. Then, from Lemma 2.2, $\langle \tilde{v}' \otimes \tilde{v}, A(s, \tau' \otimes \tau, w_0)h(e)\rangle$ is equal to

$$\int_{(Y,X)} \psi_{\tau'}(\varepsilon(Y))f_\tau(I - X'Y^{-1}X)|\det Y|^{-s-\langle \rho_{\mathbf{P}}, \tilde{\alpha}\rangle}\xi(X, Y)\, d(X, Y), \tag{2.4}$$

where the integral is over the collection of $F$-rational solutions $(Y, X)$ of (2.1). Here, $\rho_{\mathbf{P}} = (1/2)\sum_{\alpha\in\Phi^+\setminus\Sigma^+(\Theta)} \alpha$, $\tilde{\alpha}$ is as in [20], $\xi(X, Y) = \xi_L(Y^{-1})\xi_{L'}(Y^{-1}X)$, and $d(X, Y)$ is a choice of Haar measure on $N$.

**§3. The norm correspondence and its consequences.** In this section, we study the properties of the map $(X, Y) \mapsto I_{2m} - X'Y^{-1}X$. In particular, we discuss the relation with the norm map defined by Kottwitz and Shelstad in [14]. The results here have the same flavor as Sections 4 and 5 of [22]. We let

$$\mathbf{G}^\vee(m) = \begin{cases} G(m), & \text{if } \tilde{\mathbf{G}} = Sp_{2r}; \\ O_{J_{2m}}(F), & \text{if } \tilde{\mathbf{G}} = SO_{J_{2r}}. \end{cases}$$

LEMMA 3.1. *Suppose* $(X, Y)$ *is a rational solution to (2.1). Further suppose that* $Xg(Xg)' = XX'$, *for some* $g \in GL_{2m}(F)$. *Then* $Xg = Xh$ *for some* $h \in G^\vee(m)$.

*Proof.* Consider $F^n$ and $F^{2m}$ as row vectors, and let $U = F^nX \subset F^{2m}$. Let $\langle\ ,\ \rangle$ be the symmetric (or symplectic) form given on $F^{2m}$ by $J_{2m}$. Set

$$J'_{2m} = \begin{cases} J_{2m} = u_{2m}, & \text{if } \tilde{\mathbf{G}} = Sp_{2r}; \\ -J_{2m}, & \text{if } \tilde{\mathbf{G}} = SO_{J_{2r}}, \end{cases}$$

and

$$W_n = \begin{cases} u_n & \text{if } \tilde{\mathbf{G}} = Sp_{2r}; \\ w_n & \text{if } \tilde{\mathbf{G}} = SO_{J_{2r}}. \end{cases}$$

Note that in each case, $X' = J'_{2m}{}^t X W_n$. Since $(Xg)(Xg)' = XX'$, we have $XgJ'_{2m}{}^t g^t X W_n = XJ'_{2m}{}^t X W_n$, which we rewrite as $XgJ'_{2m}{}^t gX = XJ'_{2m}{}^t X$. Let $g^* : U \hookrightarrow F^{2m}$ be given by $g^*(v) = vg$. Now, suppose that $v_1, v_2 \in U$, and $v_i = y_i X$, with $y_i \in F^n$. Then

$$\langle g^*(v_1), g^*(v_2)\rangle = \langle v_1 g, v_2 g\rangle = v_1 g J'_{2m}{}^t g^t v_2 = y_1 X g J'_{2m}{}^t g^t X y_2$$

$$= y_1 X J'_{2m}{}^t X y_2 = \langle v_1, v_2\rangle.$$

Therefore, $g^*$ is an isometry onto a subspace of $F^{2m}$. Consequently, by Witt's theorem for symmetric and symplectic forms [17], we can choose $h \in G^{\vee}(m)$, with $vg = vh$ for all $v \in U$. Now, for each $y \in F^n$, we have $yXg = yXh$, and so $Xg = Xh$. □

Let $\{Y^{-1}\}$ be the $\varepsilon$-conjugacy class of $Y^{-1}$ in $GL_n(F)$. Suppose that $(X, Y)$ satisfies (2.1), and $g \in GL_n(F)$. Let $W_n$ and $J'_{2m}$ be as in the proof of Lemma 3.1. Then $(gX)(gX)' = gXJ'_{2m}{}^t X^t g W_n = gXX'\varepsilon(g^{-1})$. Note that

$$gY\varepsilon(g^{-1}) + \tilde{\varepsilon}(gY\varepsilon(g^{-1})) = gY\varepsilon(g)^{-1} + W_n^{-1\,t}(gYW_n^{-1\,t} g W_n) W_n$$

$$= gY\varepsilon(g)^{-1} + g\tilde{\varepsilon}(Y)\varepsilon(g)^{-1} = (gX)(gX)'.$$

Thus, the orbits $\{X\} \in GL_n(F)\backslash M_{n\times 2m}(F)$ parameterize the $\varepsilon$-conjugacy classes for which (2.1) has an $F$-rational solution.

Now suppose that $\{X\} \in GL_n(F)\backslash M_{n\times 2m}(F)$ parameterizes $\{Y^{-1}\} \in \mathcal{N}$. If we replace $X$ with $gX$, then $I - X'Y^{-1}X$ is unchanged if we replace $Y^{-1}$ with $\varepsilon(g)^{-1}Y^{-1}g$, which is also in $\{Y^{-1}\}$.

If $X$ is changed to $X_1 = gX$, with $g \in GL_n(F)$, we then say that $\{X_1\}$ parameterizes $\{Y_1^{-1}\}$, rather than $\{X_1\}$ parameterizes $\{Y^{-1}\}$, to point out that $(X_1, Y_1)$ satisfies (2.1), although the classes are the same.

Now suppose $\{X_1\}$, $X_1 = Xh$, $h \in GL_{2m}(F)$, also parameterizes $\{Y^{-1}\}$. Then by Lemma 3.1, $h \in G^{\vee}(m)$ and

$$I - X_1' Y^{-1} X_1 = I - (Xh)' Y^{-1} Xh = I - (J'_{2m}{}^t h^t X W_n) Y^{-1} Xh$$

$$= I - h^{-1} J'_{2m}{}^t X W_n Y^{-1} Xh = h^{-1}(I - X'Y^{-1}X)h.$$

Thus, the conjugacy class of $I - X'Y^{-1}X$ in $G^{\vee}(m)$ is unchanged.

LEMMA 3.2. *Suppose* $\{X_1\}$, $X_1 = gXh$, $g \in GL_n(F)$, $h \in GL_{2m}(F)$, *parameterizes* $\{Y_1^{-1}\}$, $Y_1 = gY\varepsilon(g)^{-1}$. *Then* $X_1 \in GL_n(F)XG^{\vee}(m)$. □

We denote the set of all conjugacy classes $\{I - X'Y^{-1}X\}$ for all possible $X$ by $N_\varepsilon(\{Y^{-1}\})$. By Lemma 3.2 it defines a one-to-finite correspondence.

LEMMA 3.3. *Suppose that* $n = n(X, Y) \in N$, *with* $Y$ *invertible. Then*
(a) $(I - X'Y^{-1}X)X' = -X'Y^{-1}\varepsilon(Y^{-1})$, *and*
(b) $X(I - X'Y^{-1}X) = -\varepsilon(Y^{-1})Y^{-1}X$.

*Proof.* By Lemma 2.2,

$$\begin{pmatrix} 0 & 0 & I \\ 0 & I & X' \\ I & X & Y \end{pmatrix} = \begin{pmatrix} \varepsilon(Y) & -Y^{-1}X & I \\ 0 & I - X'Y^{-1}X & X' \\ 0 & 0 & Y \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ X'\varepsilon(Y) & I & 0 \\ Y^{-1} & Y^{-1}X & I \end{pmatrix}.$$

Comparing the (2,1)-entries, we see that

$$0 = (I - X'Y^{-1}X)X'\varepsilon(Y) + X'Y^{-1},$$

or

$$(I - X'Y^{-1}X)X'\varepsilon(Y) = -X'Y^{-1},$$

which gives us (a). We can rewrite this as

$$(I - X'Y^{-1}X)J'_{2m}{}^{t}XW_n = -J'_{2m}{}^{t}XW_nY^{-1}W_n{}^{t}YW_n^{-1},$$

and thus,

$$J_{2m}^{-1}(I - X'Y^{-1}X)J_{2m}{}^{t}X = -{}^{t}X\varepsilon({}^{t}Y){}^{t}Y.$$

Since $I - X'Y^{-1}X \in G(m)$, we have

$${}^{t}(I - X'Y^{-1}X)^{-1}{}^{t}X = -{}^{t}X\varepsilon({}^{t}Y){}^{t}Y,$$

or

$$X(I - X'Y^{-1}X)^{-1} = -Y\varepsilon(Y)X,$$

which proves (b). □

LEMMA 3.4 *Suppose that* $X \in M_{n\times 2m}(F)$ *and* $U = F^nX$. *Let*

$$H_X = \{h \in G(m) \mid Xh = g_hX \text{ for some } g_h \in GL_n(F)\}.$$

*If* $\{0\} \subsetneq U \subsetneq F^{2m}$, *then* $H_X \subsetneq G(m)$.

*Proof.* Suppose that $h \in H_X$. Let $u \in U$, and choose $v \in F^n$ with $u = vX$. Then $uh = vXh = vg_hX \in U$. Thus, $U$ is an $H_X$-invariant subspace of $F^{2m}$. If $H_X = G(m)$, then we know that $\{0\}$ and $F^{2m}$ are the only invariant subspaces. □

LEMMA 3.5. *Suppose that $n$ is even, and $\{X\} \in GL_n(F)\backslash M_{n\times 2m}(F)/G^\vee(m)$. If $(X, Y)$ is a rational solution to (2.1), with $Y$ invertible, then*

$$X(I - X'Y^{-1}X) = -\varepsilon(Y^{-1})Y^{-1}X,$$

*and $I - X'Y^{-1}X \in H_X$.*

*Proof.* That $X(I - X'Y^{-1}X) = -\varepsilon(Y^{-1})Y^{-1}X$ follows from Lemma 3.3. Since $X(I - X'Y^{-1}X) = gX$, for some $g \in GL_n(F)$, we see that $I - X'Y^{-1}X \in H_X$. □

LEMMA 3.6. *Fix $X$ in $M_{n\times 2m}(F)$, and let $U = F^nX$. We consider $F^{2m}$ as a symplectic or symmetric space with respect to the form $J_{2m}$. If $U$ is nondegenerate, then $H'_X$, the right stabilizer of $U$ in $G(m)$, is the stabilizer of an involution of $G^\vee(m)$. If $U$ is degenerate, then $H'_X$ is contained in a proper parabolic subgroup of $G(m)$.*

*Proof.* If $U$ is nondegenerate, then Lemma 42.4 of [17] implies that $F^{2m} = U \perp W$. Of course, $W = U^\perp$. Since $Uh = U$ and $h \in G(m)$, we know $h$ stabilizes $W$ as well. Therefore, $h$ centralizes $-1_U \perp 1_W$. Conversely, suppose $h \in \mathrm{Stab}_{-1_U\perp 1_W} \cap G(m)$. For each $u \in U$, we let $uh = u_1 + w_1$, be the decomposition of $u$ with respect to $U$ and $W$. Then $u(-1_U \perp 1_W)h = -uh = -u_1 - w_1$, while $uh(-1_U \perp 1_W) = -u_1 + w_1$. Thus, $w_1 = 0$, and so $Uh = U$; that is, $h \in H'_X$.

If $U$ is degenerate, then we let $U = W \perp \mathrm{Rad}\, U$. If $u \in \mathrm{Rad}\, U$, then, for all $u_1 \in U$, we have $\langle uh, u_1\rangle = \langle u, u_1h\rangle = 0$, so $h$ stabilizes $\mathrm{Rad}\, U$. Since $\mathrm{Rad}\, U$ is totally isotropic, $H'_X \subset P_X = M_XN_X$, where $P_X$ is the parabolic subgroup stabilizing precisely $\mathrm{Rad}\, U$. □

PROPOSITION 3.7. *Suppose that $n < 2m$, and take $X \in M_{n\times 2m}(F)$. Fix an invertible $Y$ with $(X, Y)$ an $F$-rational solution to (2.1). Then $I - X'Y^{-1}X$ belongs to a proper parabolic subgroup, or a proper centralizer of a singular elliptic element of $G^\vee(m)$. Moreover, $\{N_\varepsilon(Y)\}$ is never regular elliptic.*

*Proof.* Since $n < 2m$, we have $U = F^nX \subsetneq F^{2m}$, and the first result follows from Lemmas 3.3(b) and 3.6. The last assertion is a consequence of $I - X'Y^{-1}X$ having at least $2m - n$ eigenvalues equal to 1. □

LEMMA 3.8. *Suppose that $I + S \in G(n/2)$. Then there is some $Y \in GL_n(F)$, and a projection $X \in M_n(F)$ with*

$$S = -X'Y^{-1}X = -X'Y^{-1} = \begin{cases} Y^{-1}X, & \text{if } \tilde{\mathbf{G}} = Sp_{2r}; \\ J_nw_nY^{-1}X, & \text{if } \tilde{\mathbf{G}} = SO_{J_{2r}}. \end{cases}$$

*Proof.* First suppose that $J_{2r}$ is symmetric. For $S \in M_n(F)$, we let $S^\vee = J_n{}^t S J_n^{-1}$. Then $S \mapsto S^\vee$ is an anti-involution. Suppose $I + S \in SO_{J_n}$. Then

$$(I+S)J_n{}^t(I+S) = J_n,$$

so

$$SJ_n + J_n{}^t S + SJ_n{}^t S = 0.$$

Therefore,

$$S + S^\vee + SS^\vee = 0,$$

or

$$S^\vee = -(I+S)^{-1}S.$$

Now, applying Lemma 5.6 of [22], with $A = -(I+S)^{-1}$, we can find $Y_1 \in GL_n(F)$, and a projection $X \in M_n(F)$, with $S = X^\vee Y_1^{-1} X = X^\vee Y_1^{-1} = Y_1^{-1}X$. Let $Y = Y_1 J_n w_n^{-1}$. Then $S = -X'Y^{-1}X = -X'Y^{-1}$, and $S = Y_1^{-1}X = J_n w_n Y^{-1} X$.

