QUADRATIC UNIPOTENT ARTHUR PARAMETERS AND RESIDUAL SPECTRUM OF SYMPLECTIC GROUPS

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Abstract. The purpose of this paper is to study certain quadratic unipotent Arthur parameters in the sense of Moeglin and use them to parametrize a part of the residual spectrum of symplectic groups over number fields, coming from the conjugacy class of Borel subgroups. In particular, using certain identities satisfied by local intertwining operators, Arthur's multiplicity formula is established for them which remarkably enough appears by itself in the corresponding residue of the Eisenstein series.

Introduction. In this paper we study certain quadratic unipotent Arthur parameters in the sense of Moeglin [23] (cf. Section 3) and realize them as Arthur parameters [1, 2] for certain square integrable residues of Eisenstein series attached to the conjugacy class of Borel subgroups of a symplectic group over a number field. Consequently, using certain local identities proved in [11], we prove Arthur's multiplicity formula for them (Theorem 5.1), which remarkably enough appears by itself in the corresponding residue of the Eisenstein series.

More precisely, let $G = \mathrm{Sp}_{2n}$ over a number field F with ring of adeles $\mathbb{A}_F$. As in [23], we use $G^* = O_{2n+1}(\mathbb{C})$ to denote its dual group. Let $\mu_1, \ldots, \mu_k$ be k distinct nontrivial quadratic grössencharacters of F. Fix integers $r_1 \geq \ldots \geq r_k \geq 2$ with $r_1 + \ldots + r_k \leq n$ and choose r_0 such that $r_0 + r_1 + \ldots + r_k = n$. Then $\chi = \chi(\underbrace{\mu_1, \ldots, \mu_1}_{r_1}, \ldots, \underbrace{\mu_k, \ldots, \mu_k}_{r_k}, \underbrace{1, \ldots, 1}_{r_0})$ defines a character of $T(F)\backslash T(\mathbb{A}_F)$, where T is the subgroup of diagonal elements in G. An Eisenstein series [18] attached to a character of $T(\mathbb{A}_F)$ will contribute to the residual spectrum only if the character is of the above type (Proposition 4.6).

By [14, 16], the character χ defines a homomorphism of the Weil group W_F into a Cartan subgroup of $\mathrm{SO}_{2n+1}(\mathbb{C})$. Composing this homomorphism with the standard action of $O_{2n+1}(\mathbb{C})$ on $\mathbb{C}^{2n+1}$ will then give a completely reducible representation of W_F on $\mathbb{C}^{2n+1}$ which decomposes according to eigenvalues $\mu_1, \ldots, \mu_k$, and 1, with multiplicities $2r_1, \ldots, 2r_k$, and $2r_0 + 1$, respectively. Write $\mathbb{C}^{2n+1} = V_0 \oplus V_1 \oplus \ldots \oplus V_k$, where each $V_i, \dim V_i = 2r_i$, is the eigenspace attached to

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eigenvalue μ_i, $1 \le i \le k$, and V_0 is the trivial eigenspace of dimension $2r_0+1$. In this way we get an embedding of $\prod_{i=0}^k O(V_i) \subset O_{2n+1}(\mathbb{C})$.

Now, for each i, $1 \le i \le k$, let O_i be the unipotent orbit of $O(V_i)$ attached to the principal Jordan block $(2r_i-1, 1)$. Let O_o be the principal unipotent orbit of $O(V_0)$, i.e. the one attached to the Jordan block $(2r_0+1)$.

The Arthur parameter of interest to us is a homomorphism

$$\psi\colon\ W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to O_{2n+1}(\mathbb{C})$$

which factors through $\prod_{i=0}^k O(V_i)$, sends W_F to the center of $\prod_{i=0}^k O(V_i)$ according to $1 \otimes \mu_1 \otimes \ldots \otimes \mu_k$, is trivial on the middle $SL_2(\mathbb{C})$, and for which

$$\psi(1, 1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$$

belongs to $\prod_{i=0}^k O_i$. This is clearly a quadratic unipotent Arthur parameter in the sense of Moeglin [23] (see Section 3). To ψ, Arthur associates a Langlands' parameter ϕ_ψ (see Section 3).

In this paper, we use Langlands theory of Eisenstein series [18] to construct the representations in Π_{ϕ_ψ} as residues of the Eisenstein series associated to the character χ (Theorems 4.5 and 5.1). Using certain identities satisfied by local intertwining operators which was proved in [11], we then verify Arthur's multiplicity formula for these square integrable residues (Theorem 5.1). (See section 3 for Arthur's multiciplity formula.) It is remarkable that in fact the formula itself appears in the corresponding residue of the Eisenstein series. We note that the local R-group $C_{\phi_{\psi_v}}$ for the parameter ϕ_{ψ_v} (see Section 3) is the Knapp-Stein R-group of the unitary principal series $I_v = \mathrm{Ind}_{B_0(F_v)}^{M(F_v)} \chi_v$, where M is the Levi-subgroup whose L-group is $M^* = \mathrm{Cent}\,(im\phi_\psi^+, G^*)$.

The technical combinatorial part of dealing with the residues of Eisenstein series (Proposition 4.4 and Theorem 4.5) which is an important step in the proof is contained in Section 4. They rely on several technical lemmas about Weyl groups and normalizing factors (Lemmas 4.7 and 4.8). The final interpretation of Theorem 4.5 and the proof of Arthur's multiplicity formula (Arthur's condition in the language of Moeglin [23]) is done in Section 5 (Theorem 5.1). In particular, in Section 5 we determine Arthur parameters of the square integrable residual spectrum of $\mathrm{Sp}_4(\mathbb{A}_F)$ coming from the conjugacy class of Borel subgroups. Using [12], this implies the exhaustion, i.e., that quadratic unipotent Arthur parameters completely determine the residual spectrum of $\mathrm{Sp}_4(\mathbb{A})$ coming from this conjugacy class and conversely.

We expect that quadratic unipotent Arthur parameters completely parametrize all the residual spectrum of Sp_{2n} coming from the conjugacy class of Borel subgroups. When these residues are unramified, i.e., when $\chi = 1$, the problem has been completely solved by Moeglin [22, 20]. We expect that her results will play an important role in proving the exhaustion in general. We would like to thank her for patiently answering many of our questions [21].

Finally in view of [23], we expect that the method of the present paper can be equally well applied to the case of orthogonal groups.

1. Preliminaries. Let F be a field and let $G = SO_{2n+1}$, Sp_{2n} or SO_{2n} over F. Let J_n be the $n \times n$ matrix given by

$$J_n = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & \cdot & & \\ & & \cdot & & & \\ & \cdot & & & & \\ 1 & & & & & \end{pmatrix}.$$

Let $J'_{2n} = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$. Then

$$\mathrm{Sp}\,(2n) = \left\{ g \in GL(2n) \middle| \ {}^t g J'_{2n} g = J'_{2n} \right\},$$

and

$$\mathrm{SO}\,(n) = \left\{ g \in GL(n) \middle| \ {}^t g J_n g = J_n ; det(g) = 1 \right\}.$$

In each case we let T be the maximal split torus consisting of diagonal matrices in G. Then

$$T(F) = \left\{ \begin{pmatrix} l_1 & & & & & & & \\ & l_2 & & & & & & \\ & & \ddots & & & & & \\ & & & l_n & & & & \\ & & & & l_n^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & l_2^{-1} & \\ & & & & & & & l_1^{-1} \end{pmatrix} \middle| \ l_i \in F^* \right\},$$

if $G = \mathrm{Sp}\,(2n)$ or $\mathrm{SO}\,(2n)$, and

$$T(F) = \left\{ \begin{pmatrix} l_1 &&&&&&&& \\ & l_2 &&&&&&& \\ && \ddots &&&&&& \\ &&& l_n &&&&& \\ &&&& 1 &&&& \\ &&&&& l_n^{-1} &&& \\ &&&&&& \ddots && \\ &&&&&&& l_2^{-1} & \\ &&&&&&&& l_1^{-1} \end{pmatrix} \;\middle|\; l_i \in F^* \right\}$$

if $G = \mathrm{SO}\,(2n+1)$.

Let $\Phi(G,T)$ be the roots of G with respect to T. We choose the ordering on the roots so that the Borel subgroup B is the subgroup of upper triangular matrices in G. Let Δ be the simple roots in $\Phi(G,T)$ given by $\Delta = \{\alpha_j\}_{j=1}^n$, with $\alpha_j = e_j - e_{j+1}$ for $1 \le j \le n-1$, and

$$\alpha_n = \begin{cases} e_n & G = \mathrm{SO}\,(2n+1), \\ 2e_n & G = \mathrm{Sp}\,(2n), \\ e_{n-1} + e_n & G = \mathrm{SO}\,(2n). \end{cases}$$

We let $\langle , \rangle$ be the standard Euclidean inner product on $\Phi(G,T)$. If Φ is a root system of type B_n, C_n, or D_n, then we denote by $G(\Phi)$ the split group with root system Φ.

For $G = \mathrm{SO}\,(2n+1)$ or $\mathrm{Sp}\,(2n)$, the Weyl group $W(G/T) \simeq S_n \ltimes \mathbb{Z}_2^n$. S_n acts by permutations on the λ_i, $i = 1, \ldots, n$. We will use standard cycle notation for the elements of S_n. Thus (ij) interchanges λ_i and λ_j. If c_i is the nontrivial element in the ith copy of $\mathbb{Z}_2$ then c_i takes λ_i to λ_i^{-1}. The element c_i is called a sign change because its action on $\Phi(G,T)$ takes e_i to $-e_i$. For $G = \mathrm{SO}\,(2n)$, the Weyl group is given by $W(G/T) \simeq S_n \ltimes Z_2^{n-1}$. S_n acts by permutations on the λ_i, and $\mathbb{Z}_2^{n-1}$ acts by even numbers of sign changes. The requirement that the number of sign changes be even comes from the determinant condition in $\mathrm{SO}\,(2n)$. Note that the sign change c_i is an element of $O(2n)$ and normalizes $T(F)$. Each c_i acts on $\mathrm{SO}\,(2n)$ by conjugation, and c_n induces the nontrivial graph automorphism on the Dynkin diagram of $\Phi(G,T)$.

