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CUSPIDALITY OF SYMMETRIC POWERS WITH APPLICATIONS

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Abstract

The purpose of this paper is to prove that the symmetric fourth power of a cusp form on $\mathrm{GL}(2)$, *whose existence was proved earlier by the first author, is cuspidal unless the corresponding automorphic representation is of dihedral, tetrahedral, or octahedral type. As a consequence, we prove a number of results toward the Ramanujan-Petersson and Sato-Tate conjectures. In particular, we establish the bound* $q_v^{1/9}$ *for unramified Hecke eigenvalues of cusp forms on* $\mathrm{GL}(2)$. *Over an arbitrary number field, this is the best bound available at present.*

1. Introduction

In this paper we prove a criterion for cuspidality of the fourth symmetric powers of cusp forms on $\mathrm{GL}(2)$, whose existence was established earlier by the first author. As a consequence, we show that a cuspidal representation has a noncuspidal symmetric fourth power if and only if it is of either dihedral, tetrahedral, or octahedral type. We then prove a number of corollaries toward both the Ramanujan-Petersson and Sato-Tate conjectures for cusp forms on $\mathrm{GL}(2)$ by establishing analytic properties of several new symmetric power L-functions attached to them.

More precisely, let $\mathbb{A}$ be the ring of adeles of a number field F. Let $\pi = \bigotimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ with central character ω_π. Fix a positive integer m, and let $\mathrm{Sym}^m : \mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_{m+1}(\mathbb{C})$ be the mth symmetric power representation of $\mathrm{GL}_2(\mathbb{C})$ on symmetric tensors of rank m (cf. [28], [30]). By the local Langlands correspondence (see [4], [5], [17]), $\mathrm{Sym}^m(\pi_v)$ is well defined for every v. Then Langlands functoriality in this case is equivalent to the fact that $\mathrm{Sym}^m(\pi) = \bigotimes_v \mathrm{Sym}^m(\pi_v)$ is an automorphic representation of $\mathrm{GL}_{m+1}(\mathbb{A})$. It is convenient to introduce $A^m(\pi) = \mathrm{Sym}^m(\pi) \otimes \omega_\pi^{-1}$ (denoted by $\mathrm{Ad}^m(\pi)$ in [28]).

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If $m = 2$, $A^2(\pi) = \mathrm{Ad}(\pi)$ and it is the well-known Gelbart-Jacquet lift in [2]. If $m = 3$, we proved in [13] and [12] that $\mathrm{Sym}^3(\pi)$ is an automorphic representation of $\mathrm{GL}_4(\mathbb{A})$ and gave a criterion for when it is cuspidal. In [10], the first author proved that $\mathrm{Sym}^4(\pi)$ is an automorphic representation of $\mathrm{GL}_5(\mathbb{A})$. If $\mathrm{Sym}^3(\pi)$ is cuspidal, $\mathrm{Sym}^4(\pi)$is either cuspidal or unitarily induced from cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ and $\mathrm{GL}_3(\mathbb{A})$. In this paper, we give a criterion for when $\mathrm{Sym}^4(\pi)$ is cuspidal. More precisely, we have the following.

THEOREM 3.3.7
$A^4(\pi) = \mathrm{Sym}^4(\pi) \otimes \omega_\pi^{-1}$ is a cuspidal representation of $\mathrm{GL}_5(\mathbb{A})$, except in the following three cases:

(1) *π is monomial;*

(2) *π is not monomial and $A^3(\pi)$ is not cuspidal; this is the case when there exists a nontrivial grössencharacter μ such that $\mathrm{Ad}(\pi) \simeq \mathrm{Ad}(\pi) \otimes \mu$;*

(3) *$A^3(\pi)$ is cuspidal, but there exists a nontrivial quadratic character η such that $A^3(\pi) \simeq A^3(\pi) \otimes \eta$, or, equivalently, there exists a nontrivial grössencharacter χ of E such that $\mathrm{Ad}(\pi_E) \simeq \mathrm{Ad}(\pi_E) \otimes \chi$, where E/F is the quadratic extension determined by η and π_E is the base change of π. In this case, $A^4(\pi) = \sigma_1 \boxplus \sigma_2$, where $\sigma_1 = \pi(\chi^{-1}) \otimes \omega_\pi$ and $\sigma_2 = \mathrm{Ad}(\pi) \otimes (\omega_\pi \eta)$.*

Cases (1), (2), and (3) are equivalent to π being of dihedral, tetrahedral, and octahedral type, respectively.