Now suppose that $J_{2r}$ is symplectic. Then we repeat the above argument with $S^\vee = u_n{}^t S u_n = -\tilde{\theta}^*(S)$, to get the result of the lemma. □

DEFINITION 3.9. *Suppose the pair $(Y, X)$ satisfies the hypotheses of Lemma 3.8. Then $\{Y^{-1}\}$ is called the canonical section of the norm correspondence over $\{Z\}$, where $Z = I - X'Y^{-1}X$* (cf. [22]).

*Remark.* We have called this the canonical section since it is the natural extension of the unique section defined at every *generic* point $Z$, that is, those $Z$ for which $X = I$, through $I + Y^{-1} = Z$. Observe that this applies to almost all $Z$.

LEMMA 3.10. *Suppose that $\tilde{\mathbf{G}} = SO_{J_{2r}}$ and $g \in G(n)$. Then the dimension of the $g$-fixed vectors in $F^{2n}$ is even.*

*Proof.* We know that $\mathbf{G} = \mathbf{G}(n)$ splits over a quadratic extension $E/F$. Thus, $\mathbf{G}(E) \simeq SO_{2n}(E)$. By Lemma 5.8 of [22], we know that the dimension of the fixed-point set of $g$ in $E^{2n}$ is even. Suppose that $V$ is the fixed-point set of $g$ in $F^{2n}$. Let $v \in E^{2n}$, with $gv = v$. We write $v = \sum c_i v_i$, with $v_i \in F^{2n}$, and $c_i \in E$. We may assume that $\{v_i\}$ and $\{c_i\}$ are linearly independent over $F$. Extending $\{v_i\}$ to an $F$-basis of $F^{2n}$, we write $gv_i = \sum \alpha_{ij} v_j$, with $\alpha_{ij} \in F$. Since $gv = \sum c_i g v_i = v$, we see that $\sum c_i v_i = \sum\sum c_j \alpha_{ji} v_j$. Comparing coefficients, and using the linear independence of $\{c_i\}$ over $F$, we see that $(\alpha_{ij})$ is the identity matrix, that is, $v_i \in V$, for each $i$. Thus, $\dim_F(V) = \dim_E(V \otimes_F E)$, and this last is the dimension of the $g$ fixed points in $E^{2n}$, and hence is even. □

LEMMA 3.11. *Suppose that $n = 2m$. Then $N_\varepsilon : \mathcal{N} \to \mathcal{C}$ is surjective with finite fibers.*

*Proof.* First suppose that $J_n$ is symmetric. Suppose $I - S \in G(m)$, and choose a projection $X \in M_n(F)$ with $S = X'Y^{-1}X = X'Y^{-1} = -J_n w_n Y^{-1}X$, as in Lemma 3.8.

Now, we have

$$(I + J_n w_n Y^{-1}X)J_n(I - {}^tX^tY^tX') = J_n,$$

which implies

$$J_n w_n Y^{-1}XJ_n - J_n {}^tX^tY^{-1t}X' = J_n w_n Y^{-1}XJ_n {}^tX^tY^{-1t}X'.$$

Thus

$$Y^{-1}X + \tilde{\theta}(Y^{-1})X = -Y^{-1}XJ_n {}^tX^tY^{-1}w_nX = Y^{-1}XX'\tilde{\theta}(Y^{-1})X.$$

Note that if $v$ is in the image of $X$, then $v = Xv$, so $Y^{-1}v + \tilde{\theta}(Y^{-1})v = Y^{-1}XX'\tilde{\theta}(Y^{-1})v$, or $(Y + \tilde{\theta}(Y))v = XX'v$.

So we need to show that we can choose $(X, Y)$ satisfying (2.1) on $\ker X$. Since $\ker X \subset \ker(Y^{-1}XX'\tilde{\theta}(Y^{-1})X) = \ker(Y^{-1}XX'\tilde{\theta}(Y^{-1}))$, we need to show that we can choose $(Y, X)$ so that $Y + \tilde{\theta}(Y) = 0$ on $\ker X$. We know that $J_n w_n Y^{-1}X = -X'Y^{-1}$, and $J_n w_n \tilde{\theta}(Y^{-1})X = -X'\tilde{\theta}(Y^{-1})X$. Note that $\tilde{\theta}(X)$ is also a projection, and $\tilde{\theta}(X) = -w_n J_n^{-1}X'$. Thus, both $Y^{-1}|_{\ker X}$ and $\tilde{\theta}(Y^{-1})|_{\ker X}$ are isomorphisms onto $\ker\tilde{\theta}(X)$. (Note that $-\tilde{\theta}(X)$ is the matrix denoted by $X'$ in [22].) For $v \in F^n$, we let $\tilde{\theta}(v) = {}^t(w_n v)$. Note that $\tilde{\theta}(v)\tilde{\theta}(Y^{-1}) = {}^tv^tY^{-1}w_n = \tilde{\theta}(Y^{-1}v)$. Choose bases $\mathcal{B}$ and $\mathcal{B}'$ for $\ker X$ and $\ker\tilde{\theta}(X)$, respectively. Then we see that the matrix of $Y^{-1}|_{\ker X}$ with respect to $\mathcal{B}$, $\mathcal{B}'$, is equal to the matrix of $\tilde{\theta}(Y^{-1})|_{\ker X}$, with respect to the bases $\tilde{\theta}(\mathcal{B})$, and $\tilde{\theta}(\mathcal{B}')$. Note that $\ker X$ is precisely the fixed-point space of $I - X'Y^{-1}X$ in $F^n$, and thus, by Lemma 3.10, we can choose a unique (up to $\theta$-conjugacy) $Y$ with $Y^{-1}|_{\ker X}$ a $\tilde{\theta}$-skew symmetric matrix. For such a choice of $(Y, X)$, we now have $Y^{-1} + \tilde{\theta}(Y^{-1})X = Y^{-1}XX'\tilde{\theta}(Y^{-1})$, and thus (2.1) is satisfied.

Suppose now that $J_n$ is symplectic. If $I - S \in Sp_{2n}(F)$, then we can choose a projection $X \in M_n(F)$ and $Y \in GL_n(F)$ so that $S = X'Y^{-1}X = X'Y^{-1} = -Y^{-1}X$, as in Lemma 3.8. Using an argument similar to the one above, we see that

$$Y^{-1}X + \tilde{\theta}^*(Y^{-1})X = Y^{-1}XX'\tilde{\theta}^*(Y^{-1})X.$$

Then we again have (2.1) on the image of $X$. Note that, in this case, $X' = u_n {}^tXu_n = -\tilde{\theta}^*(X)$, and $-X' = \tilde{\theta}^*(X)$ is a projection. Moreover, $-X'Y^{-1} =$

$Y^{-1}X$, and $-X'\tilde{\theta}^*(Y^{-1}) = \tilde{\theta}^*(Y^{-1})X$. Thus, both $Y^{-1}$ and $\tilde{\theta}^*(Y^{-1})$ carry $\ker X$ isomorphically onto $\ker(-X')$. Thus, the argument reduces to showing that we can choose $Y$ so that

$$Y|_{\ker X} + \tilde{\theta}^*(Y|_{\ker X}) = 0.$$

Note that $\tilde{\theta}^*(Y) = -Y$ if and only if $Yg_\theta w_n = {}^t(Yg_\theta w_n)$. However, on any vector space, one can always find a symmetric invertible transformation. For example, taking $Y|_{\ker X} = (w_n g_\theta)|_{\ker X} = u_n^{-1}|_{\ker X}$, we get the desired element of $GL_n$.

To prove that $N_\varepsilon$ has finite fibers we use the following result.

LEMMA 3.12. *If $(Y, X)$ is an F-rational solution to (2.1), then $N_\varepsilon(\{Y^{-1}\})$ determines the semisimple part of $\{\varepsilon(Y^{-1})Y^{-1}\}$ in $GL_n(F)$ uniquely.*

*Proof.* Changing $Y$ to an $\varepsilon$-conjugate, we can assume that $X$ is a projection. Suppose that $v$ is in the left image $\operatorname{Im} X$ of $X$. Then $vX = v$, so by Lemma 3.3, $v\varepsilon(Y^{-1})Y^{-1}X = -v(I - X'Y^{-1}X)$. Suppose $v$ is in the left kernel $\ker X$ of $X$. Then $v(Y + \tilde{\varepsilon}(Y)) = vXX' = 0$, so $v\varepsilon(Y^{-1})Y^{-1} = -v$. Therefore, the matrix of $\varepsilon(Y^{-1})Y^{-1}$ with respect to a basis of $F^n$ which respects $F^n = \operatorname{Im} X \oplus \ker X$ is (acting on the right)

$$\begin{pmatrix} -(I - X'Y^{-1}X)|_{\operatorname{Im} X} & * \\ 0 & -I \end{pmatrix}.$$

Thus, the semisimple part of $\{\varepsilon(Y^{-1})Y^{-1}\}$ is uniquely determined by $N_\varepsilon(\{Y^{-1}\})$, completing the lemma. □

Now, the finiteness of the fibers is as in [22]. □

COROLLARY 3.13. *If $n > 2m$, then the statement of Lemma 3.11 is true.*

*Proof* Let $j = (n/2) - m$, and consider the injection

$$h \mapsto \begin{pmatrix} I_j & & \\ & h & \\ & & I_j \end{pmatrix}$$

of $\mathbf{G}(m)$ into $\mathbf{G}(n/2)$. By Lemma 3.11, there are $X \in M_n(F)$ and $Y \in GL_n(F)$ with

$$\begin{pmatrix} I_j & & \\ & h & \\ & & I_j \end{pmatrix} = I_n - X'Y^{-1}X.$$

Let $X = (X_1 \quad X_2 \quad X_3)$, with $X_1, X_3 \in M_{n\times j}(F)$, and $X_2 \in M_{n\times 2m}(F)$. Then

$$X' = \begin{pmatrix} X_3'' \\ X_2' \\ X_1'' \end{pmatrix},$$

where $X_i'' = -W_j {}^tX_i W_n^{-1}$, and $W_\ell$ is as in the proof of Lemma 3.1. We now have

$$I - X'Y^{-1}X = I - \begin{pmatrix} X_3''Y^{-1} \\ X_2'Y^{-1} \\ X_1''Y^{-1} \end{pmatrix}(X_1 \quad X_2 \quad X_3).$$

Thus, by inspection, $I_{2m} - X_2'Y^{-1}X_2 = h$. The finiteness of the fiber follows from that of the case $n = 2m$. □

Finally, let $N_\varepsilon^{-1}$ denote the canonical section of the norm correspondence defined before. Then, by Lemma 3.11, $N_\varepsilon^{-1} : \mathscr{C} \to \mathscr{C}'$.

LEMMA 3.14. *The map $N_\varepsilon^{-1} : \mathscr{C} \to \mathscr{C}'$ is continuous.*

*Proof.* The proof is similar to the proof of Lemma 5.16 of [22]. □

Note that $\theta^*$ fixes the standard splitting of $GL_n$, while $\theta$ does not. However, $\theta^* = \mathrm{Int}(g_\theta)\theta$, where $g_\theta = \mathrm{diag}\{1, -1, 1, -1 \ldots, -1\}$. Suppose that $F = \bar{F}$. Let $\mathbf{T}_0'$ be the torus of diagonal matrices in $GL_n$. We define $\bar{N}_{\theta^*}$ by $\bar{N}_{\theta^*}(Y) = Y\theta^*(Y)$, for $Y \in \mathbf{T}_0'$.

If $Y = \mathrm{diag}\{a_1, \ldots, a_n\}$, then $\theta^*(Y) = \mathrm{diag}\{a_n^{-1}, \ldots, a_2^{-1}, a_1^{-1}\}$. Therefore, $Y\theta^*(Y) = \mathrm{diag}\{a_1a_n^{-1}, a_2a_{n-1}^{-1}, \ldots, a_na_1^{-1}\}$. Consequently,

$$\ker \bar{N}_{\theta^*} = \{\mathrm{diag}\{a_1, a_2, \ldots, a_{n/2}, a_{n/2}, \ldots, a_2, a_1\}\}.$$

Let $Y_0 = \mathrm{diag}\{a_1, \ldots, a_{n/2}, 1, 1, \ldots, 1\}$. Then

$$(I - \theta^*)(Y_0) = Y_0\theta^*(Y_0)^{-1} = \mathrm{diag}\{a_1, \ldots, a_{n/2}, a_{n/2}, \ldots, a_2, a_1\} \in \ker \bar{N}_{\theta^*}.$$

Therefore, $\ker \bar{N}_{\theta^*} = (I - \theta^*)\mathbf{T}_0'$. (Of course, this is the same as in [22], since $SO_{J_{2n}}$ is split over $\bar{F}$. For $Sp_{2n}$, it also follows from [22], since $SO_{2n}$ and $Sp_{2n}$ share the diagonal Cartan.)

Now suppose that $F$ is not necessarily algebraically closed. Let $\mathbf{T}_\mathbf{H}$ be a Cartan subgroup of $\mathbf{G}(n/2)$, defined over $F$. Choose a $\theta^*$-stable pair $(\mathbf{B}', \mathbf{T}')$ of $GL_n$, with $\mathbf{T}'$ defined over $F$, such that there is an isomorphism $\mathbf{T}_\mathbf{H} \xrightarrow{\sim} \mathbf{T}_{\theta^*}'$ is defined over $F$ [14].