2. Unipotent orbits of classical groups over $\mathbb{C}$. The theory of Jordan normal forms implies that a unipotent matrix in GL_N is conjugate to $J(p_1) \oplus J(p_2) \oplus \cdots \oplus J(p_s)$, $p_1 \ge p_2 \ge \cdots \ge p_s$, $p_1 + p_2 + \cdots + p_s = N$, where $J(p)$ is the $p \times p$ Jordan matrix with entries 1 just above the diagonal and the diagonal, and zero everywhere else. Therefore, unipotent classes in GL_N are in 1 to 1 correpondence with partitions λ of N. We use the following standard notation for λ: $\lambda = (1^{r_1}, 2^{r_2}, 3^{r_3}, \cdots)$, where r_j is the number of p_i equal to j.

Let G be a classical group, of type B_n ($O_{2n+1}(\mathbb{C})$), C_n ($\mathrm{Sp}_{2n}(\mathbb{C})$) or D_n ($O_{2n}(\mathbb{C})$). We start with the following facts:

(1) $X, X' \in G$ are conjugate in G if and only if they are conjugate in GL_N, $N = 2n+1$ or $2n$.

(2) Let $X \in GL_N$ be unipotent. Then X is conjugate to an element of G if and only if r_i is even for even i in the orthogonal case and for odd i in the symplectic case.

Therefore for $G = O_{2n+1}(\mathbb{C})$, unipotent classes are in 1 to 1 correspondence with partitions λ of $2n+1$ such that r_i is even for even i.

Let u be a unipotent element in G and let S_u be its centralizer in G. Then we have:

(3) In the orthogonal case (resp. symplectic) case, S_u/S_u° is k product of $\mathbb{Z}/2\mathbb{Z}$, where k is the number of odd (resp. even) i such that $r_i > 0$.

Here we note that for $G = GL_N(\mathbb{C})$, the centralizer $Z_G(S)$ is connected for any subset S of G.

We say that a unipotent element u is distinguished if all maximal tori of $\mathrm{Cent}(u, G)$ are contained in the center of G°, the connected component of the identity. This is equivalent to the fact that the unipotent orbit O of u does not meet any proper Levi subgroup of G (Spaltenstein [30, p. 67]; i.e., if L is a Levi subgroup of a parabolic subgroup of G and $u \in L$ for a $u \in O$, then $L^\circ = G^\circ$). If $G = O_{2n+1}(\mathbb{C})$, then $G^\circ = \mathrm{SO}_{2n+1}(\mathbb{C})$ and G° has trivial center. By Carter [5], for $G = O_{2n+1}(\mathbb{C})$ or $O_{2n}(\mathbb{C})$, if u is a unipotent element with Jordan blocks $(1^{r_1}, 2^{r_2}, \ldots)$, then the reductive part of the connected centralizer $\mathrm{Cent}(u, G)^\circ$ is of type

$$\prod_{i\ even} C_{r_i/2} \times \prod_{i\ odd,\ r_i\ even} D_{r_i/2} \times \prod_{i\ odd,\ r_i\ odd} B_{(r_i-1)/2}.$$

Therefore, O is a distinguished unipotent class if and only if it has Jordan blocks $(1^{r_1}, 3^{r_3}, 5^{r_5}, \ldots)$, where $r_i = 0$ or 1.

JACOBSON-MOROZOV THEOREM. *Suppose u is a unipotent element in a semisimple algebraic group G. Then there exists a homomorphism ϕ: $SL_2 \longmapsto G$ such that* $\phi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = u$.

Here, replacing ϕ by a conjugate under G, we can assume that $\phi\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ is in the closure of the positive Weyl chamber in the maximal torus. In fact, by the theory of weighted Dynkin diagrams (cf. Section 5.6 of [5]), $\phi\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ is uniquely determined by the unipotent orbit of u as follows (Carter [5, p. 395]):

Suppose O has Jordan blocks $(d_1, d_2, d_3, \ldots)$. For each d_i, we take the set of integers $d_i - 1, d_i - 3, \ldots, 3 - d_i, 1 - d_i$. We then take the union of these sets for all d_i and write this union as $(\xi_1, \xi_2, \xi_3, \ldots)$ with $\xi_1 \geq \xi_2 \geq \xi_3 \geq \cdots$. Then

$$\phi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \mathrm{diag}\,(a^{\xi_1}, a^{\xi_2}, a^{\xi_3}, \ldots). \tag{2.1}$$

LEMMA. ([3, Proposition 2.4]) *Let u be a unipotent element and ϕ: $SL_2 \longmapsto G$ be a homomorphism such that $\phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = u$. Let $S_\phi = \mathrm{Cent}\,(im\phi, G) \subset S_u = \mathrm{Cent}\,(u, G)$ and U^u be the unipotent radical of S_u. Then*

(1) *$S_u = S_\phi \cdot U^u$, a semi-direct product. S_ϕ is reductive.*

(2) *The inclusion $S_\phi \subset S_u$ induces an isomorphism between $S_\phi/S_\phi^\circ Z_G$ and $S_u/S_u^\circ Z_G$.*

3. Quadratic unipotent Arthur parameters. We follow Moeglin [23]. Let F be a number field and let W_F be the global Weil group of F. For $G = \mathrm{Sp}_{2n}$, we can take the dual group $G^* = O_{2n+1}(\mathbb{C})$. An Arthur parameter is a homomorphism

$$\psi\colon\ W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longmapsto O_{2n+1}(\mathbb{C}),$$

with the following properties: (The usual definition of Arthur parameter uses Langlands' hypothetical group L_F. But since we are only dealing with Langlands' quotients which come from principal series, W_F is enough.)

(1) $\psi(W_F)$ is bounded and included in the set of semi-simple elements of G^*.

(2) The restriction of ψ to the 2 copies of $SL_2(\mathbb{C})$ is algebraic.

(3) Composing $\psi|_{W_F}$ with the determinant of G^* gives a quadratic character of W_F, denoted by $det\psi$. We want $det\psi = 1$.

We call an Arthur parameter quadratic unipotent if the following two conditions are satisfied:

(4) $\psi|_{1\times SL_2(\mathbb{C})\times 1} \equiv 1$;

(5) $\psi|_{W_F}$ is trivial on the intersection of the kernels of the quadratic characters of W_F.

Because of conditions (1) and (5), the action of $\psi(W_F)$ gives an orthogonal decomposition:

$$\mathbb{C}^{2n+1} = V_0 \oplus V_1 \oplus \cdots \oplus V_k,$$

where $dim\, V_0 = 2r_0 + 1$, $dim\, V_i = 2r_i$, $2r_0 + 1 + 2r_1 + \cdots + 2r_k = 2n + 1$, $r_1 \geq \cdots \geq r_k$ and V_i is the eigenspace with eigenvalue μ_i. Here $\mu_1, \ldots, \mu_k$ are nontrivial distinct quadratic grössencharacters of F, viewed as characters of W_F (cf. [14, 16]), and $dim\, V_i$ being even comes from condition (3).

The parameter ψ factors through $\prod_{i=0}^{k} O(V_i)$:

$$\psi\colon\ W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longmapsto \prod_{i=0}^{k} O(V_i).$$

(1) W_F is mapped into the product of centers of $O(V_i)$

$$\psi|W_F\colon\ w \longmapsto 1 \times \mu_1(w) \times \cdots \times \mu_k(w) \in \{\pm 1\} \times \{\pm 1\} \cdots \times \{\pm 1\},$$

where $\{\pm 1\}$ is the center of $O(V_i)$, for $i = 0, \ldots, k$.

(2) By Jacobson-Morozov theorem, $\psi|_{1\times 1\times SL_2(\mathbb{C})}$ defines a unipotent orbit of G^* of the form

$$\prod_{i=0}^{k} O_i,$$

where O_i is a unipotent orbit of $O(V_i)$. Inside O_i we fix an element u_i such that

$$\psi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \prod_{i=0}^{k} u_i.$$

Let $S_\psi = \mathrm{Cent}\,(im\psi, G^*)$ and

$$C_\psi = S_\psi / S_\psi^\circ Z_{G^*}.$$

We know that S_ψ is a maximal reductive subgroup of $\prod_{i=0}^{k} \mathrm{Cent}\,(u_i, O(V_i))$. Therefore $S_\psi^\circ = 1$, i.e., S_ψ is finite if and only if each u_i is a distinguished unipotent element in $O(V_i)$. Especially, since $O_2(\mathbb{C})$ has no distinguished unipotent element, we have

LEMMA. *Let ψ be a quadratic unipotent Arthur parameter. Suppose $S_\psi^\circ = 1$. Then $r_k \geq 2$.*

Now it is clear that $S_\psi / S_\psi^\circ Z_{G^*}$ is equal to

$$\mathrm{Cent}\,(u_0, O(V_0)) / \mathrm{Cent}\,(u_0, O(V_0))^\circ Z_{O(V_0)} \prod_{i=1}^{k} \mathrm{Cent}\,(u_i, O(V_i)) / \mathrm{Cent}\,(u_i, O(V_i))^\circ.$$

Here $\mathrm{Cent}\,(u_i, O(V_i)) / \mathrm{Cent}\,(u_i, O(V_i))^\circ$ is t product of $\mathbb{Z}/2\mathbb{Z}$, where t is the number of i odd with $r_i > 0$ in Jordan blocks.