We give several applications of the cuspidality of third and fourth symmetric powers. First, following Ramakrishnan [23], we prove that given a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$, the set of tempered places has lower Dirichlet density of at least 34/35.

Next, we prove the meromorphic continuation and a functional equation for each of the sixth, seventh, eighth, and ninth symmetric power L-functions for cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$. An immediate corollary (cf. [27, Lemma 5.8]) is that if π_v is an unramified local component of a cuspidal representation $\pi = \bigotimes_v \pi_v$, then $q_v^{-1/9} < |\alpha_v|, |\beta_v| < q_v^{1/9}$, where $\mathrm{diag}(\alpha_v, \beta_v)$ is the Satake parameter for π_v. The archimedean analogue of 1/9, using this approach, is proved in [11]. When $F = \mathbb{Q}$, using the ideas in [19], the bound 1/9 can be improved to $7/64 + \varepsilon$, $\forall \varepsilon > 0$. This is the subject of an appendix in [10] by Kim and Sarnak. For an arbitrary number field, $1/9$ remains the best bound available at present.

Finally, we prove that partial fifth, sixth, seventh, and eighth symmetric power L-functions attached to a cuspidal representation π of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character such that $\mathrm{Sym}^4(\pi)$ is cuspidal are all invertible at $s = 1$, and we apply this fact to the Sato-Tate conjecture (see [25]), following Serre's method (see [30, Appendix]). Namely, we show that for every $\epsilon > 0$ there are sets T^+ and T^- of positive lower (Dirichlet) densities such that $a_v > 2\cos(2\pi/11) - \epsilon$ for all $v \in T^+$, and $a_v < -2\cos(2\pi/11) + \epsilon$ for all $v \in T^-$, where $a_v = \alpha_v + \beta_v$. Note that $2\cos(2\pi/11) = 1.68\ldots$.

2. Cuspidality of the symmetric cube

Suppose π is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$. We review the properties of the symmetric cube $\mathrm{Sym}^3(\pi)$ (see [13]). Recall that $A^3(\pi) = \mathrm{Sym}^3(\pi) \otimes \omega_\pi^{-1}$.

2.1. π a monomial cuspidal representation

That is, $\pi \otimes \eta \simeq \pi$ for a nontrivial grössencharacter η. Then $\eta^2 = 1$ and η determines a quadratic extension E/F. According to [14], there is a grössencharacter χ of E such that $\pi = \pi(\chi)$, where $\pi(\chi)$ is the automorphic representation whose local factor at v is the one attached to the representation of the local Weil group induced from χ_v. Let χ' be the conjugate of χ by the action of the nontrivial element of the Galois group. Then the Gelbart-Jacquet lift (adjoint) of π is given by

$$\mathrm{Ad}(\pi) = \pi(\chi\chi'^{-1}) \boxplus \eta.$$

There are two cases.

Case 1: $\chi\chi'^{-1}$ factors through the norm. That is, $\chi\chi'^{-1} = \mu \circ N_{E/F}$ for a grössencharacter μ of F. Then $\pi(\chi\chi'^{-1})$ is not cuspidal. In fact, $\pi(\chi\chi'^{-1}) = \mu \boxplus \mu\eta$. In this case,

$$A^3(\pi) = \pi(\chi\chi'^{-1}) \boxtimes \pi = (\mu \otimes \pi) \boxplus (\mu\eta \otimes \pi).$$

Case 2: $\chi\chi'^{-1}$ does not factor through the norm. In this case, $\pi(\chi\chi'^{-1})$ is a cuspidal representation. Then

$$A^3(\pi) = \pi(\chi\chi'^{-1}) \boxtimes \pi = \pi(\chi^2\chi'^{-1}) \boxplus \pi.$$

Here we use the fact that $\pi(\chi)_E = \chi \boxplus \chi'$ (see [24, Proposition 2.3.1]) and that $\pi' \boxtimes \pi = I_F^E(\pi'_E \otimes \chi)$ if $\pi = \pi(\chi)$ (see [24, §3.1]).

2.2. π not monomial

In this case, $\mathrm{Ad}(\pi)$ is a cuspidal representation of $\mathrm{GL}_3(\mathbb{A})$. We recall from [13] the following.