LEMMA 3.15. *The map* $Y \mapsto Y\theta^*(Y)$ *from* $\mathbf{T}'$ *to* $\mathbf{T}'$ *has*

$$(\mathbf{T}')^{\theta^*} = \{t | \theta^*(t) = t\}$$

*as its image, can be identified with the projection of* $\mathbf{T}'$ *onto* $\mathbf{T}'_{\theta^*}$, *and is defined over* $F$.

*Proof.* The proof follows from the above observations and the argument of Lemma 5.17 of [22]. □

LEMMA 3.16. *Assume* $n = 2m$.

(a) *Suppose that* $F$ *is algebraically closed. Let* $\{Y\} \in \mathcal{N} = \mathcal{C}'$ *be* $\varepsilon$*-semisimple, with* $Y$ *in an* $\varepsilon$*-stable Cartan subgroup of* $GL_n$. *Then, there is an* $X \in M_n(F)$ *that satisfies (2.1) with* $Y$ *so that* $I - X'Y^{-1}X$ *is semisimple in* $G(n/2)$. *Moreover,* $\varepsilon(Y^{-1})Y^{-1}$ *belongs to* $G(n/2)$, *and* $I - X'Y^{-1}X$ *is* $GL_n(F)$*-conjugate to* $-\varepsilon(Y^{-1})Y^{-1}$. *Furthermore, every* $GL_n(F)$*-conjugate of* $-\varepsilon(Y^{-1})Y^{-1}$ *belongs to the image of* $\{Y^{-1}\}$ *under* $N_\varepsilon$.

(b) *Suppose* $F$ *is not necessarily algebraically closed. If there is an* $X$ *in* $M_n(F)$ *such that* $I - X'Y^{-1}X$ *is semisimple in* $G(n/2)$, *then all the remaining statements of (a) are valid as well. If* $\{Y^{-1}\} \in \mathcal{N}$ *corresponds to* $\{X\} = \{I\}$, *that is, for almost all* $Y$, *then the* $GL_n(F)$*-conjugates of* $-\varepsilon(Y^{-1})Y^{-1}$ *exhaust the image of* $\{Y^{-1}\}$ *under the norm map* $N_\varepsilon$.

(c) *In either case, the semisimple part of every conjugacy class in* $N_\varepsilon(\{Y\})$ *is* $GL_n(F)$ *conjugate to* $\{-\varepsilon(Y^{-1})Y^{-1}\}$.

*Proof.* Assume that $F$ is algebraically closed. Suppose that $J_n$ is symmetric and that $A \in GL_n(F)$ satisfies $AJ_n{}^tA = w_n$. Since $Y$ is $\theta$-semisimple, there is an $X_1 \in M_n(F)$ with $I - X_1''Y^{-1}X_1$ semisimple in $SO_n(F)$ (the split form). Here $X'' = -w_n{}^tX_1w_n$. Let $X = X_1A$. Then

$$I - X'Y^{-1}X = I + J_n{}^t(X_1A)w_nY^{-1}X_1A = I + J_n{}^tA{}^tX_1w_nY^{-1}X_1A$$

$$= I + A^{-1}w_n{}^tX_1w_nY^{-1}X_1A = A^{-1}(I - X_1''Y^{-1}X_1)A.$$

Since $I - X_1''Y^{-1}X_1$ is semisimple in $SO_n(F)$, we see that $I - X'Y^{-1}X$ is semisimple in $SO_{J_n}(F)$.

On the other hand, if $Y$ is $\theta^*$-semisimple, then there is an $h \in Sp_{2n}(F)$ with $Y_1 = hYh^{-1} = \operatorname{diag}\{a_1, \ldots, a_n\}$. Then

$$Y_1 + \tilde{\theta}^*(Y_1) = \operatorname{diag}\{a_1 + a_n, a_2 + a_{n-1}, \ldots, a_1 + a_n\}.$$

Let $i = \sqrt{-1}$, and

$$X_1 = i\operatorname{diag}\{a_1 + a_n, a_2 + a_{n-1}, \ldots, a_{n/2} + a_{n/2+1}, 1, \ldots, 1\}.$$

Then $X_1' = -i\,\mathrm{diag}\{1, \ldots, 1, a_{n/2} + a_{n/2+1}, \ldots, a_1 + a_n\}$. Therefore, $X_1 X_1' = Y_1 + \tilde{\theta}^*(Y_1)$. Note that

$$I - X_1' Y^{-1} X_1 = -\mathrm{diag}\{a_1^{-1} a_n, a_2^{-1} a_{n-1}, \ldots, a_n^{-1} a_1\},$$

which is semisimple. Letting $X = h^{-1} X_1$, we have that

$$I - X' Y^{-1} X = I - X_1' \theta^*(h) h^{-1} Y_1 h(h^{-1} X_1) = I - X_1' Y_1^{-1} X_1$$

is semisimple.

So, in either case, we can choose $X \in M_n(F)$ with the desired property. Now, since $Y$ belongs to a $\varepsilon$-stable Cartan, we see that $\varepsilon(Y^{-1})Y^{-1}$ is semisimple in $G(n/2)$. Choose $Y_2$, which is $\varepsilon$-conjugate to $Y$, and a projection $X_2$ satisfying (2.1) with $Y$, such that $I - X_2' Y_2^{-1} X_2 = I - X' Y^{-1} X$. Then $\varepsilon(Y_2^{-1}) Y_2^{-1}$ is conjugate to $\varepsilon(Y^{-1})Y^{-1}$. By the proof of Lemma 3.12, we see that $-\varepsilon(Y^{-1})Y^{-1}$ has matrix

$$\begin{pmatrix} (I - X_2' Y_2^{-1} X_2)|_{\mathrm{Im}\, X} & * \\ 0 & I \end{pmatrix}$$

with respect to a basis that respects $F^n = \mathrm{Im}\, X_2 \oplus \ker X_2$. Thus, the eigenvalues of $-\varepsilon(Y^{-1})Y^{-1}$ different from 1 are among those of $I - X_2' Y_2^{-1} X_2$. Since $Y$ belongs to an $\varepsilon$-stable Cartan subgroup of $GL_n(F)$, we see that $Y^{-1}\varepsilon(Y^{-1}) = \varepsilon(Y^{-1})Y^{-1}$, and thus $Y_2^{-1}\varepsilon(Y_2^{-1})$ and $\varepsilon(Y_2^{-1})Y_2^{-1}$ have the same eigenvalues. Consequently, the eigenvalues of $I - X' Y^{-1} X = I - X_2 Y_2^{-1} X_2$ which are not equal to 1 are among those of $-\varepsilon(Y_2^{-1})Y_2^{-1}$, and, therefore, $I - X' Y^{-1} X$ and $-\varepsilon(Y^{-1})Y^{-1}$ are $GL_n(F)$-conjugate. This proves part (a). Parts (b) and (c) follow from part (a). □

LEMMA 3.17. *Assume $n = 2m$. Suppose that $Y + \tilde{\varepsilon}(Y) = XX'$, and $Z = I - X' Y^{-1} X$. Let $g \in G'_{\varepsilon,Y}(F)$, and suppose that there is an $h \in G(n/2)$ with $gX = Xh$. Then $h$, whose class modulo the right stabilizer of $X$ is uniquely determined, belongs to $G_Z(F)$. Conversely, suppose $h \in G_Z(F)$, and $(X, Y)$ gives the canonical section over $Z$. If there is some $g \in G'(F)$ with $gX = Xh$, then we can choose such a $g$ in $G'_{\varepsilon,Y}(F)$.*

*Proof.* We only need to prove the converse. Recall that

$$W_n = \begin{cases} u_n, & \text{if } \tilde{\mathbf{G}} = Sp_{2r}; \\ w_n, & \text{if } \tilde{\mathbf{G}} = SO_{J_{2r}}. \end{cases}$$

Then in each case, $X' = -J_n {}^t X W_n^{-1}$, and from Lemma 3.8 we have

$$X' Y^{-1} X = X' Y^{-1} = -J_n W_n^{-1} Y^{-1} X.$$

Set $X'' = -W_n {}^tXW_n^{-1} = -\tilde{\varepsilon}(X)$. Then $X'' = W_n^{-1}J_n^{-1}X'$, and thus,

$$X''Y^{-1}X = X''Y^{-1} = -Y^{-1}X.$$

Further note that $-X''$ is a projection, since $X$ is.

Take $U_1 = XF^n$, $U_1' = (-X'')F^n$, $U_2 = \{v | Xv = 0\}$, and $U_2' = \{v | X''v = 0\}$. Then $Y^{-1} : U_1 \to U_1'$, and $Y^{-1} : U_2 \to U_2'$. If we set $\tilde{\varepsilon}(v) = {}^t(W_n v) = {}^t v W_n^{-1}$, then this is an isomorphism of $F^n$ which takes $U_1$ to $U'$ and $U_1'$ to $U$, where $U = F^n X$ and $U' = F^n(-X'')$. Now $\varepsilon(Y)$ takes $U'$ to $U$.

Suppose that $h^{-1}X'Y^{-1}Xh = X'Y^{-1}X$. Then

$$J_n {}^t(h^{-1}X'Y^{-1}Xh)W_n = J_n {}^t(X'Y^{-1}X)W_n,$$

which gives

$$h^{-1}X'\varepsilon(Y)Xh = X'\varepsilon(Y)X.$$

A straightforward computation shows that

$$G'_{\varepsilon,\varepsilon(Y)} = G'_{\varepsilon,Y^{-1}} = \varepsilon\left(G'_{\varepsilon,Y}\right).$$

Note that $I + X''$ and $I - X$ are projections, and

$$(I + X'')Y^{-1}(I - X) = (I + X'')Y^{-1} = Y^{-1}(I - X).$$

We now take $g \in G'_{\varepsilon,Y}$ with $gX = Xg$. Then $\varepsilon(g)^{-1}Y^{-1}g = Y^{-1}$, and so

$$(I + X'')\varepsilon(g)^{-1}Y^{-1}g(I - X) = (I + X'')Y^{-1}(I - X), \tag{3.1}$$

which implies that

$$\varepsilon(g)^{-1}(I + X'')Y^{-1}(I - X)g = (I + X'')Y^{-1}(I - X). \tag{3.2}$$

Let $g^\vee$ be defined by $g^\vee|_{U_1} = Xh|_{U_1}$ and $g^\vee|_{U_2} = (I - X)g|_{U_2}$. If $v \in U_1$, then $g^\vee Xv = g^\vee v = Xhv$. If $v \in U_2$, then $J_n W_n Y^{-1}Xhv = hJ_n W_n Y^{-1}Xv = 0$, and so, $Xhv = 0 = g^\vee Xv$. Thus, $g^\vee X = Xh$.

If $v \in U_1$, then we know that $Y^{-1}v \in U_1'$. Note that

$$\begin{aligned} -X''\varepsilon(g^\vee)^{-1}Y^{-1}g^\vee v &= \varepsilon(h)^{-1}(-X'')Y^{-1}Xhv \\ &= -\varepsilon(h)^{-1}W_n^{-1}J_n^{-1}X'Y^{-1}Xhv \\ &= -\varepsilon(h)^{-1}W_n^{-1}J_n^{-1}hX'Y^{-1}v \end{aligned}$$

$$= -\varepsilon(h)^{-1} W_n^{-1t} h^{-1} J_n^{-1} X' Y^{-1} v$$

$$= \varepsilon(h)^{-1} \varepsilon(h)(-X'') Y^{-1} v$$

$$= -X'' Y^{-1} v.$$

Furthermore, we see that

$$(I + X'')\varepsilon(g^\vee)^{-1} = W_n(I - X) W_n^{-1} W_n^t g^\vee W_n^{-1}$$

$$= W_n^t (g^\vee (I - X)) W_n^{-1} = W_n^t ((I - X)g) W_n^{-1}$$

$$= \varepsilon(g)^{-1}(I + X'').$$

Thus, we have

$$(I + X'')\varepsilon(g^\vee)^{-1} Y^{-1} g^\vee v = \varepsilon(g)^{-1}(I + X'') Y^{-1} X h v = 0 = (I + X'') Y^{-1} v,$$

since $Y^{-1} X h v \in U_1'$.

Now suppose that $v \in U_2$. Then

$$-X''\varepsilon(g^\vee)^{-1} Y^{-1} g^\vee v = \varepsilon(h)(-X'') Y^{-1}(I - X) g v$$

$$= \varepsilon(h)(-X'')(I + X'') Y^{-1} g v = 0 = -X'' Y^{-1} v.$$

Finally, we see that

$$(I + X'')\varepsilon(g^\vee)^{-1} Y^{-1} g^\vee v = \varepsilon(g)^{-1}(I + X'') Y^{-1}(I - X) g v$$

$$= (I + X'') Y^{-1}(I - X) v = (I + X'') Y^{-1} v,$$

the next to last equality coming from (3.2), and the final one from the definition of $U_2$. So, $g^\vee$ is the desired element. □

We can now state the final result of this section. This allows us to determine the residues of the intertwining operator given in (2.4) via the theory of twisted endoscopy.