For each place v of F, we have a map $\psi_v = \psi|W_{F_v} \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. As in the global case, we can then define S_{ψ_v}. But in the local case, μ_{iv} may not be distinct. Suppose $\mu_{1v} = \mu_{2v}$. Then in the above formula,

$$\operatorname{Cent}(u_1, O(V_1))/\operatorname{Cent}(u_1, O(V_1))^\circ \times \operatorname{Cent}(u_2, O(V_2))/\operatorname{Cent}(u_2, O(V_2))^\circ$$

must be replaced by

$$\operatorname{Cent}(u_1 \times u_2, O(V \oplus V_2))/\operatorname{Cent}(u_1 \times u_2, O(V_1 \oplus V_2))^\circ.$$

To any Arthur parameter ψ, Arthur associates a Langlands' parameter $\phi_\psi\colon W_F \times SL_2(\mathbb{C}) \longmapsto G^*$ as follows:

$$\phi_\psi(w,1) = \psi\left(w, 1, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right), \quad \phi_\psi|1 \times SL_2(\mathbb{C}) \equiv 1.$$

For quadratic unipotent Arthur parameter ψ, ϕ_ψ is given by

(1) $\phi_\psi|SL_2(\mathbb{C}) \equiv 1$;

(2) $\phi_\psi(w) = \prod_{i=0}^k \phi_{\psi_i}(w) \in \prod_{i=0}^k O(V_i)$, where each $\phi_{\psi_i}(w)$ is the associated Langlands parameter for ψ_i: $W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longmapsto O(V_i)$.

Now we recall Arthur's conjecture. Let $S_{\phi_\psi} = \operatorname{Cent}(im\phi_\psi, G^*)$ and

$$C_{\phi_\psi} = S_{\phi_\psi}/S^\circ_{\phi_\psi} Z_{G^*}.$$

For each place v of F, we have local Arthur parameters $\psi_v = \psi|W_{F_v} \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$, as well as S_{ψ_v}, C_{ψ_v}, $S_{\phi_{\psi_v}}$ and $C_{\phi_{\psi_v}}$. For each v, there is also a natural map $C_\psi \longmapsto C_{\psi_v}$ and a natural surjective $C_{\psi_v} \longmapsto C_{\phi_{\psi_v}}$. The parameter ϕ_{ψ_v} gives a L-packet $\Pi_{\phi_{\psi_v}}$ which consists of Langlands' quotients.

It is a part of Arthur's local conjecture [1, 2] that for each place v of F, there is a pairing $\langle\ ,\ \rangle$ on $C_{\phi_{\psi_v}} \times \Pi_{\phi_{\psi_v}}$ and an enlargement Π_{ψ_v} of $\Pi_{\phi_{\psi_v}}$ which allows an extension of $\langle\ ,\ \rangle$ to $C_{\psi_v} \times \Pi_{\psi_v}$ such that $\pi \in \Pi_{\phi_{\psi_v}} \subset \Pi_{\psi_v}$ if and only if the function $\langle \cdot, \pi\rangle$ lies in the image of $\hat{C}_{\phi_{\psi_v}}$ in $\hat{C}_{\psi_v}$.

We define the global Arthur packet Π_ψ to be the set of irreducible representations $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ such that for each v, π_v belongs to Π_{ψ_v}. Define the global pairing on $C_\psi \times \Pi_\psi$ by

$$\langle x, \pi\rangle = \prod_v \langle x_v, \pi_v\rangle,$$

for $\pi = \otimes_v \pi_v \in \Pi_\psi$ and $x \in C_\psi$ with image x_v in C_{ψ_v}.

Arthur's conjecture (Global).

(1) The representations in the packet corresponding to ψ may occur in the discrete spectrum if and only if S_ψ is finite, i.e., $S^\circ_\psi = 1$. We call such an Arthur parameter elliptic.

(2) For an elliptic Arthur parameter ψ, there is a positive integer d_ψ and a homomorphism $\epsilon_\psi\colon C_\psi \longmapsto \{\pm 1\}$ such that the multiplicity with which any $\pi \in \Pi_\psi$ occurs discretely in $L^2(G(F)\backslash G(\mathbb{A}))$ is

$$\frac{d_\psi}{|C_\psi|}\sum_{x\in C_\psi}\epsilon_\psi(x)\langle x,\pi\rangle. \tag{3.1}$$

For quadratic unipotent Arthur parameters, we have

LEMMA. (Moeglin [23]) *For ψ quadratic unipotent, ϵ_ψ is trivial.*

In Section 4, we restrict ourselves to the case where the unipotent orbits O_i have Jordan blocks $(2r_i - 1, 1)$ for $i = 1,..,k$ and $(2r_0 + 1)$ for $i = 0$, i.e., the ones with the most weighted Dynkin diagrams (cf. [5]). We will construct representations in Π_{ϕ_ψ} as residues of Eisenstein series associated to the character $\chi = \chi(\underbrace{\mu_1,\ldots,\mu_1}_{r_1},\ldots,\underbrace{\mu_k,\ldots,\mu_k}_{r_k},\underbrace{1,\ldots,1}_{r_0})$, where $r_1 \geq r_2 \geq \cdots \geq r_k$ and μ_i's are mutually distinct and quadratic grössencharacters.

In Section 5, we interpret the result of Section 4 in terms of Arthur parameters and prove the multiplicity formula (3.1).

4. Residual spectrum of Sp_{2n}**.** We fix a nontrivial additive character $\eta = \otimes_v\eta_v$ of $\mathbb{A}/F$ and let $\xi(z,\mu)$ be the Hecke L-function with the ordinary Γ-factor so that it satisfies the functional equation $\xi(z,\mu) = \epsilon(z,\mu)\xi(1-z,\mu^{-1})$, where $\epsilon(z,\mu) = \prod_v \epsilon(z,\mu_v,\eta_v)$ is the usual ϵ-factor (see [8, p. 159]). If μ is the trivial character μ_0, then we write simply $\xi(z)$ for $\xi(z,\mu_0)$. We have the Laurent expansion of $\xi(z)$ at $z = 1$:

$$\xi(z) = \frac{c(F)}{z-1} + a + \cdots.$$

Let $\alpha^\vee$ be the coroot corresponding to $\alpha \in \Phi^+(G,T)$. Explicitly, for $\alpha = e_i - e_j$, $\alpha^\vee(\lambda) = t(1,\ldots,\underset{i}{\lambda},\ldots,\underset{j}{\lambda^{-1}},\ldots,1) \in T(F)$ for $1 \leq i < j \leq n$. For $\alpha = e_i + e_j$, $\alpha^\vee(\lambda) = t(1,\ldots,\underset{i}{\lambda},\ldots,\underset{j}{\lambda},\ldots,1)$, for $1 \leq i < j \leq n$. For $\alpha = 2e_i$, $\alpha^\vee(\lambda) = t(1,\ldots,\underset{i}{\lambda},\ldots,1)$ for $1 \leq i \leq n$. Here dots represent 1.

Let $X(T)_F$ (resp. $X^*(T)_F$) be the group of F-rational characters (resp. cocharacters) of T. There is a natural pairing $\langle,\rangle\colon X(T)_F \times X^*(T)_F \longmapsto \mathbb{Z}$. For $\alpha,\beta \in \Phi(G,T)$, $\langle\beta,\alpha^\vee\rangle = 2(\beta,\alpha)/(\alpha,\alpha)$, where $(\ ,\ )$ is the standard inner product in $\Phi(G,T)$. Let $\omega_i = e_1 + \cdots + e_i$. Then $\omega_1,\ldots,\omega_n$ are the fundamental weights of G with respect to (G,T). Since G is simply connected, $X(T)_F = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$ and $X^*(T)_F = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee$. Set $\mathfrak{a}^* = X(T)_F \otimes \mathbb{R}$, $\mathfrak{a}^*_{\mathbb{C}} = X(T)_F \otimes \mathbb{C}$, and $\mathfrak{a} = X^*(T)_F \otimes \mathbb{R} = \mathrm{Hom}\,(X(T)_F,\mathbb{R})$, $\mathfrak{a}_{\mathbb{C}} = X^*(T)_F \otimes \mathbb{C}$. The positive Weyl chamber in $\mathfrak{a}^*$ is

$$\begin{aligned} C^+ &= \{\Lambda \in \mathfrak{a}^* | \langle \Lambda, \alpha^\vee \rangle > 0, \text{ for all } \alpha \text{ positive roots}\} \\ &= \left\{ \sum_{i=1}^n a_i \omega_i | \quad a_i > 0 \right\}. \end{aligned}$$

Let $B = TU$ be the Borel subgroup, where U is the unipotent radical. Let K_∞ be the standard maximal compact subgroup of $G(\mathbb{A}_\infty)$ and for v finite, let $K_v = G(\mathcal{O}_v)$, where $\mathcal{O}_v$ is the ring of integers of F. Then $K = K_\infty \times \prod_{v finite} K_v$ is the maximal compact subgroup of $G(\mathbb{A})$ and $G(\mathbb{A}) = B(\mathbb{A})K$. The embedding $X(T)_F \hookrightarrow X(T)_{F_v}$ induces an embedding $\mathfrak{a}_v \hookrightarrow \mathfrak{a}$, where $\mathfrak{a}_v = \mathrm{Hom}\,(X(T)_{F_v}, \mathbb{R})$. There exists a homomorphism $H_B\colon\ T(\mathbb{A}) \longmapsto \mathfrak{a}$, defined by

$$\exp\langle \chi, H_B(t)\rangle = \prod_v |\chi(t_v)|_v,$$

where $\chi \in X(T)_F$ and $t = (t_v)$. We will extend H_B to G by making it trivial on U and K. If we define $H_{B_v}\colon\ T_v \longmapsto \mathfrak{a}_v$, by

$$q_v^{\langle \chi, H_{B_v}(t)\rangle} = |\chi(t)|_v,$$

where $\chi \in X(T)_{F_v}$, $t \in T_v$, and q_v is the number of elements in the residue field, when v is finite, and by

$$\exp\langle \chi, H_{B_v}(t)\rangle = |\chi(t)|_v,$$

for v infinite, then

$$\exp\langle \chi, H_B(t)\rangle = \prod_{v=\infty} \exp\langle \chi, H_{B_v}(t_v)\rangle \prod_{v<\infty} q_v^{\langle \chi, H_{B_v}(t_v)\rangle}.$$

Observe that for almost all v, $t_v \in G(\mathcal{O}_v)$ on which H_{B_v} is trivial. Thus the product is in fact finite.

