Symmetric Cube for GL_2

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This article is in connection with a talk given by the second author at the International Conference on Cohomology of Arithmetic Groups, L-functions, and Automorphic Forms, Tata Institute of Fundamental Research, December 28, 1998 through January 1, 1999. At the time of the conference all we had was [KSh2] in which, using an idea of Kim [Ki1],[Ki2] and the machinery of Eisenstein series [L1], [L4], [Sh1], [Sh2], we proved the holomorphy of symmetric cube L-functions for GL_2. Striking as this result was (see the introduction of [KSh2]), we were still far from the existence of symmetric cube of an automorphic form on GL_2 as one on GL_4. Since this is now accomplished in [KSh1], using the same general machinery and ideas, but completely different L-functions, we find it more appropriate to report on this new development rather than a result which is now an immediate corollary. The second author would like to thank Professors M. S. Raghunathan and Venkataramana for their invitation and hospitality during the conference and the rest of his month visit to Tata Institute of Fundamental Research in the winter of 1999.

1 New instances of functionality

We recall the definition of modular forms [S]. If $\mathfrak{h}$ denotes the upper half plane of complex numbers z for which $Im(z) > 0$, and given a positive integer N, Γ_N is the principal congruence subgroup

$$\Gamma_N = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| g \equiv I(\mod N) \right\},$$

then a modular form of weight k with respect to Γ, $\Gamma_N \subset \Gamma \subset SL_2(\mathbb{Z})$ is a holomorphic complex function f on $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{Q} \cup \{i\infty\}$, satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \tag{1.1}$$

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for every

$$\gamma = \begin{pmatrix} a & b \\ c & c \end{pmatrix} \in \Gamma,$$

where k is an integer, $k \geq 0$. If k is even, then Equation (1.1) is equivalent to

$$f(\gamma \cdot z)(d(\gamma \cdot z))^{\frac{k}{2}} = f(z)(dz)^{\frac{k}{2}},$$

i.e., $f(z)(dz)^{\frac{k}{2}}$ is a differential form on $\Gamma \setminus \mathfrak{h}$, justifying the term "form".

The function f is called a cusp form if f vanishes on all the cusps, i.e., the set $\mathbb{Q} \cup \{i\infty\}$.

We may basically assume $\Gamma = \Gamma_0(N) = \{\gamma | c \equiv 0 \pmod{N}\}$. Then

$$f(z) = \sum_{n>0} a_n e^{2\pi i n z} \qquad (i = \sqrt{-1}).$$

Assume $a_1 = 1$ and that f is an eigenfunction for all the Hecke operators.

Theorem 1.1 (Deligne 1973) $|a_p| \leq 2p^{\frac{k-1}{2}}$.

Suppose $k = 0$, i.e., we are interested in functions on $\Gamma \setminus \mathfrak{h}$. There are no non-constant holomorphic forms. But we relax the condition to assume f is real analytic, given as an eigenfunction for $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$, say

$$\Delta f = \frac{1}{4}(1 - s^2) f.$$

Note that holomorphic means $s = \pm 1$. We assume f is also an eigenfunction for all the Hecke operators, is bounded, and vanishes on all cusps, normalized with $a_1 = 1$. Then

$$f(x + iy) = \sum_{n \neq 0} (|n|y)^{1/2} a_n K_{s/2}(2\pi |n| y) e^{2\pi i n x},$$

with

$$z^2 \frac{d^2 K_\nu}{dz^2} + z \frac{dK_\nu}{dz} - (z^2 + \nu^2) K_\nu = 0,$$

satisfying

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \qquad (z \in \mathbb{R}),$$

as $z \to +\infty$.

The function f is called a Maass form and

Conjecture 1.2 (Ramanujan-Petersson) $|a_p| \leq 2p^{-1/2}$.

Moreover, if $\lambda_1(\Gamma) = \frac{1}{4}(1 - s^2)$, either $s \in (-1, 1)$ or $s \in i\mathbb{R}$.