LEMMA 3.18. *Suppose that $n = 2m$.*

(a) *For $\tilde{\mathbf{G}} = SO_{J_{2r}}$, the norm correspondence $N_\theta$ agrees with the norm map of Kottwitz and Shelstad on the intersection of $\mathcal{N}$ with strongly $\theta$-regular $\theta$-semisimple conjugacy classes in $GL_n$.*

(b) *For $\tilde{\mathbf{G}} = Sp_{2r}$, the norm correspondence $N_{\theta^*}$ agrees with the negative of the norm map of Kottwitz and Shelstad on the intersection of $\mathcal{N}$ with strongly $\theta^*$-regular $\theta^*$-semisimple conjugacy classes in $GL_n$.*

*Proof.* (a) follows from the argument of Lemma 5.21 of [22], since $SO_{J_{2n}}$ splits over $L$. For (b) we simply note that if $\mathbf{T_H}$ is a Cartan subgroup of $Sp_{2n}$ defined over $F$, and $\mathbf{T}'$ is a $\theta^*$-stable Cartan of $GL_n$, with $\mathbf{T_H} \to \mathbf{T}'_{\theta^*}$ defined over $F$, then $N_{\theta^*}(\{Y\}) = \{-Y^{-1}\theta^*(Y^{-1})\}$, which is the negative of the norm map defined in [14]. □

*Remark.* The appearance of the minus sign in (b) appears somewhat incongruous with the results of [22]. However, this does not affect the location of the poles of the intertwining operator, because it only has the effect of multiplying the residue by a scalar, namely, $\omega(-1)$, where $\omega$ is the central character of $\tau$. However, the sign may be of some arithmetic significance, and this question should be addressed in the future.

**§4. The poles of intertwining operators.** We return now to the computation of the poles of the intertwining operator $A(s, \tau' \otimes \tau, w_0)$ discussed in Section 2. We use the results of Section 3 to determine the residues of these integrals. We begin with the following proposition.

PROPOSITION 4.1. *Assume $n = 2m$. Suppose the $\varepsilon$-conjugacy class $\{Y^{-1}\}$ is $\varepsilon$-regular. Then $N_\varepsilon(\{Y^{-1}\})$ consists of a single conjugacy class in $G^\vee$ of a regular semisimple element in $G$. Assuming that $Y$ and $\varepsilon(Y)$ commute, that is, that $Y^{-1}\varepsilon(Y^{-1})$ is in $G$, then the converse is true, that is, if $N_\varepsilon(\{Y^{-1}\})$ is regular, then $\{Y^{-1}\}$ is $\varepsilon$-regular (and hence is $\varepsilon$-semisimple).*

*Proof.* Suppose the $\varepsilon$-conjugacy class $\{Y^{-1}\}$ is $\varepsilon$-regular. Then up to $GL_n(\bar{F})$-conjugation, $\varepsilon(Y^{-1})Y^{-1}$ is a regular semisimple element of $\mathbf{G}$. Choose $Y_2^{-1}$, $\varepsilon$-conjugate to $Y^{-1}$, and a projection $X_2$ satisfying (2.1) with $Y_2$, such that $I - X_2' Y_2^{-1} X_2 = I - X' Y^{-1} Y$. By Lemma 5.10 of [22], the eigenvalues of $\varepsilon(Y^{-1})Y^{-1}$ different from 1 are among those of the semisimple part of $I - X_2 Y_2^{-1} X_2$. Since $Y^{-1}\varepsilon(Y^{-1})$ is $GL_n(F)$-conjugate to $\varepsilon(Y^{-1})Y^{-1}$, one sees that the eigenvalues of $-\varepsilon(Y_2)^{-1}Y_2^{-1}$ and $-Y_2^{-1}\varepsilon(Y_2)^{-1}$ are the same. Therefore, one can apply the argument of Lemma 5.10 of [22] to the equation in Lemma 3.3(a) to show that the eigenvalues of the semisimple part of $I - X_2' Y_2^{-1} X_2$ that are not 1 are also among those of $-\varepsilon(Y_2)^{-1}Y_2^{-1}$. Then the semisimple parts of $I - X' Y^{-1} X$ and $-\varepsilon(Y)^{-1}Y^{-1}$ are $GL_n(\bar{F})$-conjugate. But $-\varepsilon(Y)^{-1}Y^{-1}$ is $GL_n(\bar{F})$-conjugate to a regular element in $\mathbf{G}$, and therefore, $I - X' Y^{-1} X$ must be semisimple and regular.

Suppose now that $Y + \tilde{\varepsilon}(Y) = XX'$, with $Y^{-1}\varepsilon(Y)^{-1} \in \mathbf{G}(F)$, and assume $N_\varepsilon(\{Y^{-1}\})$ contains a regular semisimple element $\{I - X' Y^{-1} X\}$. Again by Lemma 3.3(a), and the argument of Lemma 3.12, the conjugacy class of $I - X' Y^{-1} X$ is completely determined by the semisimple part of $\{-Y^{-1}\varepsilon(Y^{-1})\}$, a conjugacy class in $\mathbf{G}(F)$. That is, the eigenvalues of the first are among those of

the second. Moreover, by Lemma 3.12, the semisimple part of the conjugacy class of $-\varepsilon(Y^{-1})Y^{-1}$ is completely determined by $I - X'Y^{-1}X$. Since $\{I - X'Y^{-1}X\}$ is regular semisimple in $\mathbf{G}(F)$, and $Y^{-1}\varepsilon(Y^{-1}) \in \mathbf{G}(F)$, we conclude that $Y^{-1}\varepsilon(Y^{-1})$ is regular and semisimple. Therefore, $\{Y^{-1}\}$ is $\varepsilon$-regular. In fact, let $\tilde{Y} = (Y, \varepsilon)$ represent an element in the nonidentity component of $GL_n \rtimes \{1, \varepsilon\}$. Write $\tilde{Y} = su$, with $s$ semisimple and $u$ unipotent. Then $\tilde{Y}^2 = s^2u^2 = Y^{-1}\varepsilon(Y^{-1})$. If $Y^{-1}\varepsilon(Y^{-1})$ is semisimple, then $u^2 = u = 1$, and thus $Y$ is $\varepsilon$-semisimple and $\varepsilon$-regular. □

COROLLARY 4.2. *Suppose $n > 2m$. For almost all regular elliptic conjugacy classes $\{h\} \in G(m)$, the collection of $\varepsilon$-conjugacy classes, $N_\varepsilon^{-1}(\{h\})$ is parameterized by a unique $\varepsilon$-regular, $\varepsilon$-conjugacy class in $GL_{2m}(\bar{F})$.*

*Proof.* For almost all regular semisimple classes in $G(m)$, there is a choice of $Y_2 \in GL_{2m}(F)$ that satisfies (2.1) with $X_2 = I_{2m}$, so that $I - X_2'Y_2^{-1}X_2 \in \{h\}$. In particular, $Y_2 + \tilde{\varepsilon}(Y_2) = I_{2m}'$. By Proposition 4.1, the $\varepsilon$-conjugacy class of $Y_2$ is $\varepsilon$-regular and uniquely determined by $h$. Let $j = (n - 2m)/2$, as in Corollary 3.13. Let

$$X = \begin{pmatrix} 0_{j\times 2m} \\ I_{2m} \\ 0_{j\times 2m} \end{pmatrix} \in M_{n\times 2m}(F).$$

Then $X' = (0 \quad I' \quad 0)$, and

$$XX' = \begin{pmatrix} 0_j & 0 & 0 \\ 0 & I_{2m} & 0 \\ 0 & 0 & 0_j \end{pmatrix}.$$

Let

$$Y = \begin{pmatrix} I_j & & \\ & Y_2 & \\ & & -I_j \end{pmatrix}.$$

If $\mathbf{G}$ is orthogonal, then

$$\tilde{\varepsilon}(Y) = \begin{pmatrix} & & w_j \\ & w_{2m} & \\ w_j & & \end{pmatrix} \begin{pmatrix} I_j & & \\ & {}^tY_2 & \\ & & -I_j \end{pmatrix} \begin{pmatrix} & & w_j \\ & w_{2m} & \\ w_j & & \end{pmatrix}$$

$$= \begin{pmatrix} -I_j & & \\ & \tilde{\varepsilon}(Y_2) & \\ & & I_j \end{pmatrix}.$$

Similarly, if **G** is symplectic, then we have

$$\tilde{\varepsilon}(Y) = \begin{pmatrix} & & (-1)^j u_j \\ & (-1)^j u_{2m} & \\ u_j & & \end{pmatrix} \begin{pmatrix} I_j & & \\ & {}^t Y_2 & \\ & & -I_j \end{pmatrix} \begin{pmatrix} & & u_j^{-1} \\ & (-1)^j u_{2m} & \\ (-1)^j u_j & & \end{pmatrix}$$

$$= \begin{pmatrix} -I_j & & \\ & \tilde{\varepsilon}(Y_2) & \\ & & I_j \end{pmatrix}.$$

Therefore,

$$Y + \tilde{\varepsilon}(Y) = \begin{pmatrix} 0_j & 0 & 0 \\ 0 & I'_{2m} & 0 \\ 0 & 0 & 0_j \end{pmatrix} = XX'.$$

Moreover, we see that $I_{2m} - X'Y^{-1}X = I_{2m} - I'_{2m}Y_2^{-1} = h$.

It remains to check that for almost all $Y_2$, the class of $Y$ satisfying (2.1) is, up to $GL_n(\bar{F})$–$\varepsilon$-conjugacy, of the form given in the previous paragraph. First observe that for almost all $Y$ satisfying (2.1) with

$$X = \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix},$$

$Y$ acts semisimply on the direct sum of the image and the kernel of $XX'$, both of which are invariant under $\varepsilon$, since $Y_2$ is $\varepsilon$-regular. Moreover, $Y$ must be $\varepsilon$-skew symmetric on $\ker(XX')$ and can therefore be given in the form $\operatorname{diag}(J_1, Y_2, J_2)$ with $\operatorname{diag}(J_1, J_2)$ $\varepsilon$-symmetric (**G** symplectic) or $\varepsilon$-skew symmetric (**G** orthogonal), proving our assertion. □

LEMMA 4.3. *Suppose that* $\alpha \in F^{\times 2}\backslash F^{\times}$. *Let* $\alpha_0 = \operatorname{diag}\{\alpha, 1, \alpha, 1, \ldots, 1\} \in GSp_{2m}$. *Then*

$$N_\varepsilon(\{\alpha\gamma'\}) = \alpha_0^{-1} N_\varepsilon(\{\gamma'\})\alpha_0,$$

*for any* $\gamma' \in \mathcal{N}$. *Similarly, let* $\alpha \in NE^{\times}$. *Take* $g_0 \in GL_k(F)$ *with* $g_0 \Lambda_k {}^t g_0 = \alpha \Lambda_k$. *Let* $\alpha_0 = \operatorname{diag}\{\alpha I_{m-(k/2)}, g_0, I_{m-(k/2)}\} \in GO_{J_{2m}}$. *Then*

$$N_\varepsilon(\{\alpha\gamma'\}) = \alpha_0^{-1} N_\varepsilon(\{\gamma'\})\alpha_0,$$

*for all* $\{\gamma'\} \in \mathcal{N}$.

*Proof.* Let $\alpha^\vee = \alpha I_{2m}$. In the symplectic case, $\alpha^\vee = \alpha_0 \tilde{\varepsilon}(\alpha_0)$, while in the orthogonal case, $\alpha^\vee = J_n^{-1}\alpha_0 J_n {}^t\alpha_0$. Let $Y^{-1} \in \{\gamma'\}$. Then $X\alpha^\vee X' = \alpha Y + \tilde{\varepsilon}(\alpha Y)$, while $X\alpha^\vee X' = (X\alpha_0)(X\alpha_0)'$. Since $N_\varepsilon(\{\alpha\gamma'\})$ is the conjugacy class given by

$$I - (X\alpha_0)'(\alpha Y)^{-1}(X\alpha_0),$$

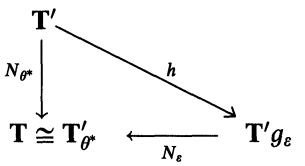
we have

$$I - (X\alpha_0)'(\alpha Y)^{-1}(X\alpha_0) = \alpha_0^{-1}(I - X'Y^{-1}X)\alpha_0. \quad \square$$

Let $\mathscr{C}$ and $\mathscr{C}'$ be the sets of conjugacy classes in $G(m)$ and $\varepsilon$-conjugacy classes in $GL_n(F)$, respectively. Suppose that $n = 2m$. Let $\mathbf{T}$ be a Cartan subgroup of $\mathbf{G}(n/2)$, defined over $F$. By Lemma 3.3.B of [14], there is a $\theta^*$-stable Cartan subgroup $\mathbf{T}'$ of $GL_n$ with an isomorphism $\mathbf{T} \tilde{\to} \mathbf{T}'_{\theta^*}$, which is defined over $F$. Furthermore, this isomorphism induces the *image map* $\mathscr{A}_{\mathbf{G}(n/2)/GL_n}$ between semisimple classes in $\mathscr{C}$ and $\theta^*$-semisimple $\theta^*$-conjugacy classes in $GL_n$. We set

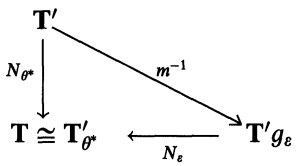
$$g_\varepsilon = \begin{cases} I, & \text{if } \mathbf{G}(n/2) \text{ is symplectic,} \\ g_\theta, & \text{if } \mathbf{G}(n/2) \text{ is orthogonal.} \end{cases}$$

(See [22, §7].) Let $h : \mathbf{T}' \to \mathbf{T}'g_\varepsilon$ be given by right multiplication by $g_\varepsilon$. Then, up to a sign, we see that the diagram




commutes on the set of strongly $\theta^*$-regular elements of $\mathbf{T}'$. (Recall that an element $t$ of $\mathbf{T}'$ is called strongly $\theta^*$-regular if $\mathbf{G}'^{\theta^*,t}$ is abelian.) Since $\mathbf{T}'$ is $\theta^*$-stable, we can, in the language of §3.1 of [14], take $\psi = 1$, to see that $h^{-1}$ is the map $m$ of [14].