4.1. Definition of Eisenstein series. For $\mu_1, \ldots, \mu_n$ grössencharacters of F, we define a character $\chi = \chi(\mu_1, \ldots, \mu_n)$ of $T(\mathbb{A})$ by

$$\chi(\mu_1, \ldots, \mu_n)(t(\lambda_1, \ldots, \lambda_n)) = \mu_1(\lambda_1) \cdots \mu_n(\lambda_n).$$

Let $I(\chi)$ be the space of functions Φ on $G(\mathbb{A})$ satisfying $\Phi(utg) = \chi(t)\Phi(g)$ for any $u \in U(\mathbb{A})$, $t \in T(\mathbb{A})$ and $g \in G(\mathbb{A})$. Then for each $\Lambda \in \mathfrak{a}_{\mathbb{C}}^*$, the representation of $G(\mathbb{A})$ on the space of functions of the form

$$g \longmapsto \Phi(g) \exp\langle \Lambda + \rho_B, H_B(g)\rangle, \quad \Phi \in I(\chi),$$

is equivalent to $I(\Lambda,\chi) = \operatorname{Ind}_{B\uparrow G}\chi \otimes \exp(\Lambda, H_B(\ ))$. We form the Eisenstein series:

$$E(g,f,\Lambda) = \sum_{\gamma\in B(F)\backslash G(F)} f(\gamma g),$$

where $f = \Phi\, e^{\langle \Lambda+\rho_B, H_B(\ )\rangle} \in I(\Lambda,\chi)$ and ρ_B is the half-sum of positive roots, i.e., $\rho_B = \omega_1 + \cdots + \omega_n$. It converges absolutely for $\operatorname{Re}\Lambda \in C^+ + \rho_B$ and extends to a meromorphic function of Λ. It is an automorphic form and the constant term of $E(g,f,\Lambda)$ along B is given by

$$E_0(g,f,\Lambda) = \int_{U(F)\backslash U(\mathbb{A})} E(ug,f,\Lambda)\,du = \sum_{w\in W} M(w,\Lambda,\chi)f(g),$$

where W is the Weyl group of T and

$$M(w,\Lambda,\chi)f(g) = \int_{wU(\mathbb{A})w^{-1}\cap U(\mathbb{A})\backslash U(\mathbb{A})} f(w^{-1}ug)\,du.$$

Then $M(w,\Lambda,\chi)$ defines an intertwining map from $I(\Lambda,\chi)$ to $I(w\Lambda, w\chi)$ and satisfies a functional equation of the form

$$M(w_1w_2,\Lambda,\chi) = M(w_1, w_2\Lambda, w_2\chi)M(w_2,\Lambda,\chi).$$

Let S be a finite set of places of F, including all the archimedean places such that for every $v \notin S$, χ_v, η_v are unramified and if $f = \otimes f_v$, for $v \notin S$, f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$. We have

$$M(w,\Lambda,\chi) = \otimes_v A(w,\Lambda,\chi_v).$$

Then by applying Gindikin-Karpelevic method, we can see that for $v \notin S$,

$$A(w,\Lambda,\chi_v)f_v = \prod_{\alpha>0, w\alpha<0} \frac{L(\langle\Lambda,\alpha^\vee\rangle, \chi_v\circ\alpha^\vee)}{L(\langle\Lambda,\alpha^\vee\rangle+1, \chi_v\circ\alpha^\vee)}\tilde{f}_v,$$

where $\tilde{f}_v$ is the K_v-fixed function in the space of $I(w\Lambda, w\chi)$ (cf. [6, 17, 18, 27, 28]). For any v, let

$$r_v(w) = \prod_{\alpha>0, w\alpha<0} \frac{L(\langle\Lambda,\alpha^\vee\rangle, \chi_v\circ\alpha^\vee)}{L(\langle\Lambda,\alpha^\vee\rangle+1, \chi_v\circ\alpha^\vee)\epsilon(\langle\Lambda,\alpha^\vee\rangle, \chi_v\circ\alpha^\vee, \eta_v)}.$$

We normalize the intertwining operators $A(w,\Lambda,\chi_v)$ for all v by

$$A(w,\Lambda,\chi_v) = r_v(w)R(w,\Lambda,\chi_v).$$

Let $R(w, \Lambda, \chi) = \otimes_v R(w, \Lambda, \chi_v)$ and

$$r(w) = \Pi_v r_v(w) = \prod_{\alpha>0, w\alpha<0} \frac{\xi(\langle \Lambda, \alpha^\vee \rangle, \chi \circ \alpha^\vee)}{\xi(\langle \Lambda, \alpha^\vee \rangle + 1, \chi \circ \alpha^\vee)\epsilon(\langle \Lambda, \alpha^\vee \rangle, \chi \circ \alpha^\vee)}.$$

$R(w, \Lambda, \chi)$ satisfies the functional equation

$$R(w_1 w_2, \Lambda, \chi) = R(w_1, w_2\Lambda, w_2\chi) R(w_2, \Lambda, \chi),$$

for any w_1, w_2. We know, by Winarsky [32] for p-adic cases and by Shahidi [26, p. 110] for real and complex cases that

$$A(w, \Lambda, \chi_v) \prod_{\alpha>0, w\alpha<0} L_v(\langle \Lambda, \alpha^\vee \rangle, \chi_v \circ \alpha^\vee)^{-1} \tag{4.1}$$

is holomorphic for any v. So for any v, $R(w, \Lambda, \chi_v)$ is holomorphic for Λ with $\mathrm{Re}\,(\langle \Lambda, \alpha^\vee \rangle) > -1$, for all positive α with $w\alpha < 0$. For $\chi = \chi(\mu_1, \ldots, \mu_n)$,

$$\chi \circ \alpha^\vee = \begin{cases} \mu_i \mu_j^{-1}, & \text{for } \alpha = e_i - e_j \\ \mu_i \mu_j, & \text{for } \alpha = e_i + e_j \text{ and } i < j \\ \mu_i, & \text{for } \alpha = 2e_i. \end{cases}$$

For $\alpha \in \Phi^+$, let $S_\alpha = \{\Lambda \in \mathfrak{a}_{\mathbb{C}}^* | \langle \Lambda, \alpha^\vee \rangle = 1\}$. We call S_α a singular hyperplane. We say that $E(g, f, \Lambda)$ has a pole of order l at Λ_0 if Λ_0 is the intersection of l singular hyperplanes in general position on which the Eisenstein series has a simple pole.

Langlands' theory [18, 25] says that $L_d^2(B)$ is generated by square integrable iterated residues of $E(g, f, \Lambda)$ at poles of order n.

We recall Langlands' square integrability criterion for autormorphic forms through their constant terms in our case ([18, p. 104] or [9, p. 187]). We write the intertwining operator $M(w, \Lambda, \chi)$ as follows:

$$M(w, \Lambda, \chi) f(g) = T(w, \Lambda, \chi) \Phi(g) e^{\langle w\Lambda + \rho_B, H_B(g) \rangle}.$$

Suppose the iterated residue of $E_0(g, f, \Lambda)$ at $\Lambda = \beta$ is given by

$$\mathrm{Res}_\beta\, E_0(g, f, \Lambda) = \sum_{w \in \Omega} \mathrm{Res}_\beta\, T(w, \Lambda, \chi) \Phi(g) e^{\langle w\beta + \rho_B, H_B(g) \rangle}.$$

Here Ω is the set of all $w \in W$ which contribute a nonzero residue. Then we have

LEMMA. (Langlands) $\mathrm{Res}_\beta\, E(g, f, \Lambda)$ *is square integrable if and only if* $\mathrm{Re}\,(w\beta)$ *is in* $-\{\sum_{i=1}^{2n} a_i \alpha_i | a_i > 0\}$ *for all* $w \in \Omega$.

For $\Psi \subset \Phi^+$, we define $r(w, \Lambda, \Psi)$ by

$$r(w, \Lambda, \Psi) = \prod_{\alpha \in \Psi, w\alpha < 0} \frac{\xi(\langle \Lambda, \alpha^\vee \rangle, \chi \circ \alpha^\vee)}{\xi(\langle \Lambda, \alpha^\vee \rangle + 1, \chi \circ \alpha^\vee) \epsilon(\langle \Lambda, \alpha^\vee \rangle, \chi \circ \alpha^\vee)}.$$

Observe that we have suppressed the dependence of $r(w, \Lambda, \Psi)$ on χ.

4.2. Residues of the Eisenstein series.

We start with

PROPOSITION 4.1. *Let $E(g, f, \Lambda)$ be the Eisenstein series associated to the trivial character. Its constant term $E_0(g, f, \Lambda)$ is given by*

$$E_0(g, f, \Lambda) = \sum_{w \in W} r(w, \Lambda, \Phi^+) R(w, \Lambda, 1) f.$$

Let $\Lambda_0 = \rho_B$ be the half-sum of positive roots. Then only $w = w_0$, the longest element of the Weyl group, contributes a pole of order n, and the residue of $E(g, f, \Lambda)$ at Λ_0 is constant.

Proof. Note that $\{\alpha | \langle \rho_B, \alpha^\vee \rangle = 1\}$ is the set of simple roots. Therefore, ρ_B is the intersection of the n singular hyperplanes S_α for simple roots α. But

$$\{w | w\alpha < 0, \text{for all simple roots } \alpha\} = \{w_0, \text{the longest Weyl group elements in } W\}.$$

Therefore, the residue at $\lambda = \rho_B$ is

$$(*) \otimes_v R_v(w_0, \rho_B, \chi_v) f_v,$$

where $f_v \in I_v(\rho_B, \chi_v)$. But $R_v(w_0, \rho_B, \chi_v) I_v(\rho_B, \chi_v)$ is the Langlands' quotient, which is constant. Therefore, the residue is constant.

Remark 1. Here the half-sum of positive roots corresponds, by (2.1), to the unipotent orbits with Jordan blocks $(2n + 1)$ for $G^* = O_{2n+1}(\mathbb{C})$, $(2n - 1, 1)$ for $G^* = O_{2n}(\mathbb{C})$, resp. i.e., $\rho_B = \frac{\xi_1}{2} e_1 + \frac{\xi_2}{2} e_2 + \cdots$. We note that Λ_0 and w_0 satisfy $(\Lambda_0, e_n) = 0$ and $w_0 \Lambda_0 = -\Lambda_0$, the first only valid for $G = \mathrm{SO}_{2n}$.