Conjecture 1.3 (Selberg) $\lambda_1(\Gamma) \geq \frac{1}{4}$, *i.e.*, $s \in i\mathbb{R}$.

There is a well-known way of realizing f as an irreducible subrepresentation of $L^2(\mathrm{GL}_2(\mathbb{Q})\mathbb{A}_Q^* \setminus \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}))$, using adeles of $\mathbb{Q}$ (cf. [G]). More generally, one wants to study $L^2(\mathrm{GL}_2(F)\mathbb{A}_F^* \setminus \mathrm{GL}_2(\mathbb{A}_F))$ for an arbitrary number field F, where we are considering those which transform according to a fixed character of $F^* \setminus \mathbb{A}_F^*$, center of $\mathrm{GL}_2(\mathbb{A}_F)$. We will further assume that they are infinite dimensional which amounts to being cuspidal, i.e., for each φ in the subrepresentation

$$\int_{F\backslash\mathbb{A}_F} \varphi(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g)\, dx = 0$$

for almost all $g \in \mathrm{GL}_2(\mathbb{A}_F)$. If π is such a constituent, then $\pi = \otimes_v \pi_v$ as v runs over all the places of F, and for almost all finite places v, π_v is parametrized by a conjugacy class $\{t_v\} = \{\mathrm{diag}(\alpha_v, \beta_v)\} \subset \mathrm{GL}_2(\mathbb{C})$. The representation π_v is then induced from a pair of unramified (quasi)-character μ_{v1} and μ_{v2} of F_v^*, the completion of F at v. Then $\alpha_v = \mu_{v1}(\varpi_v)$ and $\beta_v = \mu_{v2}(\varpi_v)$, where ϖ_v is a generator for the maximal ideal P_v of the ring of integers O_v of F_v. The absolute value at v is normalized so as to satisfy $|\varpi_v| = q_v^{-1}$, where q_v is the cardinality of O_v/P_v.

The Ramanujan-Petersson Conjecture then requires $|\alpha_v| = |\beta_v| = 1$, i.e. π_v is tempered, where Selberg demands similarly that $s = 2s_{\infty 1} = -2s_{\infty 2}$ be pure imaginary, i.e., each $\pi_\infty = \mathrm{Ind}(|\ |^{s_{\infty 1}}, |\ |^{s_{\infty 2}})$ is also tempered.

Theorem 1.4 (Kim-Shahidi [KSh1])

a) *If* $v < \infty$, *then*

$$q_v^{-5/34} \leq |\alpha_v| \text{ and } |\beta_v| \leq q_v^{5/34} \qquad (\frac{1}{7} < \frac{5}{34} < \frac{1}{7} + 0.004).$$

b) *If* $v = \infty$, *then* $|Re(s_{v\iota})| \leq 5/34$; *i.e.*,

$$\lambda = \frac{1}{4}(1 - s^2) \geq 0.22837.$$

Let us now look at some examples of Langlands functoriality which appear in the process of proving the theorem. They are extremely important. Consider the map

$$\begin{aligned} \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C}) &\longrightarrow \mathrm{GL}_6(\mathbb{C}), \\ (g_1, g_2) &\longrightarrow g_1 \otimes g_2, \end{aligned}$$

or more generally

$$\begin{aligned} \rho : GL_m(\mathbb{C}) \times GL_n(\mathbb{C}) &\longrightarrow GL_{mn}(\mathbb{C}), \\ (g_1, g_2) &\longrightarrow g_1 \otimes g_2. \end{aligned}$$

Langlands [L2] predicts the existence of a map

$$\begin{aligned} \rho_* : \mathrm{Aut}(GL_m(\mathbb{A}_F)) \times \mathrm{Aut}(GL_n(\mathbb{A}_F)) &\longrightarrow \mathrm{Aut}(GL_{mn}(\mathbb{A}_F)) \\ (\pi_1, \pi_2) &\longrightarrow \pi_1 \boxtimes \pi_2. \end{aligned}$$

This is very important, because it allows us to multiply automorphic forms on two different GL-groups. Of course we have the usual addition $\pi_1 \boxplus \pi_2$ which is the usual induction from a parabolic subgroup with Levi component $GL_m(\mathbb{A}_F) \times GL_n(\mathbb{A}_F)$ to $GL_{m+n}(\mathbb{A}_F)$, of the representation $\pi_1 \otimes \pi_2$.