LEMMA 4.4. *Let* $\mathbf{T}$ *be a Cartan subgroup of* $\mathbf{G}(n/2)$ *defined over* $F$. *Then there is a* $\theta^*$*-stable Cartan* $\mathbf{T}'$ *of* $\mathbf{G}'(n)$, *such that the diagram*



*commutes up to a sign on all strongly $\theta^*$-strongly regular $\theta^*$-semisimple elements of $\mathbf{T}'(F)$. Furthermore, $\mathbf{T} \simeq \mathbf{T}'_{\theta^*}$ induces the image map $\mathscr{A}_{\mathbf{G}/\mathbf{G}'}$. If $\delta^* \in \mathbf{T}'$ is $\theta^*$-strongly regular, then* $\mathrm{Cent}_{\theta^*}(\delta^*, \mathbf{G}') = (\mathbf{T}')^{\theta^*}$.

*Proof.* We only need to prove the last statement. If $\delta^*$ is strongly $\theta^*$-regular, then $\mathrm{Cent}_{\theta^*}(\delta^*, \mathbf{G}')$ is a maximal torus in $\mathbf{G}'$, stable under $\mathrm{Int}(\delta^*) \circ \theta^*$. Since $\mathbf{T}'_{\delta^*} = \mathbf{T}'$ is a maximal torus and $\mathbf{T}'$ is $\theta^*$-stable, we have $\mathrm{Cent}_{\theta^*}(\delta^*, \mathbf{G}') = (\mathbf{T}')^{\theta^*}$. □

We must eventually integrate over twisted conjugacy classes in $\mathscr{N}$. By Proposition 4.1 and the surjectivity of the norm correspondence, up to a set of measure zero, these twisted conjugacy classes are parameterized by regular semisimple conjugacy classes in $G$. There may be two regular conjugacy classes in $G$ which parameterize the same class in $\mathscr{N}$. This will be taken into account as we integrate over all of the Cartan subgroups of $G$. We can therefore fix a representative $\mathbf{T}$ for each conjugacy class of Cartan subgroups of $\mathbf{G}$ that are defined over $F$. Let $d\gamma$ be a Haar measure for $T = \mathbf{T}(F)$. Then by Lemma 3.15, the last part of Lemma 4.4, Proposition 4.1, and upon computing the Jacobian for the corresponding open immersion in page 227 of [1] (or Theorem 3.2 of [23]), the measure $|W(T)|^{-1}|D_{\theta^*}(\gamma')|\,d\gamma$ as $\mathbf{T}$ ranges provides us with a measure for $\mathscr{N}$. Here $\{\gamma\}$ is in $N_{\theta^*}(\{\gamma'\})$ for each $\varepsilon$-regular $\{\gamma' g_\varepsilon\}$ in $\mathscr{N}$ and

$$D_{\theta^*}(\gamma') = \det(\mathrm{Ad}(\gamma') \circ \theta^* - 1)|_{\mathfrak{g}/\mathfrak{g}_{\theta^*,\gamma'}},$$

as described in [14]. We introduce the positive multiple $|W(T)|^{-1}$ since, because of surjectivity of the norm correspondence, we may and do transfer the integration to one on $G$ for which the measure for integration over semisimple conjugacy classes of $G$ coming from the conjugacy class of $\mathbf{T}$ is, in fact, $|W(T)|^{-1}|D(\gamma)|\,d\gamma$, as suggested by the Weyl integration formula.

By Lemma 4.5.A of [14],

$$\kappa_1(\{\gamma\}, \{\gamma'\}) = |D_{\theta^*}(\gamma')|/|D(\gamma)|$$

is bounded on $\{(\{N_{\theta^*}(\gamma')\}, \{\gamma'\})\}$. Suppose $\{\gamma\}$ is regular and semisimple. Define

$$\kappa(\{\gamma\}, \{\gamma'\}) = \begin{cases} \kappa_1(\{\gamma\}, \{\gamma' g_\varepsilon^{-1}\}), & \text{if } \{\gamma\} \in N_\varepsilon(\{\gamma'\}) \text{ and } \gamma' \text{ is } \varepsilon\text{-regular;} \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $|D(\gamma)| = |D(\gamma^{-1})|$.

For each regular semisimple conjugacy class $\{\gamma\} \in \mathscr{C}$, define

$$\mathscr{A}(\{\gamma\}) = \{\{\alpha\gamma'\} | \alpha \in F^{\times^2} \backslash F^\times, \{\gamma\} \in N_\varepsilon(\{(\gamma')^{-1}\})\}.$$

We set

$$\Delta(\{\gamma\},\{\alpha\gamma'\}) = \begin{cases} \omega'(\alpha)\kappa(\{\gamma\},\{\gamma'\}), & \text{if } \{\gamma\} \in N_\varepsilon(\{(\gamma')^{-1}\}); \\ 0, & \text{otherwise.} \end{cases}$$

Finally, let

$$\Phi_\varepsilon(\mathscr{A}(\{\gamma\}), f') = \sum_{\{\gamma'\} \in \mathscr{A}(\{\gamma\})} \Delta(\gamma,\gamma')\Phi_\varepsilon(\gamma', f').$$

The main part of the residue of the intertwining operator at $s = 0$ comes from the regular elliptic elements in the form of the Weyl integration formula applied to the class function $\Phi_\varepsilon(\mathscr{A}(\{\gamma\}), f')$, that is, as a pairing between characters and twisted characters (cf. Section 5). More precisely, the contribution from these regular elliptic classes has the form

$$R_{\mathbf{G}}(f_\tau, f') = \sum_{\{\mathbf{T}_i\}} \mu(T_i)\,|W(T_i)|^{-1} \int_{T_i} \Phi(\{\gamma\}, f_\tau)\Phi_\varepsilon(\mathscr{A}(\{\gamma\}), f')\,|D(\gamma)|\,d\gamma,$$

where $\{\mathbf{T}_i\}$ runs over the conjugacy classes of elliptic Cartan subgroups of $\mathbf{G} = \mathbf{G}(m)$ and $T_i = \mathbf{T}_i(F)$. For each $i$, $\mu(T_i)$ is the measure of $T_i$. Observe that by the way that our transfer factor $\Delta$ is defined; this is, in fact, an integration over $\varepsilon$-conjugacy classes in $\mathscr{N}$ (since the norm correspondence is surjective).

Let us first check the convergence of $R_{\mathbf{G}}$. It is enough to show that

$$\int_{\mathbf{T}(F)} \Phi(\gamma, f_\tau)\Phi_\varepsilon(\mathscr{A}(\{\gamma\}), f')|D(\gamma)|\,d\gamma,$$

converges for any elliptic torus $\mathbf{T}$ of $\mathbf{G}(n/2)$. By Theorem 14 of [10], the function $|D(\gamma)|^{1/2}\Phi(\gamma, f_\tau)$ is bounded on the intersection of $\mathbf{T}(F)$, with the regular set of $G$. Therefore, we are reduced to proving the convergence of

$$\int_{\mathbf{T}(F)} \Phi_\varepsilon(\mathscr{A}(\{\gamma\}), f')|D(\gamma)|^{1/2}\,d\gamma.$$

It is enough to show that each integral in the sum converges absolutely, that is, that

$$\int_{\substack{\mathbf{T}(F) \\ \mathbf{N}_{\theta^*}(\{\gamma'\})=\{\gamma\}}} |\Phi_\varepsilon(\alpha\gamma' g_\varepsilon, f')|\,\kappa_1(\{\gamma\},\{\gamma'\})\,|D(\gamma)|^{1/2}\,d\gamma$$

converges for every $\alpha \in F^{\times^2}\backslash F^\times$. We are therefore reduced to showing the con-

vergence of

$$\int_{N_{\theta^*}(\{\gamma'\})=\{\gamma\}} |\Phi_{\theta^*}(\gamma', R_{g_\varepsilon\alpha} f')|\,|D_{\theta^*}(\gamma')|\,|D(\gamma)|^{-1/2}\,d\gamma'. \tag{4.1}$$

Since $\Phi_{\theta^*}(-,-)$ is a tempered distribution, the function $\Phi_{\theta^*}(\gamma', f')|D_{\theta^*}(\gamma')|^{1/2}$ is bounded on the intersection of $\mathbf{T}_{\theta^*}(F)$ with the $\theta^*$-regular set [6], [10]. Now the convergence of (4.1) follows from the boundedness of $\kappa_1(\{\gamma\},\{\gamma'\})^{1/2}$ (Lemma 4.5.A of [14]).

Let $M^\vee = GL_n(F) \times G^\vee$ act on $N$ by the adjoint action, where $\mathbf{G}^\vee = \mathbf{G}^\vee(m) = O_{J_{2m}}$ if $\mathbf{G}(m) = SO_{J_{2m}}$ and $\mathbf{G}^\vee(m) = Sp_{2m}$, otherwise. Then the orbit of $n(X,Y) \in N$ consists of pairs $(gXh^{-1}, gY\varepsilon(g)^{-1}) \in N$, with $g \in G'$ and $h \in G^\vee$. The stabilizer $\Delta^\vee$ of the action at $n(X,Y)$ consists of all those pairs $(g,h)$ with $g \in \mathbf{G}'_{\varepsilon,Y}(F)$ for which $gX = Xh$. Then $h \in \mathbf{G}^\vee_Z$ where $Z = I - X'Y^{-1}X$ and $\mathbf{G}^\vee_Z$ is the centralizer of $Z$ in $\mathbf{G}^\vee(m)$. By abuse of both notation and terminology, identify $\Delta^\vee$ as a subgroup of both $\mathbf{G}^\vee_Z(F)$ and $\mathbf{G}'_{\varepsilon,Y}(F)$ through projection onto its components.

To compute (2.4) we first integrate over each orbit of $N$ under $M^\vee$. The measure $d^*(X,Y) = |\det Y|^{-\langle \rho_{\mathbf{P}}, \tilde{\alpha}\rangle} d(X,Y)$ is an invariant measure on these orbits. In fact, it can be easily checked that

$$d^*(gXh^{-1}, gY\varepsilon(g)^{-1}) = d^*(X,Y),$$

since

$$d(gXh^{-1}, gY\varepsilon(g)^{-1}) = |\det g|^{\langle 2\rho_{\mathbf{P}}, \tilde{\alpha}\rangle} d(X,Y).$$

One can then write $d^*(X,Y)$ as a product of measures $d_1^*(X,Y)$ and $d_2^*(X,Y)$, the first one for integration over the orbit of $(X,Y)$ under $M^\vee$, and the second to run over all such orbits. (They both are specified later.) Then $d_2^*(Y^{-1}X, \varepsilon(Y)) = d_2^*(X,Y)$. Since $d^*(X,Y)$ and $d_2^*(X,Y)$ are both invariant under the action of $M^\vee$, so is $d_1^*(X,Y)$. Let $d\delta$ be a measure on $\Delta^\vee$. Fix measures $dg$ and $dh$ on $GL_n(F)$ and $\mathbf{G}^\vee(F)$ so that the measure $d_1^*(X,Y)$ on each orbit is given by the quotient of $dh\,dg$ by $d\delta$. Consider the map $(g,h) \mapsto (gXh^{-1}, gY\varepsilon(g)^{-1})$ from $GL_n(F) \times \mathbf{G}^\vee(F)$ to the orbit of $(X,Y)$. Changing our representative for the orbit from $(X,Y)$ to $(Y^{-1}X, \varepsilon(Y))$ changes $(g,h)$ to $(gY^{-1}, h)$. Thus, the measures $dg$ and $dh$ are unchanged by this change of representative. Also, changing $(X,Y)$ to $(Y^{-1}X, \varepsilon(Y))$ changes $\Delta^\vee$ to $(Y,I)\Delta^\vee(Y,I)^{-1}$, and therefore leaves $d\delta$ invariant. Thus, $d_1^*(X,Y) = d_1^*(Y^{-1}X, \varepsilon(Y))$, and therefore $d^*(Y^{-1}X, \varepsilon(Y)) = d^*(X,Y)$.

Making this change of variables, the integral in (2.4) can be expressed as

$$\int_{(Y,X)} \psi_{\tau'}(Y) f_\tau(I - X'Y^{-1}X)\xi_L(\varepsilon(Y)^{-1})\xi_{L'}(X)|\det Y|^s\,d^*(X,Y). \tag{4.2}$$

Let $\omega'$ be the central character of $\tau'$. Since we are assuming that $\tau'$ is self-dual, $\omega'^2$ is trivial. We choose $f' \in C_c^\infty(G')$ so that

$$\psi_{\tau'}(g') = \int_{Z(G')} f'(zg')\omega'(z^{-1})\, d^\times z.$$

Making this substitution, we rewrite (4.2) as

$$\int_{F^\times} \int_{(Y,X)} f'(zY) f_\tau(I - X'Y^{-1}X)\omega'(z)|\det Y|^s \xi_L(\varepsilon(Y)^{-1})\xi_{L'}(X)\, d^*(X,Y)\, d^\times z. \tag{4.3}$$

There is a map from the orbit of $n(X,Y)$ under $M^\vee$ onto $G'/\Delta^\vee \times \Delta^\vee \backslash G^\vee$ whose fiber is homeomorphic to $X\Delta^\vee$. The integration over the orbit is then equal to the integration over the product of $G'/\Delta^\vee \times \Delta^\vee \backslash G^\vee$ with $X\Delta^\vee$. The contribution to (4.3) from the orbit of $n(X,Y)$ under the action of $M^\vee$ is then

$$\begin{aligned}\tilde{\psi}(s,Z) = &\sum_{\alpha \in (F^{\times 2})\backslash F^\times} \omega'(\alpha) \\ &\cdot \int_{g \in G'/\Delta^\vee} \int_{h \in \Delta^\vee \backslash G^\vee} \int_{X\Delta^\vee} f'(\alpha g Y \varepsilon(g)^{-1}) f_\tau(h^{-1}Zh)\, |\det(gY\varepsilon(g)^{-1})|^s\, dg\, dh\, d(Xh_0) \\ &\cdot \int_{Z(G')} \xi_{\tilde{L}}(z^{-2} gY\varepsilon(g)^{-1}) \xi_{L'}(z^{-1} gXh_0 h)\, |\det z|^{-2s}\, d^\times z,\end{aligned}$$

where $\tilde{L} = \tilde{\varepsilon}(L)$. Observe that we have suppressed the dependence of $\tilde{\psi}$ on other parameters $X$, $Y, f, f_\tau$, $L$, and $L'$. Breaking the variables further,

$$\begin{aligned}\tilde{\psi}(s,Z) = &\sum_{\alpha} \omega'(\alpha) \\ &\cdot \int_{g \in G'/G'_{\varepsilon,Y}} \int_{h \in G_Z^\vee \backslash G^\vee} \int_{g_0 \in G'_{\varepsilon,Y}/\Delta^\vee} \int_{XG_Z^\vee} f'(\alpha g Y \varepsilon(g)^{-1}) f_\tau(h^{-1}Zh) \\ &\cdot |\det(gY\varepsilon(g)^{-1})|^s\, dg\, dh\, dg_0\, dXh_0 \\ &\cdot \int_{Z(G')} \xi_{\tilde{L}}(z^{-2} gY\varepsilon(g)^{-1}) \xi_{L'}(z^{-1} gg_0 Xh_0 h)\, |\det z|^{-2s}\, d^\times z,\end{aligned} \tag{4.4}$$

where $G'_{\varepsilon,Y} = \mathbf{G}'_{\varepsilon,Y}(F)$ and $G_Z^\vee = \mathbf{G}_Z^\vee(F)$.