For χ a nontrivial character, we can assume, after conjugation, that $\chi = \chi(\underbrace{\mu_1, \ldots, \mu_1}_{r_1}, \ldots, \underbrace{\mu_k, \ldots, \mu_k}_{r_k}, \underbrace{1, \ldots, 1}_{r_0})$, $r_0 + \cdots + r_k = n$, $r_1 \geq \cdots \geq r_k$.

Let $E(g, f, \Lambda)$ be the Eisenstein series associated to the character χ.

PROPOSITION 4.2. *The Eisenstein series has a pole of order n only if $r_k \geq 2$ and μ_i is a quadratic grössencharacter for $i = 1, \ldots, k$.*

We divide the set of positive roots Φ^+ as follows:
$\Phi_1 = \{e_i \pm e_j, \quad 1 \le i < j \le r_1\}$,
$\Phi_2 = \{e_{r_1+i} \pm e_{r_1+j}, \quad 1 \le i < j \le r_2\}$,
$\vdots$
$\Phi_k = \{e_{r_1+\cdots+r_{k-1}+i} \pm e_{r_1+\cdots+r_{k-1}+j}, \quad 1 \le i < j \le r_k\}$,
$\Phi_0 = \{e_{r_1+\cdots+r_k+i} \pm e_{r_1+\cdots+r_k+j}, \quad 1 \le i < j \le r_0, \quad 2e_{r_1+\cdots+r_k+i}, \quad i = 1, \dots, r_0\}$,
$\Phi_D = \Phi^+ - \bigcup_{i=0}^{k} \Phi_k$.

$\Phi_1, \dots, \Phi_k$ are root systems of type D_n and Φ_0 is a root system of type C_n. This corresponds to the decomposition $O(V_1) \times \cdots \times O(V_k) \times O(V_0) \subset O_{2n+1}(\mathbb{C})$. Let W_i be the Weyl group corresponding to Φ_i for $i = 0, \dots, k$. Let $\Lambda = \Lambda_1 + \cdots + \Lambda_k + \Lambda_0$, where $\Lambda_i = a_{r_1+\cdots+r_{i-1}+1} e_{r_1+\cdots+r_{i-1}+1} + \cdots + a_{r_1+\cdots+r_i} e_{r_1+\cdots+r_i}$ for $i = 1, \dots, k$ and $\Lambda_0 = a_{r_1+\cdots+r_k+1} e_{r_1+\cdots+r_k+1} + \cdots + a_n e_n$.

We recall the following well-known result (Carter [5, p. 47]).

PROPOSITION 4.3. *Let Δ be a set of simple roots and W be the associated Weyl group. Let w_α be the simple reflection with respect to $\alpha \in \Delta$. Then W is generated by the w_α, $\alpha \in \Delta$. Let θ be a subset of Δ and W_θ be the subgroup of W generated by the w_α, $\alpha \in \theta$. Then each coset wW_θ has a unique element d_θ characterized by any of the following equivalent properties:*

(1) $d_\theta \theta > 0$;

(2) *d_θ is of minimal length in wW_θ; and*

(3) *For any $x \in W_\theta$, $l(d_\theta x) = l(d_\theta) + l(x)$.*

We apply Proposition 4.3 to $\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n\}$ and $\theta = \Delta - \{e_{r_1} - e_{r_1+1}, e_{r_1+r_2} - e_{r_1+r_2+1}, \dots, e_{r_1+\cdots+r_k} - e_{r_1+\cdots+r_k+1}\}$. Let D be the set of such distinguished coset representatives. Then we have

PROPOSITION 4.4. *The constant term $E_0(g, f, \Lambda) = \sum_{w \in W} r(w, \Lambda, \Phi^+) R(w, \Lambda, \chi) f$ can be written as*

$$\prod_{i=1}^{k} \sum_{w_i \in W_i} r(w_i, \Lambda_i, \Phi_i) \sum_{w_0 \in W_0} r(w_0, \Lambda_0, \Phi_0) \cdot \sum_{d \in D} \sum_{c \in C} r(dcw_1 \cdots w_k w_0, \Phi_D) R(dcw_1 \cdots w_k w_0, \Lambda, \chi) f,$$

where C is the set spanned by $c_{r_1}, c_{r_1+r_2}, \dots, c_{r_1+\cdots+r_k}$. Here c_i's are sign changes in the Weyl group: its action on $\Phi(G, T)$ takes e_i to $-e_i$.

Let $\Lambda_0 = \Lambda_{1,0} + \cdots + \Lambda_{k,0} + \Lambda_{0,0}$, where $\Lambda_{i,0}$ is the half-sum of positive roots in Φ_i for $i = 0, \dots, k$. Then

THEOREM 4.5. *The residue of* $E_0(g, f, \Lambda)$ *at* Λ_0 *is given by*

$$\sum_{d\in D}(*)R(dw_0,\Lambda_0,\chi)\prod_{i=1}^{k}(1+R(c_{r_1+\cdots+r_i}))f, \tag{4.2}$$

where w_0 *is determined by* Λ_0 *and* $(*)$ *signifies a constant. It is square integrable.*

It is instructive to consider first a simple case to illustrate our method, namely, $\chi = \chi(\mu, \ldots, \mu)$, μ nontrivial and quadratic. The main idea of our proof is already contained in this simple case.

Let $\Phi_1 = \{e_i \pm e_j, \quad 1 \le i < j \le n\}$ and $\Phi_2 = \{2e_i, \quad i = 1, \ldots, n\}$. Then for $\alpha \in \Phi_1$, $\chi \circ \alpha^\vee = 1$ and for $\alpha \in \Phi_2$, $\chi \circ \alpha^\vee$ is nontrivial. Let W_1 be the Weyl group associated to Φ_1. It is a Weyl group of type D_n. Here $\#(W/W_1) = 2$ and the nontrivial coset has a distinguished coset representative, i.e., c_n. It is the unique element which satisfies $c_n\Phi_1 > 0$. Here for $w \in W_1$,

$$\{\alpha > 0| \quad c_n w\alpha < 0\} = \{\alpha \in \Phi_1| \quad w\alpha < 0\} \cup \{\alpha \in \Phi_2| \quad c_n w\alpha < 0\}.$$

Therefore the constant term of the Eisenstein series is

$$\sum_{w_1\in W_1} r(w_1,\Lambda,\Phi_1)(r(w_1,\Lambda,\Phi_2)R(w_1,\Lambda,\chi) + r(c_n w_1,\Lambda,\Phi_2)R(c_n w_1,\Lambda,\chi)).$$

We consider the residue at $\Lambda = \Lambda_0$, the half-sum of positive roots of Φ_1. Since $(\Lambda_0, e_n) = 0$, the last term is holomorphic on every singular hyperplane. The first term has a pole of order n at $w = w_0$, the longest element of W_1. Since $w_0\Lambda_0 = -\Lambda_0$, $w_0 e_i = \pm e_k$, for some $k < n$ depending on i. Therefore, for $\alpha = 2e_i$, $i = 1, .., n-1$, $w_0\alpha < 0$ if and only if $c_n w_0 \alpha < 0$. So $r(w_0, \Lambda_0, \Phi_2) = r(c_n w_0, \Lambda_0, \Phi_2)$, and the residue is

$$(*)R(c_n w_0,\Lambda_0,\chi)(1 + R(c_n,\Lambda_0,\chi)),$$

since $c_n w_0 c_n = w_0$.

4.3. Proof of Proposition 4.2.

We need

PROPOSITION 4.6. *If one of* μ *is not a quadratic grössencharacter or* $r_i = 1$ *for some* $i > 0$, *then the Eisenstein series has no pole of order* n. *In particular, for* $\chi = \chi(\underbrace{\mu, \ldots, \mu}_{r_1}, \nu_1, \ldots, \nu_{r_2})$ *(if* $r_1 > 1$, μ *is not quadratic), the Eisenstein series has no pole of order* n.

Proof. Let

$$\begin{aligned}\Phi_1 &= \{e_i - e_j, \quad 1 \le i < j \le r_1\},\\ \Phi_2 &= \{e_{r_1+i} \pm e_{r_1+j}, \quad 1 \le i < j \le r_2, \quad 2e_{r_1+i}, \quad i = 1, \dots, r_2\},\\ \Phi_3 &= \Phi^+ - \Phi_1 \cup \Phi_2.\end{aligned}$$

Then Φ_1 is a root system of type A_{r_1-1} and Φ_2, a root system of type C_{r_2}. For $\alpha \in \Phi_3$, $\chi \circ \alpha^\vee$ is nontrivial. Let W_i be the Weyl group of Φ_i for $i = 1, 2$. Let $\Lambda = \Lambda_1 + \Lambda_2$, where $\Lambda_1 = a_1 e_1 + \cdots + a_{r_1} e_{r_1}$ and $\Lambda_2 = a_{r_1+1} e_{r_1+1} + \cdots + a_n e_n$. Then $W_1 W_2$ is W_θ in Proposition 3 with $\theta = \Delta - \{e_{r_1} - e_{r_1+1}\}$. Let D be the set of distinguished coset representatives for W/W_1W_2. For $d \in D$, $w_1 \in W_1$ and $w_2 \in W_2$,

$$\begin{aligned}\{\alpha > 0\colon\ d w_1 w_2 \alpha < 0\} = \{&\alpha \in \Phi_1 |\, w_1\alpha < 0\}\\ \cup\{&\alpha \in \Phi_2 |\, w_2\alpha < 0\}\\ \cup\{&\alpha \in \Phi_3 |\, d w_1 w_2 \alpha < 0\}.\end{aligned}$$

Using (4.1), we write the constant term of the Eisenstein series

$$E_0(g, f, \Lambda) = \sum_{w \in W} r(w, \Lambda, \Phi^+) R(w, \Lambda, \chi) f$$

as follows:

$$E_0(g, f, \Lambda) = \sum_{w \in W} \tilde{r}(w, \Lambda, \Phi^+) \otimes_{v \notin S} \tilde{f}_v \otimes \otimes_{v \in S} \tilde{A}(w, \Lambda, \chi) f_v,$$

where $\tilde{A}(w, \Lambda, \chi)$ is the expression (4.1) and

$$\tilde{r}(w, \Lambda, \Phi^+) = \prod_{\alpha \in \Phi^+, w\alpha < 0} \frac{\xi_S(\langle \Lambda, \alpha^\vee \rangle, \chi)}{\xi_S(\langle \Lambda, \alpha^\vee \rangle + 1, \chi)},$$

where $\xi_S(z, \chi) = \prod_{v \notin S} L(z, \chi_v)$ is the partial Hecke L-function. Then we have

$$\begin{aligned}E_0(g, f, \Lambda) = \sum_{w_1 \in W_1} \tilde{r}(w_1, \Lambda_1, \Phi_1) \sum_{w_2 \in W_2} \tilde{r}(w_2, \Lambda_2, \Phi_2)&\\ \sum_{d \in D} \tilde{r}(d w_1 w_2, \Lambda, \Phi_3) \otimes_{v \notin S} \tilde{f}_v \otimes \otimes_{v \in S} \tilde{A}(d w_1 w_2, \Lambda, \chi) f_v&.\end{aligned}$$

$\tilde{A}(w, \Lambda, \chi)$ is entire and $\tilde{r}(w, \Lambda, \Phi_3)$ is holomorphic on any singular hyperplane. Any pole of the first term is an intersection of $\le r_1 - 1$ singular hyperplanes in general position and any pole of the second term is an intersection of $\le r_2$ singular hyperplanes in general position. Therefore, any pole of the Eisenstein series is an intersection of $\leq r_1 + r_2 - 1 < n$ singular hyperplanes in general position. This proves Proposition 4.6.