We can therefore operate with automorphic forms as if they are Weil group representations and the result would be the global Langlands correspondence.

The map must be functorial in the sense that, if π_{1v} corresponds to $\{t_{1v}\} \subset GL_m(\mathbb{C})$ and π_{2v} corresponds to $\{t_{2v}\} \subset GL_n(\mathbb{C})$, then

$$\pi_{1v} \boxtimes \pi_{2v} \longleftrightarrow \{t_{1v} \otimes t_{2v}\} \subset GL_{mn}(\mathbb{C}).$$

In fact, more generally, the map ρ_* must respect the local Langlands correspondence of Harris-Taylor [HT] and Henniart [He]. More precisely, if ρ_{1v} and ρ_{2v} are representations of Deligne-Weil group which parametrize π_{1v} and π_{2v}, respectively, then $\rho_{1v} \otimes \rho_{2v}$ must parametrize $\pi_{1v} \boxtimes \pi_{2v}$, and therefore $(\rho_*(\pi_1, \pi_2))_v = \pi_{1v} \boxtimes \pi_{2v}$, where $\pi_{1v} \boxtimes \pi_{2v}$ corresponds to $\rho_{1v} \otimes \rho_{2v}$. Let us call $\rho_*(\pi_1, \pi_2)$, satisfying these properties, the *functorial product* of π_1 and π_2.

Theorem 1.5 (Kim-Shahidi [KSh1]) *Suppose $m = 2$ and $n = 3$. Then ρ_* exists and is functorial except possibly at places $v|2$ for which π_{1v} is extraordinary supercuspidal while π_{2v} is a supercuspidal representation of $GL_3(F_v)$ defined by a non-normal cubic extension of F_v. In this case, $\Pi_v = (\pi_{1v} \boxtimes \pi_{2v}) \otimes \eta \cdot \det$, with η at most a quadratic character of F_v^* and $\Pi = \otimes_v \Pi_v = \rho_*(\pi_1, \pi_2)$. Moreover, Π is an isobaric (cf. [JS], [L2]) automorphic representation of $GL_6(\mathbb{A}_F)$. More precisely, there exist (unitary) cuspidal representations σ_i of $GL_{n_i}(\mathbb{A}_F)$, $1 \leq i \leq r$, $\sum_i n_i = 6$, such that $\Pi = \sigma_1 \boxplus \sigma_2 \boxplus \cdots \boxplus \sigma_r$.*

Remark For the last statement one needs the weak Ramanujan conjecture for GL(2) and GL(3) which is proved in [Ra]. Assuming this conjecture for all GL groups, the fact that $\Pi = \sigma_1 \boxplus \cdots \boxplus \sigma_r$, must be true for all m and n.

Here is a sketch of how Theorem 1.4 follows from Theorem 1.5.

Proof (of Theorem 1.4) Take a cusp form $\pi_1 = \pi$ on $\mathrm{GL}_2(\mathbb{A}_F)$. Let $\pi_2 = Ad(\pi)$ be the Gelbart-Jacquet lift [GJ] of π. This is a cuspidal representation of $\mathrm{GL}_3(\mathbb{A}_F)$, if π is not monomial, whose unramified components are given by

$$(Ad(\pi))_v \leftrightarrow \{\mathrm{diag}(\alpha_v\beta_v^{-1}, 1, \alpha_v^{-1}\beta_v)\},$$