Let $Z_j$'s be representatives for the $G$-orbits inside the $G^\vee$-orbit of $Z$. Note that there are either one or two $G$-orbits in each $G^\vee$-orbit. Denote by

$$\Phi(Z_j, f_\tau) = \int_{\mathbf{G}_{Z_j}(F)\backslash G} f_\tau(h^{-1}Z_j h)\, dh$$

and

$$\Phi(Z, f_\tau) = \int_{\mathbf{G}^\vee(F)\backslash G^\vee} f_\tau(h^{-1}Zh)\, dh$$

the corresponding orbital integrals. Finally, let

$$\Phi_{\varepsilon,s}(\alpha Y, f') = \int_{G'/\mathbf{G}'_{\varepsilon,Y}(F)} f'(\alpha g Y \varepsilon(g)^{-1})\, |\det(gY\varepsilon(g)^{-1})|^s\, dg.$$

Clearly,

$$\lim_{s\to 0} \Phi_{\varepsilon,s}(\alpha Y, f') = \Phi_\varepsilon(\alpha Y, f') = \int_{G'/\mathbf{G}'_{\varepsilon,Y}(F)} f'(\alpha g Y \varepsilon(g)^{-1})\, dg$$

is the corresponding twisted orbital integral.

Suppose first that $n = 2m$. Assume $(Y, I)$ satisfies (2.1) and $Z = I - I'Y^{-1}$. Note that $I' = -I$ if $\mathbf{G}$ is symplectic and is $-J_n w_n$, otherwise. (If $\mathbf{G}$ is a split orthogonal group then, again, $I' = -I$ and $Z = I + Y^{-1}$.) Then $\mathbf{G}_Z(F) \simeq \mathbf{G}'_{\varepsilon,Y}(F) \simeq \Delta^\vee$. Let

(4.5)

$$\psi(s, Z) = \sum_\alpha \omega'(\alpha)$$

$$\cdot \int_{g\in G'/\mathbf{G}'_{\varepsilon,Y}(F)} \int_{h\in G_Z^\vee\backslash G^\vee} \int_{h_0\in G_Z^\vee} f'(\alpha g Y \varepsilon(g)^{-1}) f_\tau(h^{-1}Zh)\, |\det(gY\varepsilon(g)^{-1})|^s\, dh dg\, dh_0$$

$$\cdot \int_{Z(G')} \xi_{\tilde{L}}(z^{-2} g Y \varepsilon(g)^{-1}) \xi_{L'}(z^{-1} g h_0 h)\, |\det z|^{-2s}\, d^\times z.$$

Then $\psi(s, Z) = \tilde{\psi}(s, Z)$.

Let $L(\mathbf{1}, s) = (1 - q^{-s})^{-1}$ denote the local Hecke $L$-function attached to the trivial character $\mathbf{1}$. We have the following result.

LEMMA 4.5. *Let $n = 2m$. Assume that $Y$ is $\varepsilon$-regular. Then $\tilde{\psi}(s, Z)$ converges absolutely for* $\operatorname{Re} s > 0$. *Suppose further that $(Y, I)$ satisfies (2.1), that is, for*

*almost all twisted conjugacy classes in $\mathcal{N}$. Then $\psi(s, Z)$ also converges absolutely for $\operatorname{Re} s > 0$, and there exists a function $E_Z(s) = E(s, Z, Y, f_\tau, f', L, L')$, which as a function of $s$ is entire, such that $\psi(s, Z) = E_Z(s)$ if $Z$ is regular and nonelliptic, and*

$$\psi(s, Z) = E_Z(s) + \left( \sum_{\alpha \in (F^\times)^2 \backslash F^\times} \sum_j \Phi_{\varepsilon,s}(\alpha Y, f') \Phi(Z_j, f_\tau) \mu(\mathbf{G}_{Z_j}(F)) \right) q^{b(Y,Z)s} L(\mathbf{1}, 2ns)$$

*for $\operatorname{Re} s > 0$ if $Z$ is regular elliptic. Here $b(Y, Z)$ is an integer depending on the data $f_\tau, f', L, L'$, as well as $Y$ and $Z$. In particular,*

$$\operatorname*{Res}_{s=0} \psi(s, Z) = (2n \log q)^{-1} \sum_{\alpha \in (F^\times)^2 \backslash F^\times} \sum_j \Phi_\varepsilon(\alpha Y, f') \Phi(Z_j, f_\tau) \mu(\mathbf{G}_{Z_j}(F))$$

*if $Z$ is regular elliptic, and*

$$\operatorname*{Res}_{s=0} \psi(s, Z) = 0$$

*if $Z$ is regular but nonelliptic.*

*Proof.* We prove the lemma when $X = I$. The convergence of $\tilde{\psi}(s, Z)$ can be proved the same way. We may assume $L$ (therefore, $\tilde{L}$) and $L'$ are both basic neighborhoods of 0. Clearly $f'(\alpha g Y \varepsilon(g)^{-1}) \xi_{\tilde{L}}(z^{-2} g Y \varepsilon(g)^{-1}) = 0$ unless $g Y \varepsilon(g)^{-1} \in z^2 \tilde{L} \cap \alpha^{-1} \operatorname{supp}(f')$ for some $\alpha$. A standard argument using the support of $f'$ then implies that $|\det z|$ must be bounded below with the bound depending only on $L$ and $f'$. Since $Y$ and $Z$ are $\varepsilon$-semisimple and semisimple, respectively, we may assume $g$ and $h$ in (4.5) belong to compact sets $\operatorname{supp}(g)$ and $\operatorname{supp}(h)$ of $G'/\mathbf{G}'_{\varepsilon,Y}(F)$ and $\mathbf{G}_Z^\vee(F) \backslash G^\vee$, depending upon $Y$, $Z$, $f_\tau$, and $f'$, respectively.

To study the convergence of (4.5), we may assume $h_0$ is in the $F$-points of the split component of $\mathbf{G}_Z^\vee$ (i.e., the connected component of $\mathbf{G}_Z$), which we may further assume are given by diagonal matrices. Write $z = \operatorname{diag}(z, z, \ldots, z)$, and

$$h_0 = \operatorname{diag}(a_1, a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_2^{-1}, \ldots, a_2^{-1}, a_1^{-1}, \ldots, a_1^{-1}).$$

By the compactness of $\operatorname{supp}(g)$ and $\operatorname{supp}(h)$, $\xi_{L'}(z^{-1} g h_0 h) = 0$, unless $z^{-1} h_0$ belongs to a compact subset of $M_n(F)$, that is, $|z a_i| \geqslant \kappa$ and $|z a_i^{-1}| \geqslant \kappa$, for some $\kappa > 0$. Consequently, $|z| \geqslant \kappa$. Moreover, there exists a $\kappa_1 > 0$ such that if $|z^{-1} a_i^{-1}| \leqslant \kappa_1$ and $|z^{-1} a_i| \leqslant \kappa_1$, then $\operatorname{supp}(g) z^{-1} h_0 T \operatorname{supp}(h) \subset L'$, since both $\operatorname{supp}(g)$ and $\operatorname{supp}(h)$ are compact. Here $T$ is the compact part of $G_Z^\vee$. Observe

that $\kappa_1 \geqslant \kappa$. Consequently, with such a $z$ and $h_0$, $h$ is free to change over all its compact support leading to $\Phi(Z, f_\tau)$, which vanishes if the split component of $\mathbf{G}_Z^\vee$ is nontrivial (cf. [22]). Otherwise $\mathbf{G}_Z^\vee(F)$ is compact.

Let $\eta$ be the lower bound given by $f'$ and $L$ on $|z|$. We use $\kappa$ instead of $\eta$ if $\kappa \geqslant \eta$. We may assume $\kappa_1/\kappa$ is a nonnegative integral power of $q$. Choose a nonnegative integer $m$ so that $|\varpi^{-m}| = \kappa_1/\kappa$. Then $|\varpi^{-m} z a_i| \geqslant \kappa_1$, $|\varpi^{-m} z a_i^{-1}| \geqslant \kappa_1$, and $|\varpi^{-m} z| \geqslant \eta\kappa_1/\kappa \geqslant \eta$. Consequently, for all $z$ with $|z| \geqslant \eta\kappa_1/\kappa$, we can drop both $\xi_{\tilde{L}}$ and $\xi_{L'}$ and integrating over all $z = \beta\varpi^{-m}$, with $\beta \in \mathfrak{o}^\times$, and all of $\mathbf{G}_Z^\vee(F)$, we get

$$\sum_{\alpha \in (F^\times)^2 \backslash F^\times} \sum_j \Phi(Z_j, f_\tau) \Phi_{\varepsilon,s}(\alpha Y, f') \sum_{m \geqslant \ell} q^{-2mns} \mu(m). \tag{4.6}$$

Here $|\varpi^{-\ell}| = \eta\kappa_1/\kappa$ and $\mu(m)$ is equal to

$$\prod_i \int_{q^{-(m-d)} \leqslant |a_i| \leqslant q^{(m-d)}} d^\times a_i$$

times the measure of the compact part of $\mathbf{G}_Z^\vee(F)$. Moreover, $d$ is given by $\kappa_1 = q^d$. The series clearly converges for $\operatorname{Re} s > 0$. If $\mathbf{G}_Z^\vee(F)$ is not compact, then (4.6) vanishes. The remaining values of $z$, that is, $\eta\kappa_1/\kappa > |z| \geqslant \eta$, only lead to an entire function in $s$. The proof is now complete. □

COROLLARY 4.6. *Let $\mathbf{T}_i$ be a Cartan subgroup of $\mathbf{G}$. Denote by $T_i'$ the subset of regular elements of $T_i = \mathbf{T}_i(F)$. Let $\omega_i$ be a compact subset of $T_i'$. Then given $f_\tau, f', L$, and $L'$, $b = b(Y, Z)$ and $E_Z$ can be chosen independent of $Y$ and $Z$ for all $Z \in \omega_i$.*

*Proof.* This follows from the corollary to Lemma 19 of [10], which implies that the compact sets $\operatorname{supp}(g)$ and $\operatorname{supp}(h)$ of the proof of Lemma 4.5 can in fact be chosen to be the same for all $Z \in \omega_i$. □

To calculate the residue for the intertwining operator, we must now integrate over all the orbits of the action of $M^\vee$ on $N$. We do this by integrating over all the $\varepsilon$-regular $\varepsilon$-conjugacy classes in $\mathcal{N}$.

First assume $n = 2m$. We must integrate $\tilde{\psi}(s, Z)$ over orbits of $N$ under the action of $M^\vee$. Almost all of these orbits are parameterized by $\varepsilon$-regular conjugacy classes in $\mathcal{N}$. Therefore removing a set of measure zero from these orbits, we may integrate $\psi(s, Z)$ over $\varepsilon$-regular (and $\varepsilon$-semisimple) $\varepsilon$-conjugacy classes $\{Y\}$ in $\mathcal{N}$. Then, by Proposition 4.1, $\{Z\} = N_\varepsilon(\{Y^{-1}\})$ is regular and semisimple in $\mathbf{G}^\vee(F)$.