THEOREM 2.2.1
Let σ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$. Then the triple L-function $L_S(s, \mathrm{Ad}(\pi) \times \pi \times \sigma)$ has a pole at $s = 1$ if and only if $\sigma \simeq \pi \otimes \chi$ and $\mathrm{Ad}(\pi) \simeq \mathrm{Ad}(\pi) \otimes (\omega_\pi\chi)$ for some grössencharacter χ.

By Theorem 2.2.1, we have the following.

THEOREM 2.2.2
Let π be a nonmonomial cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$. *Then* $A^3(\pi)$ *is not cuspidal if and only if there exists a nontrivial grössencharacter* μ *such that* $\mathrm{Ad}(\pi) \simeq \mathrm{Ad}(\pi) \otimes \mu$. *In that case,*

$$A^3(\pi) = (\pi \otimes \mu) \boxplus (\pi \otimes \mu^2).$$

3. Cuspidality of the symmetric fourth

Suppose π is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Let $A^4(\pi) = \mathrm{Sym}^4(\pi) \otimes \omega_\pi^{-1}$. We review the properties of $\mathrm{Sym}^4(\pi)$ (see [10]).

3.1. π monomial

Suppose π is a monomial cuspidal representation given by $\pi = \pi(\chi)$. Then the Gelbart-Jacquet lift of π is given by $\mathrm{Ad}(\pi) = \pi(\chi\chi'^{-1}) \boxplus \eta$.

Case 1: $\chi\chi'^{-1}$ factors through the norm. Then (see Section 2.1) since $\wedge^2(A^3(\pi)) = A^4(\pi) \oplus \omega_\pi$,

$$A^4(\pi) = (\pi \boxtimes \pi) \boxplus \omega_\pi = \omega_\pi \boxplus \omega_\pi \boxplus \mu\omega_\pi \boxplus \eta\omega_\pi \boxplus \mu\eta\omega_\pi.$$

We used the fact that η and μ are quadratic grössencharacters.

Case 2: $\chi\chi'^{-1}$ does not factor through the norm. Then (see Section 2.1)

$$A^4(\pi) = \big(\pi(\chi^2\chi'^{-1}) \boxtimes \pi\big) \boxplus \omega_\pi = \pi(\chi^3\chi'^{-1}) \boxplus \pi(\chi^2) \boxplus \omega_\pi.$$

3.2. π a nonmonomial representation such that $\mathrm{Sym}^3(\pi)$ *is not cuspidal*

This is the case when there exists a nontrivial grössencharacter μ such that $\mathrm{Ad}(\pi) \simeq \mathrm{Ad}(\pi) \otimes \mu$. Note that $\mu^3 = 1$. Then $A^3(\pi) = (\pi \otimes \mu) \boxplus (\pi \otimes \mu^2)$. Hence

$$\wedge^2(A^3\pi) = \mathrm{Sym}^2(\pi) \boxplus \omega_\pi \boxplus \omega_\pi\mu \boxplus \omega_\pi\mu^2.$$

So

$$A^4(\pi) = \mathrm{Sym}^2(\pi) \boxplus \omega_\pi\mu \boxplus \omega_\pi\mu^2.$$

PROPOSITION 3.2.1
Suppose π is a nonmonomial representation such that $A^3(\pi)$ is not cuspidal; that is, $\mathrm{Ad}(\pi) \simeq \mathrm{Ad}(\pi) \otimes \mu$. *Then* $L(s, \pi, \mathrm{Sym}^4 \otimes \omega_\pi^{-1})$ *has a pole at $s = 1$ if and only if $\omega_\pi = \mu$ or μ^2. In particular, if $\omega_\pi = 1$,* $L(s, \pi, \mathrm{Sym}^4)$ *is holomorphic at $s = 1$.*

3.3. Both $\mathrm{Ad}(\pi)$ *and* $\mathrm{Sym}^3(\pi)$ *cuspidal*

Throughout this paper S is always a set of places of F such that for $v \notin S$ every representation is unramified.