whenever π_v is given by $\{\mathrm{diag}(\alpha_v, \beta_v)\}$. Let $\Pi = \pi_1 \boxtimes \pi_2$. Then an argument using L-functions shows that $\Pi = \sigma_1 \boxplus \sigma_2$, where $\sigma_1 = \pi_1 = \pi$ and σ_2 is an automorphic form on $\mathrm{GL}_4(\mathbb{A}_F)$. Suppose π_v is unramified. Then σ_{2v} corresponding to $\{\mathrm{diag}(\alpha_v^2\beta_v^{-1}, \alpha_v, \beta_v, \alpha_v^{-1}\beta_v^2)\}$. The worst situation happens if σ_2 is a cuspidal representation of $\mathrm{GL}_4(\mathbb{A}_F)$. Then by results of Luo-Rudnick-Sarnak [LRS]: $|\alpha_v^2\beta_v^{-1}|$ and $|\beta_v^2\alpha_v^{-1}| \le q_v^{\frac{1}{2}-\frac{1}{17}}$. But $|\alpha_v| = |\beta_v^{-1}|$ and therefore $|\alpha_v|^{\pm 3} \le q_v^{\frac{1}{2}-\frac{1}{17}}$ or $q_v^{-(\frac{1}{6}-\frac{1}{51})} \le |\alpha_v|$ and $|\beta_v| \le q_v^{\frac{1}{6}-\frac{1}{51}}$. But $\dfrac{1}{6} - \dfrac{1}{51} = \dfrac{5}{34}$. Similarly at the archimedean places.

□

Corollary 1.6 ([KSh1]) *$\sigma_2 \otimes \omega_\pi = Sym^3(\pi)$, i.e., symmetric cubes exist. It is functorial everywhere. Moreover, it is cuspidal unless either π or $Ad(\pi)$ is monomial, i.e., there exist non-trivial grossencharacters η and η' such that* $\mathrm{Ad}(\pi) \otimes \eta \cong \mathrm{Ad}(\pi)$ *or* $\pi \otimes \eta' \cong \pi$.

This is very important. We must therefore recall what Sym^3_* is. This time consider the map

$$\mathrm{Sym}^3 : \ \mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_4(\mathbb{C})$$

defined by action of $\mathrm{GL}_2(\mathbb{C})$ on symmetric tensors of rank 3. In other words, if $P(x, y)$ is a homogeneous cubic form in two variable, $\mathrm{Sym}^3(g)$, $g \in \mathrm{GL}_2(\mathbb{C})$, is the matrix in $\mathrm{GL}_4(\mathbb{C})$ which gives the change of coefficients in $P(x, y)$, if we consider the form $P_g(x, y) = P((x, y)g)$. It is a homomorphism and therefore a 4-dimensional irreducible representation of $\mathrm{GL}_2(\mathbb{C})$, called the symmetric cube representation of (the standard representation of) $\mathrm{GL}_2(\mathbb{C})$.

Similar maps can be defined for any m and it is very important to define Sym^m_*. The map Sym^2_* was established by Gelbart-Jacquet [GJ] in 1978. Since then, many experts have been interested in getting Sym^3_*. There are serious and important applications. For example Sym^2_* has been very important to Langlands-Tunnel and therefore Wiles' proof of Fermat's last problem. We expect similar influence when Wiles' program starts seriously for Siegel modular forms of rank 2. In fact, the image of Sym^3 lies irreducibly inside $GSp(4, \mathbb{C})$ and will allow us for example to study Siegel modular forms of weight 3 as images of modular forms of weight 2 under Sym^3_* and so on ....

As yet another example, let n be a positive integer and consider the natural embedding of

$$i : Sp_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}).$$

We expect a map

$$i_* : \mathrm{Aut}(SO_{2n+1}(\mathbb{A}_F)) \longrightarrow \mathrm{Aut}(\mathrm{GL}_{2n}(\mathbb{A}_F)),$$

such that if $\pi = \otimes_v \pi_v \in \mathrm{Aut}(SO_{2n+1}(\mathbb{A}_F))$ and for unramified places v, π_v corresponds to $\{t_v\} \subset Sp_{2n}(\mathbb{C})$, then for each such v, $i_*(\pi)_v$ corresponds to $\{i_*(t_v)\} \subset \mathrm{GL}_{2n}(\mathbb{C})$.