Let $\{\mathbf{T}_i\}$ denote a complete set of conjugacy classes of Cartan subgroups of $\mathbf{G}$ defined over $F$. We now must integrate over $\mathcal{N}$. As we discussed before, using Proposition 4.1 and the surjectivity of the norm correspondence, we instead

integrate over $\cup_i T_i$, using measures

$$|W(T_i)|^{-1}\kappa_1(\{\gamma_i\},\{\gamma_i'\})|D(\gamma_i)|\,d\gamma_i = |W(T_i)|^{-1}|D_{\theta^*}(\gamma_i')|\,d\gamma_i.$$

Now suppose $n > 2m$. Then for almost all $\{Y\} \in \mathcal{N}$, we can choose a representative $\mathrm{diag}(J_1, g, J_2)$, with $g \in GL_{2m}(F)$ $\varepsilon$-regular in $GL_{2m}(F)$ as in Corollary 4.2. More precisely, we may assume $X \in M_{n\times 2m}(F)$ has the form

$$X = \begin{pmatrix} 0_{j\times 2m} \\ I_{2m} \\ 0_{j\times 2m} \end{pmatrix}$$

with $j = (n/2) - m$. Moreover, as in Corollary 4.2, we may assume $Y$ preserves the kernel and the image of $XX'$ for almost all $Y$, and therefore $Y$ must be $\varepsilon$-skew symmetric on $\ker(XX')$. Consequently, $\mathrm{diag}(J_1, J_2)$ must be either symmetric (**G** symplectic) or skew symmetric (**G** orthogonal) on this kernel. Outside of a set of measure zero, the twisted conjugacy classes in $\mathcal{N}$ now form a fiber bundle with finite fibers coming from the twisted conjugacy classes of possible $\mathrm{diag}(J_1, J_2)$. The base of the fiber bundle is parameterized by $\varepsilon$-conjugacy classes $\{Y^{-1}\}$ in $GL_{2m}(F)$ such that $Y$ is the second component of a rational solution to (2.1) whenever $GL_{2m} \times \mathbf{G}(m)$ is considered as a Levi subgroup of $\mathbf{G}(3m)$, and we may use $\theta^*$-stable Cartan subgroups of $GL_{2m}$ and their $F$-isomorphisms with $\mathbf{T}_i$'s as in the case $n = 2m$. Here $\{\mathbf{T}_i\}_i$ is a complete set of representatives for the conjugacy classes of Cartan subgroups of **G**. We then get measures $\kappa_1(\gamma_i, \gamma_i')|D(\gamma_i)|\,d\gamma_i$ on the $T_i$'s, and by surjectivity we need to integrate over $\cup_i T_i$ for each fiber. The integral over the whole fiber bundle is then achieved by using the image correspondence $\mathscr{A}$ defined precisely as in the case $n = 2m$, but still integrating over $\cup_i T_i$. Still assuming $n \geqslant 2m$ and choosing

$$X = X_0 = \begin{pmatrix} 0_{j\times 2m} \\ I_{2m} \\ 0_{j\times 2m} \end{pmatrix},$$

equation (4.4) can now be written as

$$\begin{aligned}\tilde{\psi}(s, Z) = \sum_{\alpha} \omega'(\alpha) \\ \cdot \int_{g\in G'/\mathbf{G}'_{\varepsilon,Y}(F)} \int_{h\in \mathbf{G}_Z^{\vee}(F)\backslash G^{\vee}} \int_{h_0\in \mathbf{G}_Z^{\vee}(F)} f'(\alpha g Y \varepsilon(g)^{-1}) f_\tau(h^{-1} Z h) \\ \cdot |\det(gY\varepsilon(g)^{-1})|^s\, dg\, dh\, dh_0 \\ \cdot \int_{Z(G')} \xi_{\tilde{L}}(z^{-2} gY\varepsilon(g)^{-1}) \xi_{L'}(z^{-1} g X_0 h_0 h) |\det z|^{-2s}\, d^{\times} z.\end{aligned}$$

Lemma 4.5 is now valid for $\tilde{\psi}(s, Z) = \psi(s, Z)$ even if $n > 2m$, and in what follows we may assume $n \geqslant 2m$.

For each $G$-conjugacy class of Cartan subgroups of $\mathbf{G}$, choose a representative $\mathbf{T}_i$. Given $i$, let $T_i'$ be the subset of regular elements in $T_i = \mathbf{T}_i(F)$. For $\gamma \in T_i'$, let

$$\psi_{\mathscr{A}}(s, \gamma) = \sum \psi(s, \gamma),$$

where the sum is over all $\{Y\}$ for which $N_\varepsilon(\{Y^{-1}\})$ contains $\{\gamma\}$. Let $\omega_i$ be a compact subset of $T_i'$. By the absolute convergence of intertwining operators for $\operatorname{Re} s > 0$ (cf. [24]) we see that (4.3) is equal to

$$\sum_i |W(T_i)|^{-1} \int_{\omega_i} \psi_{\mathscr{A}}(s, \gamma)\, |D(\gamma)|\, d\gamma + \sum_i |W(T_i)|^{-1} \int_{T_i' \setminus \omega_i} \psi_{\mathscr{A}}(s, \gamma)\, |D(\gamma)|\, d\gamma.$$

Let

$$R_{\omega_i}(s) = |W(T_i)|^{-1} \int_{\omega_i} \psi_{\mathscr{A}}(s, \gamma)\, |D(\gamma)|\, d\gamma.$$

By Corollary 4.6,

$$R_{\omega_i}(s) = h_i(s) + |W(T_i)|^{-1} \int_{\omega_i} \Phi(\gamma, f_\tau) \Phi_\varepsilon(\mathscr{A}(\{\gamma\}), f')\, |D(\gamma)|\, d\gamma \cdot \mu(T_i) q^{bs} L(\mathbf{1}, 2ns),$$

if $\mathbf{T}_i$ is anisotropic, where $h_i(s)$ is an entire function of $s$. Otherwise, $R_{\omega_i}(s)$ is entire. Clearly,

$$\operatorname*{Res}_{s=0} \sum_i R_{\omega_i}(s)$$

$$= (2n \log q)^{-1} \sum_i{}' \mu(T_i) |W(T_i)|^{-1} \int_{\omega_i} \Phi(\gamma, f_\tau) \Phi_\varepsilon(\mathscr{A}(\{\gamma\}), f') |D(\gamma)|\, d\gamma,$$

where the sum $\sum_i'$ is understood to be over conjugacy classes of anisotropic Cartan subgroups $\mathbf{T}_i$ of $\mathbf{G}$. Thus, the residue of the operator at $s = 0$ equals

$$(2n \log q)^{-1} \sum_i{}' \mu(T_i) |W(T_i)|^{-1} \int_{\omega_i} \Phi(\gamma, f_\tau) \Phi_\varepsilon(\mathscr{A}(\{\gamma\}), f')\, |D(\gamma)|\, d\gamma$$

$$+ \operatorname*{Res}_{s=0} \sum_i |W(T_i)|^{-1} \int_{T_i' \setminus \omega_i} \psi_{\mathscr{A}}(s, \gamma)\, |D(\gamma)|\, d\gamma.$$

Letting $\omega_i \to T_i'$ for all $i$, the residue is then equal to

$$cR_{\mathbf{G}}(f_\tau, f') + \sum_i |W(T_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname{Res}_{s=0} \int_{T_i' \setminus \omega_i} \psi_{\mathscr{A}}(s, \gamma)\, |D(\gamma)|\, d\gamma,$$

where $c = (2n \log q)^{-1}$. Here $R_{\mathbf{G}}$ is defined the same way as in the case $n = 2m$.

Now suppose that $n < 2m$, and consider the injection

$$h \mapsto \begin{pmatrix} I_{m-(n/2)} & & \\ & h & \\ & & I_{m-(n/2)} \end{pmatrix}$$

of $\mathbf{G}(n/2)$ into $\mathbf{G}(m)$. Let $\mathbf{N}_\varepsilon : \mathscr{N} \to \mathscr{C}$ be the $\varepsilon$-norm correspondence from $\varepsilon$-conjugacy classes $\mathscr{N}$ in $GL_n(F)$ to conjugacy classes in $G(m)$. Suppose that $X \in M_{n\times 2m}(F)$ and $Y \in GL_n(F)$ satisfy $Y + \tilde{\varepsilon}(Y) = XX'$. Note that $\operatorname{rank} X'Y^{-1}X \leqslant n$, and so at most $n$ eigenvalues of $I - X'Y^{-1}X$ are different from 1. Thus, the semisimple part of the conjugacy class $\{I - X'Y^{-1}X'\}$ has a representative in $G(n/2)$. Now let $\mathscr{C}^\vee$ be the subset of $\mathscr{C}$ consisting of those conjugacy classes of $G(m)$ whose semisimple parts meet $G(n/2)$. Then we see that $N_\varepsilon : \mathscr{N} \to \mathscr{C}^\vee$.

LEMMA 4.7. *If $n < 2m$, then the norm correspondence $N_\varepsilon$ has finite fibers.*

*Proof.* We only need to show that if $Y \in GL_n(F)$ and $X \in M_{n\times 2m}(F)$ satisfy (2.1), then $I - X'Y^{-1}X$ determines the semisimple part of $\varepsilon(Y^{-1})Y^{-1}$. But, by Lemma 3.3, we have $\varepsilon(Y^{-1})Y^{-1}X = -X(I - X'Y^{-1}X)$. We may suppose that $X$ is in row echelon form, with the last $n - r$ rows of $X$ identically zero. We thus have the decomposition $F^n = F^r \oplus F^{n-r}$, with $X|_{F^r}$ an injection into $F^{2m}$, and $X|_{F^{n-r}} = 0$. Consequently, the matrix of $\varepsilon(Y^{-1})Y^{-1}$ with respect to a basis respecting the above decomposition of $F^n$ is $\left(\begin{smallmatrix} A & * \\ 0 & -I \end{smallmatrix}\right)$, with $A$ determined by $I - X'Y^{-1}X$. Thus, $N_\varepsilon$ has finite fibers. □

Continuing with our study of the case $n < 2m$, we notice that again almost all of the $\varepsilon$-conjugacy classes in $\mathscr{N}$ can be parameterized by regular semisimple conjugacy classes in $\mathscr{C}^\vee$ or by regular semisimple conjugacy classes in $G(n/2)$. More precisely, for $\mathbf{G} = SO_J$, we set $X_1 = (0_{n\times j} \quad I_n \quad 0_{n\times j})$, with $j = (2m - n)/2$. Then

$$X_1' = -\begin{pmatrix} & & w_j \\ & J_n & \\ w_j & & \end{pmatrix} \begin{pmatrix} 0_{j\times n} \\ I_n \\ 0_{j\times n} \end{pmatrix} w_n$$

$$= \begin{pmatrix} 0_{j\times n} \\ -J_n w_n \\ 0_{j\times n} \end{pmatrix}.$$

Thus, $X_1X_1' = -J_nw_n = Y + \tilde{\varepsilon}(Y)$. If $\mathbf{G} = Sp_{2r}$, and $j$ is even, then we take $X_1 = (0 \quad I_n \quad 0)$, as above. We have $X_1X_1' = -I_n = Y + \tilde{\varepsilon}(Y)$. If $j$ is odd, then we take $X_1 = (0 \quad u_n \quad 0)$. Then again we see that $X_1X_1' = -I_n = Y + \tilde{\varepsilon}(Y)$. In all but this last case, the identity

$$I_{2m} - X_1'Y^{-1}X_1 = Z = \begin{pmatrix} I_j & & \\ & Z_1 & \\ & & I_j \end{pmatrix},$$

with $Z_1$ in the image of the norm correspondence for $Y^{-1}$ in the case $n = 2m$, is now obvious. Thus, the parameterization of almost all the $\varepsilon$-conjugacy classes in $\mathcal{N}$ is as claimed in these cases. In the case where $\mathbf{G} = Sp_{2r}$ and $j$ is odd, then

$$I_{2m} - X_1'Y^{-1}X_1 = \begin{pmatrix} I_j & & \\ & I_n - u_nY^{-1}u_n & \\ & & I_j \end{pmatrix}$$

$$= \begin{pmatrix} I_j & & \\ & \tilde{\varepsilon}({}^t(I_n + Y^{-1})) & \\ & & I_j \end{pmatrix}.$$

Since $Y \in \mathcal{N}$ satisfies (2.1) with $X = I_n$, we know that $Z_1 = I_n + Y^{-1} \in Sp_{2n}$, and thus $\varepsilon(Z_1) = Z_1$, or $\tilde{\varepsilon}(Z_1) = Z_1^{-1}$. Therefore, $\tilde{\varepsilon}({}^t(I + Y^{-1})) = {}^t Z_1^{-1}$. Thus,

$$I - X_1'Y^{-1}X_1 = \begin{pmatrix} I_j & & \\ & {}^tZ_1^{-1} & \\ & & I_j \end{pmatrix},$$

with $Z_1$ as before.

We again choose a complete set of representatives $\{\mathbf{T}_i\}_i$ for the conjugacy classes of Cartan subgroups in $\mathbf{G}(n/2)$. None of the $\mathbf{T}_i$ is elliptic in $\mathbf{G} = \mathbf{G}(m)$. By Lemma 3.2, the contribution from almost all the orbits is given by

$$\psi(s, Z) = \sum_{\alpha} \omega'(\alpha)$$

$$\cdot \int_{g \in G'/\mathbf{G}'_{\varepsilon,Y}(F)} \int_{h \in \mathbf{G}_Z^{\vee}(F)\backslash G^{\vee}} \int_{X_1\mathbf{G}^{\vee}(F)} f'(\alpha g Y \varepsilon(g)^{-1}) f_{\tau}(h^{-1}Zh)$$

$$\cdot |\det(gY\varepsilon(g)^{-1})|^s \, dg \, dh \, d(X_1h_0)$$

$$\cdot \int_{Z(G')} \xi_{\tilde{L}}(z^{-2}gY\varepsilon(g^{-1}))\xi_{L'}(z^{-1}gX_1h_0h)|\det z|^{-2s} \, d^{\times}z,$$

where $X_1$ is as above and $Z = I - X_1' Y^{-1} X_1$. Moreover, $X_1 \mathbf{G}_Z^{\vee}(F) \simeq \mathbf{G}'_{\varepsilon,Y}(F)$, and therefore the formula for $\psi(s, Z)$ can be written as

$$\psi(s,Z) = \sum_{\alpha} \omega'(\alpha)$$

$$\cdot \int_{g \in G'/\mathbf{G}'_{\varepsilon,Y}(F)} \int_{h \in \mathbf{G}_Z^{\vee}(F)\backslash G^{\vee}} \int_{g_0 \in \mathbf{G}'_{\varepsilon,Y}(F)} f'(\alpha g Y \varepsilon(g^{-1})) f_\tau(h^{-1} Z h)$$

$$\cdot |\det(g Y \varepsilon(g^{-1}))|^s \, dg \, dh \, dg_0$$

$$\cdot \int_{Z(G')} \xi_{\tilde{L}}(z^{-2} g Y \varepsilon(g^{-1})) \xi_{L'}(z^{-1} g g_0 X_1 h) |\det z|^{-2s} d^{\times} z,$$

which is exactly the same expression as the one given in the case $n > 2m$, with the roles of $\mathbf{G}^{\vee}(F)$ and $\mathbf{G}'_{\varepsilon,Y}(F)$, as well as $h_0$ and $g_0$, interchanged. Using the image correspondence, again we define $\psi_{\mathscr{A}}(s, Z)$ as in the other cases. The integration of $\psi_{\mathscr{A}}(s, \gamma)$ now is over $\cup_i T_i$, and an argument similar to those used before applies. But this time $R_{\mathbf{G}}(f_\tau, f') = 0$, since $\Phi(\gamma, f_\tau) = 0$ for all elliptic $\gamma$. Observe that although the integration is only over regular semisimple conjugacy classes in $G(n/2)$, the orbital integrals $\Phi(-, f_\tau)$ are computed over all of $G = G(m)$.