Proposition 4.2 is now immediate.

4.4. Proof of Proposition 4.4. First we prove the assertion for

$$\chi = \chi(\underbrace{\mu, \ldots, \mu}_{r_1}, \nu_1, \ldots, \nu_{n-r_1}),$$

where μ is nontrivial, quadratic and μ and ν_j are distinct for all j.

Let $\chi_1 = \chi(\underbrace{\mu, \ldots, \mu}_{r_1})$ and $\chi_2 = \chi(\nu_1, \ldots, \nu_{n-r_1})$. Let

$\Phi_1 = \{e_i \pm e_j, \quad 1 \leq i < j \leq r_1\}$,

$\Phi_2 = \{e_{r_1+i} \pm e_{r_1+j}, \quad 1 \leq i < j \leq n - r_1, \quad 2e_{r_1+i}, i = 1, \ldots, n - r_1\}$,

$\Phi_3 = \Phi - \Phi_1 \cup \Phi_2 = \{e_i \pm e_{r_1+j}, \quad i = 1, \ldots, r_1, j = 1, \ldots, n - r_1, \quad 2e_i, i = 1, \ldots, r_1\}$.

Then for $\alpha \in \Phi_1$, $\chi \circ \alpha^\vee = 1$. For $\alpha \in \Phi_2$, $\chi \circ \alpha^\vee = \chi_2 \circ \alpha^\vee$ and for $\alpha \in \Phi_3$, $\chi \circ \alpha^\vee$ is nontrivial. Let W_i be the Weyl group associated to Φ_i, $i = 1, 2$. Let $\Lambda = \Lambda_1 + \Lambda_2$, $\Lambda_1 = a_1e_1 + \cdots + a_{r_1}e_{r_1}$ and $\Lambda_2 = a_{r_1+1}e_{r_1+1} + \cdots + a_ne_n$. We apply Proposition 4.3 to $\Delta = \{e_1 - e_2, \ldots, e_{n-1} - e_n\}$ and $\theta = \Delta - \{e_{r_1} - e_{r_1+1}\}$. Let D_θ be the set of distinguished coset representatives. Then we need

LEMMA 4.7. *$D = D_\theta \cup D_\theta c_{r_1}$ is the set of distinguished coset representatives for W/W_1W_2, i.e., $d \in D$ if and only if $d(\Phi_1 \cup \Phi_2) > 0$.*

Proof. Since D_θ contains no sign changes, it follows immediately that $d(\Phi_1 \cup \Phi_2) > 0$ for all $d \in D$. It can be easily checked that $\#D = \#(W/W_1W_2)$. We therefore only need to show that each coset has a unique coset representative in D. Suppose $d_i \in D$ for $i = 1, 2$ and $d_1^{-1}d_2 = w_1w_2 \in W_1W_2$. Then $d_2 = d_1w_1w_2$. Using $d_2\alpha > 0$ for all $\alpha \in \Phi_2$ implies that $d_1w_1w_2\alpha > 0$. Here w_1 and w_2 commute and $w_1\alpha = \alpha$ for $\alpha \in \Phi_2$. Therefore, we have $d_1w_2\alpha > 0$ for all $\alpha \in \Phi_2$ or $w_2\alpha > 0$ for all $\alpha \in \Phi_2$. This implies that $w_2 = 1$. In the same way, we have $w_1 = 1$. This proves the lemma.

For $d \in D$, $w_1 \in W_1$ and $w_2 \in W_2$,

$$\begin{aligned}\{\alpha > 0\colon\ dw_1w_2\alpha < 0\} = {} & \{\alpha \in \Phi_1 |\, w_1\alpha < 0\} \\ & \cup \{\alpha \in \Phi_2 |\, w_2\alpha < 0\} \\ & \cup \{\alpha \in \Phi_3 |\, dw_1w_2\alpha < 0\}.\end{aligned}$$

Then the constant term of $E(g, f, \Lambda)$ is given by

$$(4.3)\quad \sum_{w_1 \in W_1} r(w_1, \Lambda_1, \Phi_1) \sum_{w_2 \in W_2} r(w_2, \Lambda_2, \Phi_2)$$
$$\cdot \sum_{d \in D_\theta} (r(dw_1w_2, \Lambda, \Phi_3)R(dw_1w_2, \Lambda, \chi) + r(dc_{r_1}w_1w_2, \Lambda, \Phi_3)R(dc_{r_1}w_1w_2, \Lambda, \chi)).$$

In order to apply induction, let $\chi = \chi(\underbrace{\mu_1, \ldots, \mu_1}_{r_1}, \underbrace{\mu_2, \ldots, \mu_2}_{r_2}, \nu_1, \ldots, \nu_l)$, $r_1 + r_2 + l = n$. Let $\chi_2 = \chi(\underbrace{\mu_2, \ldots, \mu_2}_{r_2}, \nu_1, \ldots, \nu_l)$. We repeat the above for χ_2 and divide Φ_2 as follows:

$$\begin{aligned}
\Phi_4 &= \{e_{r_1+i} \pm e_{r_1+j}, \quad 1 \le i < j \le r_2\},\\
\Phi_5 &= \{e_{r_1+r_2+i} \pm e_{r_1+r_2+j}, \quad 1 \le i < j \le l, \quad 2e_{r_1+r_2+i}, \quad i = 1, .., l\},\\
\Phi_6 &= \Phi_2 - \Phi_4 \cup \Phi_5.
\end{aligned}$$

Let W_i be the Weyl group of Φ_i for $i = 4, 5$. Then $D' = D_{\theta'} \cup D_{\theta'}c_{r_1+r_2}$ is the set of distinguished coset representatives for W_2/W_4W_5, where $D_{\theta'}$ is the set of distinguished coset representatives for

$$\theta' = \{e_{r_1+1} - e_{r_1+2}, \ldots, e_{n-1} - e_n\} - \{e_{r_1+r_2} - e_{r_1+r_2+1}\}.$$

Then one can show that $D_\theta D_{\theta'}$ is the set of distinguished coset representatives in Proposition 4.3 for $\{e_1 - e_2, \ldots, e_{n-1} - e_n\} - \{e_{r_1} - e_{r_1+1}, e_{r_1+r_2} - e_{r_1+r_2+1}\}$ and

$$DD' = D_\theta D_{\theta'} \cup D_\theta D_{\theta'}c_{r_1} \cup D_\theta D_{\theta'}c_{r_1+r_2} \cup D_\theta D_{\theta'}c_{r_1}c_{r_1+r_2}$$

is the set of distinguished coset representatives for $W/W_1W_4W_5$, i.e., $d \in DD'$ if and only if $d(\Phi_1 \cup \Phi_4 \cup \Phi_5) > 0$. Proposition 4.4 now follows by induction.

4.5. Proof of Theorem 4.5. We apply induction and start with the equation (4.3). Suppose the first term has a pole at $\Lambda_{1,0}$ and $w_1 = w_{\Lambda_{1,0}}$ contributes the pole. Let $\Lambda = \Lambda_{1,0} + \Lambda_2$. We need

LEMMA 4.8. *For each $w_2 \in W_2$,*

$$r(dw_{\Lambda_{1,0}}w_2, \Lambda, \Phi_3) = r(dc_{r_1}w_{\Lambda_{1,0}}w_2, \Lambda, \Phi_3).$$

Proof. Recall the properties of w_1 and $\Lambda_{1,0}$: $(\Lambda_{1,0}, e_{r_1}) = 0$ and $w_1\Lambda_{1,0} = -\Lambda_{1,0}$.

Therefore, $\langle \Lambda, \alpha^\vee \rangle = 0$ for $\alpha = 2e_{r_1}$. For $i < r_1$, $w_1e_i = \pm e_k$, $k < r_1$ and

$w_1 e_{r_1} = \pm e_{r_1}$. So for $\alpha = e_i \pm e_{r_1+j}, 2e_i$, $i = 1, \ldots, r_1 - 1$, $j = 1, \ldots, n - r_1$, $dw_1 w_2 \alpha < 0$ if and only if $dc_{r_1} w_1 w_2 \alpha < 0$ since $w_1 w_2 \alpha = c_{r_1} w_1 w_2 \alpha$.

For $\alpha = e_{r_1} \pm e_{r_1+j}$, we have

LEMMA 4.9. *Only one of the following is possible: either α satisfies both $dw_1 w_2 \alpha < 0$ and $dc_{r_1} w_1 w_2 \alpha < 0$, or α and $e_{r_1} \mp e_{r_1+j}$ satisfy only one inequality.*

Proof. Suppose $dw_1 w_2(e_{r_1} \pm e_{r_1+j}) = d(-e_{r_1} \mp e_{r_1+k})$. Then $dc_{r_1} w_1 w_2(e_{r_1} \pm e_{r_1+j}) = d(e_{r_1} \mp e_{r_1+k})$. Since d is a permutation, we have our assertion. The other case is similar, completing the lemma.