The first author showed in [10] that $\wedge^2(A^3(\pi))$ is an automorphic representation of $\mathrm{GL}_6(\mathbb{A})$ and that $\wedge^2(A^3(\pi)) = A^4(\pi) \boxplus \omega_\pi$. Hence $A^4(\pi)$ is an automorphic representation of $\mathrm{GL}_5(\mathbb{A})$, either cuspidal or induced from cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ and $\mathrm{GL}_3(\mathbb{A})$. We want to give a criterion for when $A^4(\pi)$ is cuspidal. First we note that

$$L_S\big(s, \sigma \otimes A^3(\pi), \rho_2 \otimes \wedge^2 \rho_4\big) = L_S\big(s, \sigma \times A^4(\pi)\big) L_S(s, \sigma \otimes \omega_\pi). \qquad (1)$$

LEMMA 3.3.1

$A^4(\pi)$ is not cuspidal if and only if $L_S(s, \sigma \otimes A^3(\pi), \rho_2 \otimes \wedge^2 \rho_4)$ has a pole at $s = 1$ for some cuspidal representation σ of $\mathrm{GL}_2(\mathbb{A})$.

Proof

Since $A^4(\pi)$ is either cuspidal or induced from cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ and $\mathrm{GL}_3(\mathbb{A})$, $A^4(\pi)$ is not cuspidal if and only if $L_S(s, \sigma \times A^4(\pi))$ has a pole at $s = 1$ for some cuspidal representation σ of $\mathrm{GL}_2(\mathbb{A})$. Our assertion follows from (1) since $L_S(s, \sigma \otimes \omega_\pi)$ is invertible at $s = 1$. □

In order to find a criterion for the pole of $L(s, \sigma \otimes A^3(\pi), \rho_2 \otimes \wedge^2 \rho_4)$, we need the following unpublished result of H. Jacquet, I. Piatetski-Shapiro, and J. Shalika (cf. [22]).

THEOREM 3.3.2 (Jacquet, Piatetski-Shapiro, and Shalika)

Let π be a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A})$ such that there exist a grössencharacter χ and a finite set S of places as above for which $L_S(s, \pi, \wedge^2 \otimes \chi^{-1})$ has a pole at $s = 1$. Then there exists a globally generic cuspidal automorphic representation τ of $\mathrm{GSp}_4(\mathbb{A})$ with central character χ such that π is the Langlands functorial lift of τ under the natural embedding ${}^L\,\mathrm{GSp}_4 = \mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C})$.

The following paragraph is a brief sketch of the steps of the proof of the theorem, which we are including at a referee's suggestion. We thank Dinakar Ramakrishnan for helping us with its preparation.

Theorem 3.3.2 is proved using the dual reductive pair $(\mathrm{GO}_6(\mathbb{A}), \mathrm{GSp}_4(\mathbb{A}))$. More precisely, one considers the low-rank isogeny of SL_4 and SO_6 to lift a cuspidal representation $\pi = \bigotimes_v \pi_v$ of $\mathrm{GL}_4(\mathbb{A})$ to one, still denoted by π, on $\mathrm{GO}_6(\mathbb{A})$, provided that the central character of π_v is trivial on $\pm I$ for each v. One can then compute the theta lift of π to an automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ by integrating functions in the space of π in the usual way against the θ-function, a function on the two-fold cover of $\mathrm{GSp}_{12}(\mathbb{A})$. It is this integral that, in view of [6], is in fact equal to the residue of $L_S(s,\pi,\wedge^2\otimes\chi^{-1})$ at $s=1$, where χ is the central character of the theta lift. Hence the nonvanishing of the theta lift of π to $\mathrm{GSp}_4(\mathbb{A})$ is equivalent to the existence of a pole for $L_S(s,\pi,\wedge^2\otimes\chi^{-1})$ at $s=1$. Here S is a finite set of places for which $v\notin S$ implies that π_v is unramified. The lift is irreducible and globally generic.

We now look at how the Satake parameter behaves under the map ${}^L\mathrm{GSp}_4=\mathrm{GSp}_4(\mathbb{C})\hookrightarrow\mathrm{GL}_4(\mathbb{C})$. Suppose τ_v is an unramified representation given by $\mathrm{Ind}_B^{\mathrm{GSp}_4}\ \mu\otimes\nu\otimes\lambda$, where μ,ν,λ are unramified quasicharacters of $F_v^\times$ and $\mu\otimes\nu\otimes\lambda$ is the character of the torus which assigns to $\mathrm{diag}(x,y,ty^{-1},tx^{-1})$ the value $\mu(x)\nu(y)\lambda(t)$. Note that the central character is $\chi_v=\mu\nu\lambda^2$.