We state our result as follows.

THEOREM 4.8. *Let $2\ell = \min(n, 2m)$ and denote by $\{\mathbf{T}_i\}$ a complete set of representatives for the conjugacy classes of Cartan subgroups of $\mathbf{G}(\ell)$. For each $i$, let $\omega_i$ denote a compact subset of $T_i'$, the set of regular elements of $T_i$. Then the intertwining operator $A(s, \tau' \otimes \tau, w_0)$ has a pole at $s = 0$ if and only if*

$$cR_{\mathbf{G}}(f_\tau, f') + \sum_i |W(T_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname*{Res}_{s=0} \int_{T_i' \backslash \omega_i} \psi_{\mathscr{A}}(s, \gamma) |D(\gamma)| \, d\gamma \neq 0,$$

*for some choice of the data $f_\tau, f', L$, and $L'$. Here $c = (2n \log q)^{-1}$. If $n < 2m$, then $R_{\mathbf{G}}(f_\tau, f') = 0$ for all $f_\tau$ and $f'$, and therefore it is the nonvanishing of*

$$\sum_i |W(T_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname*{Res}_{s=0} \int_{T_i' \backslash \omega_i} \psi_{\mathscr{A}}(s, \gamma) |D(\gamma)| \, d\gamma$$

*that determines the pole of the operator at $s = 0$.*

COROLLARY 4.9. *Suppose that $\tau' \simeq \tilde{\tau}'$.*

(a) *The induced representation $I(\tau' \otimes \tau)$ is irreducible if and only if*

$$cR_{\mathbf{G}}(f_\tau, f') + \sum_i |W(T_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname*{Res}_{s=0} \int_{T_i' \backslash \omega_i} \psi_{\mathscr{A}}(s, \gamma) |D(\gamma)| \, d\gamma \neq 0$$

*for some choice of the data.*

(b) *Assume* $\tau$ *is generic. If* $I(\tau' \otimes \tau)$ *is irreducible, then* $I(s, \tau' \otimes \tau)$, $s \in \mathbb{R}$, *is reducible exactly at* $s = \pm 1/2$ *or* $s = \pm 1$, *and at only one of these pairs.*

*Remark 4.10.* When $n \geqslant 2m$, the residue is the sum of $cR_{\mathbf{G}}$ and the more complicated term

$$\sum_i |W(T_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname{Res}_{s=0} \int_{T_i' \backslash \omega_i} \psi_{\mathscr{M}}(s, \gamma) |D(\gamma)| \, d\gamma.$$

In the next section, we relate the nonvanishing of the second term to the simple nonvanishing condition of Proposition 5.1 of [21], that is, in terms of the theory of twisted endoscopy that discusses the equivalence of this nonvanishing with $\tau'$ coming from $SO_{n+1}(F)$.

*Remark 4.11.* Clearly Theorem 4.8 and Corollary 4.9 are valid for the split even special orthogonal groups and are the correct versions of what was originally stated as Theorems 7.8 and 8.1 in [22]. In fact, although the main term $R_{\mathbf{G}}$ of Theorem 7.8 of [22] is correct, the singular terms given there are too optimistic and are consequences of a gap in the proof of Theorem 7.8 of [22]. However, in Proposition 5.2 of the next section, using $L$-functions, we relate the singular terms from the two different versions to each other. We refer the reader to the introduction for further comments.

**§5. The connection with twisted endoscopy and $L$-functions.** We now show how the results from Section 4 can be related to the theory of twisted endoscopy and $L$-functions.

Let $\chi_\tau$ be the distribution character of $\tau$. By the work of Harish-Chandra, [10], [11], we know that $\chi_\tau$ is given by a locally integrable function, which we also denote by $\chi_\tau$. From [12] and [7], we can choose a matrix coefficient $f_\tau$ for $\tau$ such that $\Phi(\gamma, f_\tau) = \chi_\tau(\gamma)$, for all regular semisimple $\gamma \in G$. The matrix coefficient $f_\tau$ is then called a *pseudocoefficient.*

Since $\tau' \simeq \tilde{\tau}' \simeq (\tau')^\varepsilon$, we see that $\tau'$ extends to a representation of the disconnected group $GL_n(F) \ltimes \{1, \varepsilon\}$. This comes from fixing an equivalence $\tau'(\varepsilon)$ between $\tau'$ and $(\tau')^\varepsilon$. The $\varepsilon$-twisted distribution character of $\tau'$ is defined by $\chi^\varepsilon_{\tau'}(f') = \operatorname{trace}(\tau'(f)\tau'(\varepsilon))$. In [6] Clozel showed that there is a locally integrable function, also denoted by $\chi^\varepsilon_{\tau'}$, on the $\varepsilon$-regular elements such that

$$\chi^\varepsilon_{\tau'}(f') = \int_{Z' \backslash G'} \chi^\varepsilon_{\tau'}(g) f'(g) \, dg,$$

with $Z'$ the center of $G'$. Kottwitz and Rogawski [15] proved the existence of $\varepsilon$-twisted pseudocoefficients. That is, there exists a matrix coefficient $\psi_{\tau'}$ of $\tau'$ such that $\Phi_\varepsilon(\gamma', \psi_{\tau'}) = \chi^\varepsilon_{\tau'}(\gamma')$ for every $\varepsilon$-regular element $\gamma' \in GL_n(F)$. Thus, by

choosing $f' \in C_c^\infty(GL_n(F))$, which defines $\psi_{\tau'}$, we have

$$\Phi_\varepsilon(\mathscr{A}(\{\gamma\}), f') = \sum_{\gamma' \in \mathscr{A}(\{\gamma\})} \Delta(\gamma, \gamma')\chi_{\tau'}^\varepsilon(\gamma'),$$

and we denote this by $\chi_{\tau'}^\varepsilon(\mathscr{A}(\{\gamma\}))$.

Let $n = 2m$. Then, by the above observations, we can choose $f_\tau$ and $f'$ so that the regular term becomes

$$R_{\mathbf{G}}(f_\tau, f') = \sum_{\{\mathbf{T}_i\}} \mu(T_i)|W(T_i)|^{-1} \int_{\mathbf{T}_i} \chi_\tau(\gamma)\chi_{\tau'}^\varepsilon(\mathscr{A}(\{\gamma\}))|D(\gamma)|\, d\gamma.$$

Thus, $R_{\mathbf{G}}$ defines a pairing between the character $\chi_\tau$ of $\tau$, and the $\varepsilon$-twisted character $\chi_{\tau'}^\varepsilon$ of $\tau'$. Consequently, we expect that the nonvanishing of $R_G(f_\tau, f')$ must in part point towards $\tau'$ coming from $\tau$ via twisted endoscopy [2], [3], [13], [14]. For our purposes, we define the notion of endoscopic transfer below. Observe that $R_{\mathbf{G}}$ makes sense and is convergent for any discrete series $\tau$.

DEFINITION 5.1. *A self-dual irreducible supercuspidal representation $\tau'$ of $G'$ is said to be the $\varepsilon$-twisted endoscopic transfer of a discrete series representation $\tau$ of $G(n/2)$ if $R_{\mathbf{G}}(f_\tau, f') \neq 0$, for some matrix coefficient $f_\tau$ of $\tau$, and some $f' \in C_c^\infty(GL_n(F))$ defining one for $\tau'$.*

Assume $n \geqslant 2m$, and resume our assumption that $\tau$ and $\tau'$ are supercuspidal. We expect that the Rankin-Selberg product $L$-function $L(s, \tau' \times \tau)$, which was formally defined in [20], must satisfy the following (defining) condition:

> *$L(s, \tau' \times \tau)$ has a pole at $s = 0$ if and only if $R_{\mathbf{G}}(f_\tau, f') \neq 0$ for some $f_\tau$ and $f'$, or equivalently, if and only if $\tau'$ comes by twisted endoscopy from $\tau$ (Definition 5.1).*

As discussed in the next few paragraphs, this seems to be in complete agreement with definitions given in [21].

We must now study the singular contributions, and we therefore continue with our assumption $n \geqslant 2m$, and let

$$R_{\mathrm{sing}}(f_\tau, f') = \sum_i |W(T_i)|^{-1} \lim_{\omega_i \to T_i} \operatorname*{Res}_{s=0} \int_{T_i' \backslash \omega_i} \psi_{\mathscr{A}}(s, \gamma)\, |D(\gamma)|\, d\gamma$$

so that the residue of the intertwining operator $A(s, \tau' \otimes \tau, w_0)$ at $s = 0$ can be written as

$$cR_{\mathbf{G}}(f_\tau, f') + R_{\mathrm{sing}}(f_\tau, f'),$$

where $c = (2n \log q)^{-1}$. Observe that we have suppressed the dependence of $R_{\text{sing}}$ on $L$ and $L'$.

Our goal in what follows is to separate out the poles coming from the two $L$-functions $L(s, \tau' \times \tau)$ and $L(s, \tau', \wedge^2 \rho_n)$ by means of the nonvanishing of $R_{\mathbf{G}}$ and $R_{\text{sing}}$, respectively, when $n \geqslant 2m$.

By Lemma 4.5, $R_{\text{sing}}(f_\tau, f') = 0$ if and only if the process of taking the residue and integration can be interchanged. On the other hand, the theory of $L$-functions discussed earlier demands the existence of poles coming from those of $L(s, \tau', \wedge^2 \rho_n)$ at $s = 0$. Such poles depend on $\tau'$ alone and cannot be reflected in the nonvanishing of $R_{\mathbf{G}}$. Consequently, one must expect $R_{\text{sing}} \neq 0$ in general, as we discuss below. Therefore, the fact that interchanging the process of taking the residue at $s = 0$ and integration must be checked is not just an analytic impediment, but rather a fact that reflects deep arithmetic connections that govern the problem.

Now, let us assume that $\tau'$ comes from $SO_{n+1}(F)$ as explained in Proposition 5.1 and Theorem 7.6 of [21], or in other words, $L(s, \tau', \wedge^2 \rho_n)$ has a pole at $s = 0$. Then, by the simplicity of the pole of $A(s, \tau' \otimes \tau, w_0)$ at $s = 0$, $L(s, \tau' \times \tau)$ must be holomorphic there. (See below and the discussion after Definition 5.1.) Moreover, assume that $\tau$ is such that $R_{\mathbf{G}}(f_\tau, f') = 0$ for all $f_\tau$ and $f'$. We expect this to be true for all $\tau$ anyway, if $\tau'$ comes from $SO_{n+1}(F)$. Then, by the theory of $L$-functions [20] as explained before,

$$L(s, \tau' \times \tau)^{-1} L(2s, \tau', \wedge^2 \rho_n)^{-1} A(s, \tau' \otimes \tau, w_0)$$

must be nonzero and holomorphic. This implies that $R_{\text{sing}}(f_\tau, f') \neq 0$ for some data. We therefore have the following result.

PROPOSITION 5.2. *Suppose $n \geqslant 2m$. Assume that $\tau'$ comes from $SO_{n+1}(F)$. Then $R_{\text{sing}} \neq 0$ for any irreducible unitary supercuspidal representation $\tau$ of $G(m)$ from which $\tau'$ does not come by twisted endoscopy (Definition 5.1). (We expect this to be the case for every irreducible unitary supercuspidal representation $\tau$ of $G(m)$ if $\tau'$ comes from $SO_{n+1}(F)$.)*

When $n < 2m$, the term $R_{\mathbf{G}} = 0$, and therefore the control of the poles of both $L$-functions lies within $R_{\text{sing}}$, which now constitutes the whole residue. Further analysis of the term $R_{\text{sing}}$ is now necessary to distinguish the two $L$-functions.

We conclude the paper by stating a result about $L(s, \tau' \times \tau)$ for any even $n$ and any $m$ with no further assumption on the relation between $\tau$ and $\tau'$. We may and do assume that $\tau' \simeq \tilde{\tau}'$.

PROPOSITION 5.3. (a) *Suppose $\tau'$ comes from $SO_{n+1}(F)$ (nonvanishing condition (5.2) of [21]), or equivalently, $L(s, \tau', \wedge^2 \rho_n)$ has a pole at $s = 0$. Then $L(s, \tau' \times \tau)$ is holomorphic at $s = 0$.*

(b) *If $\tau'$ does not come from $SO_{n+1}(F)$, then $L(s, \tau' \times \tau)$ has a pole at $s = 0$ if and only if $cR_{\mathbf{G}} + R_{\text{sing}} \neq 0$ for some data.*

*Proof.* The proposition is a consequence of the simplicity of the poles of $A(s, \tau' \otimes \tau, w_0)$, Theorem 4.8, and the holomorphy and nonvanishing of

$$L(s, \tau' \times \tau)^{-1} L(2s, \tau', \wedge^2 \rho_n)^{-1} A(s, \tau' \otimes \tau, w_0).$$

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