Definition An automorphic representation $\Pi = \otimes\Pi_v$ of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ is called a weak lift of an irreducible automorphic representation $\pi = \otimes\pi_v$ of $SO_{2n+1}(\mathbb{A}_F)$, if at almost all unramified places v, Π_v corresponds to $i(t_v)$. We usually require that for every $v = \infty$, $\Pi_v = i_*(\pi_v)$ according to local Langlands correspondence [L3].

Theorem 1.7 (Cogdell-Kim-Piatetski-Shapiro-Shahidi [CKPSS]) *Let* π *be an irreducible globally generic cuspidal automorphic representation of* $SO_{2n+1}(\mathbb{A}_F)$. *Then* π *has a weak lift to* $\mathrm{GL}_{2n}(\mathbb{A}_F)$.

Remark This can also be approached using the trace formula (Arthur). But one needs the fundamental lemmas for regular and weighted orbital integrals of classical groups.

Remark There is another case of functoriality obtained by Kim which when combined with Sym^3_* leads to existence of Sym^4_*. In a joint work we have obtained important applications and better estimates. That will be discussed in another occasion.

It is therefore clear that functoriality requires that every homomorphism ρ between two L-groups

$$\rho : {}^LG_1 \longrightarrow {}^LG_2$$

of connected reductive algebraic groups over a number field F, should lead to a map (in loose terms)

$$\rho_* : \mathrm{Aut}(\mathbf{G}_1(\mathbb{A}_F)) \longrightarrow \mathrm{Aut}(\mathbf{G}_2(\mathbb{A}_F))$$

so that if $\pi = \otimes_v \pi_v \in \mathrm{Aut}(\mathbf{G}_1(\mathbb{A}_F))$, then for each unramified v, $(\rho_*(\pi))_v$ corresponds to $\{\rho(t_v)\} \subset {}^LG_2$, if π_v corresponds to $\{t_v\} \subset {}^LG_1$.

We finally point out how these new cases are proved. One applies an appropriate version of converse theorems of Cogdell-Piatetski-Shapiro [CP] to L-functions obtained from the method of Eisenstein series initiated by Langlands [L1], [L4] and developed by Shahidi [Sh1], [Sh2], ... In the case of $GL_2 \times GL_3$, one needs to prove that the triple L-functions for $\pi_1 \otimes \pi_2 \otimes \sigma$ on $GL_2(\mathbb{A}_F) \times GL_3(\mathbb{A}_F) \times GL_k(\mathbb{A}_F)$, $k = 1, 2, 3, 4$, where σ is an irreducible cuspidal representation of $GL_k(\mathbb{A}_F)$, are nice. In view of this converse theorem, this means that when twisted by a highly ramified (at a finite set of finite unramified places of F) grössencharacher of F, they are:

1) entire,

2) are bounded in vertical strips of finite width, and

3) satisfy a standard functional equation.

The machinery of Eisenstein series allows us to consider these L-functions as coming from triples $(\mathbf{G}, \mathbf{M}, \pi)$, where $\mathbf{G}$ is a connected reductive group and $\mathbf{M}$ a maximal Levi subgroup, both defined over F (cf. [Sh1], [Sh2]). Here π is a globally generic cuspidal representation of $M = \mathbf{M}(\mathbb{A}_F)$. In the case at hand, $\mathbf{G}$ is the simply connected group of either type A_4, D_5, E_6, or E_7 (cf. [Sh2]). The derived group of $\mathbf{M}$ is isomorphic to $SL_2 \times SL_3$, $SL_2 \times SL_3 \times SL_2$, $SL_2 \times SL_3 \times SL_3$, or $SL_2 \times SL_3 \times SL_4$, respectively. π is closely related to $\pi_1 \otimes \pi_2 \otimes \sigma$.

In a general setting including these cases and using the machinery of Eisenstein series [L4], [Sh1], [Sh2], 1) follows from an important and crucial observation of Kim [Ki1], [Ki2], 2) is proved in Gelbart-Shahidi [GSh] (subtle), and 3) follows from the general theory [Sh1], [Sh2].

A good amount of local analysis is necessary and theory of base change [AC] is required to prove the lift is functorial.

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