Then the Satake parameter corresponding to τ_v is (see, e.g., [31, p. 95])

$$\mathrm{diag}(\mu\nu\lambda,\mu\lambda,\lambda,\nu\lambda).$$

Here we identify μ with $\mu(\varpi)$; we do the same with ν and λ. The Satake parameter for $\wedge^2(\pi_v)$ is

$$\begin{aligned}\mathrm{diag}(&\mu^2\nu\lambda^2,\mu\nu\lambda^2,\mu\nu^2\lambda,\mu\lambda^2,\nu\lambda^2,\mu\nu\lambda^2)\\&=\mathrm{diag}(\chi_v\mu,\chi_v\nu,\chi_v,\chi_v\mu^{-1},\chi_v\nu^{-1},\chi_v).\end{aligned}$$

Hence, if σ_v is an unramified representation of $\mathrm{GL}_2(F_v)$ given by $\pi(\eta_1,\eta_2)$, then

$$\begin{aligned}L(s,&\sigma_v\otimes\pi_v,\rho_2\otimes\wedge^2\rho_4)^{-1}\\&=\prod_{i=1}^{2}(1-\chi_v\eta_i\mu^{\pm1}q_v^{-s})(1-\chi_v\eta_i\nu^{\pm1}q_v^{-s})(1-\chi_v\eta_iq_v^{-s})\\&\quad\times\prod_{i=1}^{2}(1-\chi_v\eta_iq_v^{-s})\\&=L\big(s,(\sigma_v\otimes\chi_v)\times\tau_v\big)^{-1}L(s,\sigma_v\otimes\chi_v)^{-1}.\end{aligned}\tag{2}$$

Here $L(s,(\sigma_v\otimes\chi_v)\times\tau_v)$ is the degree 10 Rankin-Selberg L-function for $\mathrm{GL}_2\times\mathrm{GSp}_4$. Note that if τ' is any irreducible constituent of $\tau|_{\mathrm{Sp}_4(\mathbb{A})}$, then

$$L\big(s,(\sigma_v\otimes\chi_v)\times\tau_v\big)=L\big(s,(\sigma_v\otimes\chi_v)\times\tau_v'\big).$$

We now apply the above observation to $A^3(\pi)$, where π is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$. Since $\wedge^2(A^3(\pi))=A^4(\pi)\boxplus\omega_\pi$, $L_S(s,A^3(\pi),\wedge^2\otimes\omega_\pi^{-1})$ has a pole at $s=1$. So there exists a generic cuspidal representation τ of $\mathrm{GSp}_4(\mathbb{A})$ with central character ω_π. Let τ' be any irreducible constituent of $\tau|_{\mathrm{Sp}_4(\mathbb{A})}$. Then by (1) and (2), we have

$$L_S\big(s,\sigma\times A^4(\pi)\big)=L_S\big(s,(\sigma\otimes\omega_\pi)\times\tau'\big).$$

Recall that if we consider the maximal Levi subgroup $\mathrm{GL}_2 \times \mathrm{Sp}_4 \subset \mathrm{Sp}_8$ with the cuspidal representation $(\sigma \otimes \omega_\pi) \otimes \tau'$ (see [27]), we obtain the following L-function as a normalizing factor in the constant term of the Eisenstein series:

$$L_S\big(s, (\sigma \otimes \omega_\pi) \times \tau'\big) L_S(2s, \omega_\sigma \omega_\pi^2).$$

Note that if $(\omega_\sigma \omega_\pi^2)^2 \neq 1$, $(\sigma \otimes \omega_\pi) \otimes \tau'$ is not self-contragredient. Hence we have the following.

PROPOSITION 3.3.3
If $(\omega_\sigma \omega_\pi^2)^2 \neq 1$, then $L_S(s, (\sigma \otimes \omega_\pi) \times \tau')$ is holomorphic at $s = 1$.