If $\alpha = e_{r_1} \pm e_{r_1+j}$ satisfy $dw_1 w_2 \alpha < 0$, then $\langle \Lambda, \alpha^\vee \rangle = \pm(\Lambda_2, e_{r_1+j})$ and

$$\frac{\xi((\Lambda_2, e_{r_1+j}), \mu\nu_j)}{\xi((\Lambda_2, e_{r_1+j}) + 1, \mu\nu_j)\epsilon((\Lambda_2, e_{r_1+j}), \mu\nu_j)} \times \frac{\xi(-(\Lambda_2, e_{r_1+j}), \mu\nu_j)}{\xi(-(\Lambda_2, e_{r_1+j}) + 1, \mu\nu_j)\epsilon(-(\Lambda_2, e_{r_1+j}), \mu\nu_j)} = 1,$$

using the functional equation $\xi(z, \mu) = \epsilon(z, \mu)\xi(1-z, \mu)$ for μ a nontrivial quadratic grössencharacter.

This proves Lemma 4.8.

It now follows that the residue at Λ, as a function of Λ_2, is

$$\sum_{w_2 \in W_2} r(w_2, \Lambda_2, \Phi_2)\Big(\sum_{d \in D} (*) R(dw_2 w_{1,0}, \Lambda, \chi)(1 + R(c_{r_1})),$$

where $w_{1,0} = c_{r_1} w_{\Lambda_{1,0}}$ since $c_{r_1} w_{\Lambda_{1,0}} c_{r_1} = w_{\Lambda_{1,0}}$. Theorem 4.5 now follows by applying induction. We only need:

PROPOSITION 4.10. *The residue in Theorem 4.5 is square integrable.*

Proof. By Langlands' Lemma and the fact that $w_0 \Lambda_0 = -\Lambda_0$, it is enough to show that $d\Lambda_0$ is a linear combination of simple roots with positive coefficients for any $d \in D$.

First of all, it is easy to see that any linear combination of e_i's with non-negative coefficients which contains e_1 is a linear combination of simple roots with positive coefficients. Λ_0 satisfies this property. Since d is a permutation, it is enough to show that $d\Lambda_0$ contains e_1.

Recall the property of Λ_0 that Λ_0 contains $e_1, e_{r_1+1}, \ldots, e_{r_1+\ldots+r_k+1}$. Also recall the property of $d \in D$ from Proposition 4.3: $d\theta > 0$ where $\theta = \{e_1 - e_2, \ldots, e_{r_1-1} - e_{r_1}\} \cup \{e_{r_1+1} - e_{r_1+2}, \ldots, e_{r_1+r_2-1} - e_{r_1+r_2}\} \cup \ldots \cup \{e_{r_1+r_2+\ldots+r_k+1} - e_{r_1+r_2+\ldots+r_k+2,\ldots}, e_{n-1} - e_n\}$. Hence, one of $e_1, e_{r_1+1}, \ldots, e_{r_1+r_2+\ldots+r_k+1}$ is sent to e_1 by $d \in D$. So $d\Lambda_0$ contains e_1.

5. Arthur parameters for the residual spectrum. In this section we interpret Theorem 4.5 in terms of Arthur parameters. Recall the quadratic Arthur parameters in our case: we have a decomposition $\mathbb{C}^{2n+1} = V_0 \oplus V_1 \oplus \cdots \oplus V_k$, where $\dim V_0 = 2r_0 + 1$, $\dim V_i = 2r_i$ for $i = 1, \ldots, k$, $r_1 \geq r_2 \geq \cdots \geq r_k \geq 2$, $r_0 + \cdots + r_k = n$, and embedding $\prod_{i=0}^{k} O(V_i) \subset O_{2n+1}(\mathbb{C})$. Let $\mu_1, \ldots, \mu_k$ be distinct quadratic grössencharacters and O_i be the unipotent orbits with Jordan blocks $(2r_i - 1, 1)$ for $i = 1, \ldots, k$ and $(2r_0 + 1)$ for $i = 0$ (see the remark after Proposition 4.1). Then the Arthur parameter of interest to us is the homomorphism

$$\psi\colon\ W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longmapsto \prod_{i=0}^{k} O(V_i) \subset O_{2n+1}(\mathbb{C}),$$

satisfying:

(1) $\psi|W_F\colon\ w \longmapsto 1 \times \mu_1(w) \times \cdots \times \mu_k(w) \in \{\pm 1\} \times \{\pm 1\} \times \cdots \times \{\pm 1\}$, where $\{\pm 1\}$ is the center of $O(V_i)$ for $i = 0, \ldots, k$;

(2) $\psi|_{1 \times SL_2(\mathbb{C}) \times 1} \equiv 1$; and

(3) by Jacobson-Morozov theorem, $\psi|_{1 \times 1 \times SL_2(\mathbb{C})}$ defines the unipotent orbit $\prod_{i=0}^{k} O_i$ of G^*.

Recall that we are considering the residue of the Eisenstein series at $\Lambda_0 = \Lambda_{1,0} + \ldots + \Lambda_{k,0} + \Lambda_{0,0}$, where each $\Lambda_{i,0}$ is the half-sum of (positive) roots in $\Phi_i, i = 0, 1, \ldots, k$. The character χ and the quasicharacter $\exp\langle \Lambda_0, H_B(\ )\rangle$ of T may be viewed as homomorphisms from W_F into ${}^{L}T$ (cf. [14, 16]). The unipotent orbits O_i are determined by Λ_0 through Jacobson-Morozov's theorem. Then the associated Langlands' parameter ϕ_ψ, i.e., the homomorphism

$$\phi_\psi\colon\ W_F \times SL_2(\mathbb{C}) \to O_{2n+1}(\mathbb{C})$$

defined by $\phi_\psi|SL_2(\mathbb{C}) = 1$ and

$$\phi_\psi(w) = \psi\left(w, 1, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right),$$

is $\phi_\psi = \chi \otimes \exp\langle \Lambda_0, H_B(\ )\rangle$ (cf. [1]). Its nontempered part is $\phi_\psi^+ = \exp\langle \Lambda_0, H_B(\quad)\rangle$.

Let $M^* = \mathrm{Cent}\,(im\phi_\psi^+, G^*)$. Since $(\Lambda_0, e_i) = 0$ for $i = r_1, r_1+r_2, \ldots, r_1+\ldots+r_k$, the Levi subgroup M which has M^* as its L-group, will be, up to isomorphism, of the form $GL_{n_1} \times \ldots \times GL_{n_r} \times \mathrm{Sp}_{2k}$, where $n_1, \ldots, n_r$ are determined by Λ_0.

The parameter Λ_0 may not be in the positive Weyl chamber of the split component of M. But one can choose an element w' in the Weyl group of shortest length so that $\lambda_0 = w'\Lambda_0$ belongs there. Then, using the functional equation, the Eisenstein series attached to λ and $\chi' = w'\chi$ will have a pole of order n at $\lambda = \lambda_0$. The Arthur parameter which is determined only up to conjugacy will not change. From now on we shall assume that Λ_0 is in the positive Weyl chamber of the split component of M.

For each place v, decompose ϕ_{ψ_v} as $\phi_{\psi_v} = \phi^{\circ}_{\psi_v} \cdot \phi^{+}_{\psi}$ as in [1]. The parameter $\phi^{\circ}_{\psi_v}$ factors through M^* and is the Langlands parameter for the (tempered) constituents of the unitary principal series $I_v = \mathrm{Ind}_{B_0(F_v)}^{M(F_v)} \chi_v = \oplus_i \pi_{v,i}$, of $M(F_v)$, where $B_0 = B \cap M$. For each $\pi_{v,i}$, let $\Pi_{v,i} = J(\pi_{v,i} \otimes \exp\langle \Lambda_0, H_P(\ )\rangle)$ be the corresponding Langlands quotient [4, 19], where $P = MN$. Then for each v the L-packet parametrized by ϕ_{ψ_v} is $\Pi_{\phi_{\psi_v}} = \{\Pi_{v,i}\}$ (cf. [1, 19]). The R-group for the parameter ϕ_{ψ_v}, i.e., $C_{\phi_{\psi_v}}$ is the same as the R-group of I_v for each v in the sense of Knapp-Stein (cf. [7, 10, 11, 13]).

By theorem C_n of [10], the R-group $C_{\phi_{\psi_v}}$ of I_v is a subgroup of the group generated by the sign changes $c_i, i = r_1, r_1+r_2, r_1+\ldots+r_k$, a product of 2-groups. Moreover, if the sign change $c_{r_1+\ldots+r_i}$ in (4.2) does not belong to $C_{\phi_{\psi_v}}$ for some i, then the normalized operator $R(c_{r_1+\ldots+r_i})$ acts like identity.

Let $\pi(\chi_v) = \{\pi_{v,i}\}$. Then, given a place v, in [11] Keys and Shahidi defined a pairing $\langle\ ,\ \rangle$ on $C_{\phi_{\psi_v}} \times \pi(\chi_v)$. We extend the pairing $\langle\ ,\ \rangle$ to $C_{\phi_{\psi_v}} \times \Pi_{\phi_{\psi_v}}$ as in Arthur [1, p. 9] by setting $\langle \tau_v, \Pi_{v,i}\rangle = \langle \tau_v, \pi_{v,i}\rangle$. This can further be extended to $C_{\psi_v} \times \Pi_{\phi_{\psi_v}}$, using the surjection $C_{\psi_v} \to C_{\phi_{\psi_v}}$ for each v ([1, p. 11]). Let $\Pi = \otimes_v \Pi_{v,i}$ where almost all $\Pi_{v,i}$ are spherical. Then $\Pi \in \Pi_{\phi_\psi}$, the (global) L-packet of ϕ_ψ. Finally set $\langle \tau, \Pi\rangle = \prod_v \langle \tau_v, \Pi_{v,i}\rangle$, where τ_v is the image of τ under the map $C_\psi \to C_{\psi_v}$. We need to be more precise since this is an infinite product. For each place v, the corresponding pairing in [11] is defined by means of a nontrivial additive character of F_v. Fix a nontrivial additive character $\eta = \otimes_v \eta_v$ of $\mathbb{A}/F$. At each place v, there is a unique representation $\pi_{v,0} \in \{\pi_{v,i}\}$ which is generic with respect to η_v and for which $\langle \tau_v, \pi_{v,0}\rangle = 1$ for all $\tau_v \in C_{\phi_{\psi_v}}$. If η_v is unramified and $\{\pi_{v,i}\}$ contains a spherical representation (which is equivalent to χ_v being unramified), then it is $\pi_{v,0}$. Consequently the product $\prod_v \langle \tau_v, \Pi_{v,i}\rangle$ is a finite product. Moreover $\langle \cdot, \Pi\rangle$ does not depend on the choice of η.

We should finally mention that by theorem 5.1 of [11] the action of each normalized operator $R(\tau)$ in (4.2) on a component Π is according to the pairing $\langle \tau, \Pi\rangle$. (The rank one characters coming from χ in the normalized operators $R(c_i), i = r_1, r_1+r_2, \ldots, r_1+\ldots+r_k$, are all nontrivial and therefore the global sign of theorem 5.1 of [11] is 1 for all of them.)

Applying (4.2) to $\Pi = \otimes_v \Pi_{v,i} \in \Pi_{\phi_\psi}$ now implies that the residue is equal to

$$\sum_{d \in D} (*) R(dw_0, \Lambda_0, \chi) \sum_{x \in C_{\phi_\psi}} \langle x, \Pi\rangle \Pi. \tag{5.1}$$

It is now clear that since C_{ϕ_ψ} is abelian, (5.1) is nonzero if and only if $\langle \cdot, \Pi\rangle$ is the trivial character. We can therefore reformulate Theorem 4.5 as:

THEOREM 5.1. Π *appears in* $L^2(G(F)\backslash G(\mathbb{A}))$ *if and only if* $\langle \cdot, \Pi\rangle$ *is the trivial character, i.e., the Arthur condition (cf. [23]) holds.*

Since C_ψ is abelian, this is equivalent to the fact that there exists a positive integer d_ψ such that $\Pi \in \Pi_{\phi_\psi}$ appears with multiplicity

$$\frac{d_\psi}{|C_\psi|} \sum_{x \in C_\psi} \langle x, \Pi \rangle = \frac{d_\psi}{|C_{\phi_\psi}|} \sum_{x \in C_{\phi_\psi}} \langle x, \Pi \rangle.$$

This proves the global Arthur conjecture on the multiplicity formula for the residual spectrum.

Remark 2. We remark that, in the sum (5.1), $w_0\Lambda_0 = -\Lambda_0$ and $d\Lambda_0 = \Lambda_0$ and $d\chi = \chi$ imply $d = 1$. Therefore there is no cancellation among the summands and the residue is a sum of isomorphic images of the Langlands' quotient.

5.1. Special case of $G = \mathrm{Sp}_4$**.** The dual group $G^* = O_5(\mathbb{C})$ has only one distinguished orbit, namely the principal one given by the Jordan block (5). The corresponding unipotent parameter then parametrizes the constants, the only spherical residue of Sp_4 (cf. [22]). To parametrize the rest of the residual spectrum, we construct quadratic unipotent Arthur parameters in the sense of Moeglin as follows: There is a natural embedding of $O_4(\mathbb{C})$ into $O_5(\mathbb{C})$ by sending $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_4(\mathbb{C})$ into $\begin{pmatrix} A & 0 & B \\ 0 & 1 & 0 \\ C & 0 & D \end{pmatrix}$ in $O_5(\mathbb{C})$.

For μ a nontrivial quadratic grössencharacter of F, we define ψ as follows:

(1) $\psi | W_F\colon\ w \longmapsto \mu(w) \in \{\pm 1\}$ = Center of $O_4(\mathbb{C})$;

(2) $\psi | 1 \times SL_2(\mathbb{C}) \times 1 \equiv 1$; and

(3) $\psi | 1 \times 1 \times SL_2(\mathbb{C})$ determines a unipotent orbit with Jordan blocks (3,1) in $O_4(\mathbb{C})$.

By conjugation and by (2.1), we can assume that

$$\psi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \mathrm{diag}\,(a^2, 1, 1, 1, a^{-2}).$$

The associated Langlands parameter ϕ_ψ is, by definition, $\phi_\psi\colon\ W_F \times SL_2(\mathbb{C}) \longmapsto O_5(\mathbb{C})$:

(1) $\phi_\psi | SL_2(\mathbb{C}) \equiv 1$,

(2) $\phi_\psi(w) = \psi(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}) = \mathrm{diag}\,(\mu(w)|w|, \mu(w), 1, \mu(w), \mu(w)|w|^{-1})$.

The nontempered part of ϕ_ψ is

$$\phi_+(w) = \mathrm{diag}\,(|w|, 1, 1, 1, |w|^{-1}).$$

Therefore $\mathrm{Cent}\,(im\phi_+, O_5(\mathbb{C})) = \mathbb{C}^\times \times O_3(\mathbb{C}) = M^*$ with $M = F^\times \times SL_2(F)$, the Levi subgroup of the non-Siegel parabolic subgroup of Sp_4.

The tempered part of ϕ_ψ is

$$\phi_\psi^\circ(w) = \operatorname{diag}(\mu(w), \mu(w), 1, \mu(w), \mu(w)) \in M^*.$$

Therefore

$$\operatorname{Cent}(im\phi_\psi, O_5(\mathbb{C})) = \operatorname{Cent}(im\phi_\psi^\circ, M^*) = \mathbb{C}^\times \times O_2(\mathbb{C}) \times \{\pm 1\}.$$

So

$$\operatorname{Cent}(im\phi_\psi, O_5(\mathbb{C}))/Z_{G^*} \operatorname{Cent}(im\phi_\psi, O_5(\mathbb{C}))^\circ \simeq \{\pm 1\}.$$

For each place v, we can see that $\phi_{\psi_v}^\circ$ is the Langlands parameter for tempered representations $\{\pi_\pm(\mu)\}$, where $\pi_\pm(\mu)$ is the irreducible constituents of $\operatorname{Ind}_B^M \chi(\mu, \mu)$ and ϕ_{ψ_v} is the Langlands parameter for the Langlands' quotients $\{J_\pm(\mu_v)\}$ of $\operatorname{Ind}_P^G \pi_\pm(\mu) \otimes e^{\langle e_1, H_P(\ )\rangle}$ (see [12] for more details).

Now we calculate $S_\psi = \operatorname{Cent}(im\psi, O_5(\mathbb{C}))$. Let u be a distinguished unipotent element with Jordan block (3,1) in $O_4(\mathbb{C})$. Since μ is a nontrivial quadratic grössencharacter, $S_\psi^\circ = 1$ and

$$S_\psi / S_\psi^\circ Z_{G^*} \simeq \operatorname{Cent}(u, O_4(\mathbb{C}))/\operatorname{Cent}(u, O_4(\mathbb{C}))^\circ.$$

Here $\operatorname{Cent}(u, O_4(\mathbb{C}))/\operatorname{Cent}(u, O_4(\mathbb{C}))^\circ \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Among the remaining automorphic forms in the Arthur packet of ψ are the cuspidal representations studied by Howe and Piatetski-Shapiro (cf. [1], example 2.4.2).

This parametrizes the residual spectrum of Sp_4 obtained in [12], where Kim obtained this as the residue of the Eisenstein series associated to $\chi = \chi(\mu, \mu)$ at $\Lambda_0 = e_1$. It was also proved there that quadratic unipotent Arthur parameters exhaust the whole residual spectrum of Sp_4, coming from Borel subgroups. We observe that since $O_4(\mathbb{C})$ has only one distinguished unipotent orbit, namely the orbit $(3, 1)$, our result covers the result of [12], coming from this conjugacy class, except for the constants.

5.2. One extreme case. We give an example which was our motivation for the general result: Let $G = \mathrm{Sp}_{4n}$ and $\chi = \chi(\mu_1, \ldots, \mu_n, \mu_1, \ldots, \mu_n)$, where μ's are mutually distinct and quadratic. Then the Eisenstein series has a pole at $\Lambda_0 = e_1 + \cdots + e_n$ and the residue is given by

$$\sum_{d \in D} (*) R(dc_1 \ldots c_n, \Lambda_0, \chi) \prod_{i=n+1}^{2n} (1 + R(c_i, \Lambda_0, \chi)) f,$$

where D is the set of permutations s which satisfy $s(i) < s(i+n)$ for $i = 1, \ldots, n$.

Remark 3. The above technique can be used for GL_n to prove that the Eisenstein series associated to $\chi = \chi(\underbrace{\mu, \ldots, \mu}_{r_1}, \nu_1, \ldots, \nu_{r_2})$ has no pole of order $n-1$ if μ and ν_i are distinct. This is a very special case of the remarkable result proved by Moeglin and Waldspurger [24].

We divide Φ^+ as follows:

$$\begin{aligned}
\Phi_1 &: \quad e_i - e_j, \quad 1 \le i < j \le r_1 \\
\Phi_2 &: \quad e_{r_1+i} - e_{r_1+j}, \quad 1 \le i < j \le r_2 \\
\Phi_3 &: \quad e_i - e_{r_1+j}, \quad i = 1, \ldots, r_1,\ j = 1, \ldots, r_2
\end{aligned}$$

For $\alpha \in \Phi_3$, $\chi \circ \alpha^\vee$ is nontrivial. Let W_i be the Weyl group associated to Φ_i for $i = 1, 2$. We apply Proposition 4.3 to $\theta = \Delta - \{e_{r_1} - e_{r_1+1}\}$. Then $W_\theta = W_1 W_2$. Let D be the set of distinguished coset representatives for $W/W_1 W_2$. Then as in the proof of Proposition 4.6, we can show that the constant term of the Eisenstein series $E_0(g, f, \Lambda)$ attached to $f \in I(\Lambda, \chi)$ has at most poles of order $< n-1$.

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