Proof
By following the proof of [8, Theorem 3.2], we can show that the completed L-function $L(s, (\sigma \otimes \omega_\pi) \times \tau')$ is entire. Since $(\sigma \otimes \omega_\pi) \otimes \tau'$ is not self-contragredient, by [7, Proposition 2.1] the global intertwining operator $M(s, (\sigma \otimes \omega_\pi) \otimes \tau', w_0)$ is holomorphic for $\operatorname{Re} s > 0$. Using [8, Proposition 3.4], the local normalized intertwining operators $N(s, (\sigma_v \otimes \omega_{\pi_v}) \otimes \tau_v', w_0)$ are holomorphic and nonzero for $\operatorname{Re} s \geq 1/2$. Hence we see that the L-function $L(s, (\sigma \otimes \omega_\pi) \times \tau') L(2s, \omega_\sigma \omega_\pi^2)$ is holomorphic for $\operatorname{Re} s \geq 1/2$. Since $L(2s, \omega_\sigma \omega_\pi^2)$ has no zeros for $\operatorname{Re} s \geq 1/2$, $L(s, (\sigma \otimes \omega_\pi) \times \tau')$ is holomorphic for $\operatorname{Re} s \geq 1/2$. Our assertion follows from the functional equation. $\square$

Using the integral representation, we have the following.

THEOREM 3.3.4 (Ginzburg-Rallis-Soudry [3])
If $\omega_\sigma \omega_\pi^2 = 1$, then $L_S(s, (\sigma \otimes \omega_\pi) \times \tau')$ is holomorphic at $s = 1$.

Proof
This follows immediately from the integral representation for $L_S(s, (\sigma \otimes \omega_\pi) \times \tau')$. The possible poles come from the poles of the Eisenstein series attached to $(\mathrm{GL}_2, \sigma \otimes \omega_\pi)$ for the split group SO_5, which in turn come from the poles of symmetric square L-function $L_S(s, \sigma \otimes \omega_\pi, \mathrm{Sym}^2)$. The last L-function is entire if $\omega_\sigma \omega_\pi^2 = 1$. $\square$

COROLLARY 3.3.5
If $\omega_\sigma \omega_\pi^2 = 1$, or if $(\omega_\sigma \omega_\pi^2)^2 \neq 1$, then $L_S(s, \sigma \times A^4(\pi))$ is holomorphic at $s = 1$.

PROPOSITION 3.3.6
Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ such that both $\Pi = \mathrm{Ad}(\pi)$ and $A^3(\pi)$ are cuspidal. Then the following are equivalent.
(1) *$A^4(\pi)$ is not cuspidal.*

(2) *There exists a quadratic extension E/F such that $A^3(\pi) \simeq A^3(\pi) \otimes \eta$, where η is the quadratic grössencharacter attached to E/F via class field theory. This means that the base change $(A^3(\pi))_E$ is not cuspidal. (Note that $A^3(\pi_E) \simeq (A^3(\pi))_E$.)*

(3) *There exists a quadratic extension E/F such that $\Pi_E \simeq \Pi_E \otimes \chi$ for a nontrivial grössencharacter χ of E, where Π_E is the base change of Π to E.*

Proof

Consider

$$L_S\big(s, (\sigma \otimes \omega_\pi) \times \Pi \times \Pi\big) = L_S\big(s, (\sigma \otimes \omega_\pi) \times \Pi\big) L_S\big(s, \sigma \otimes A^3(\pi), \rho_2 \otimes \wedge^2 \rho_4\big).$$

Since $L_S(s, (\sigma \otimes \omega_\pi) \times \Pi)$ does not have a pole or a zero at $s = 1$, $L_S(s, (\sigma \otimes \omega_\pi) \times \Pi \times \Pi)$ has a pole at $s = 1$ if and only if $L_S(s, \sigma \otimes A^3(\pi), \rho_2 \otimes \wedge^2 \rho_4)$ has a pole at $s = 1$.

Statement (1) implies statement (2). Since $A^4(\pi)$ is not cuspidal, $L_S(s, \sigma \otimes A^3(\pi), \rho_2 \otimes \wedge^2 \rho_4)$ has a pole at $s = 1$ for some σ by Lemma 3.3.1. Hence, by Corollary 3.3.5, $\omega^2 = 1$, $\omega \neq 1$, where $\omega = \omega_\sigma \omega_\pi^2$. Let E/F be the quadratic extension attached to ω via class field theory. Let $(A^3(\pi))_E$ be the base change of $A^3(\pi)$. Consider the equality

$$\begin{aligned} &L_S\Big(s, \sigma_E \otimes \big(A^3(\pi)\big)_E, \rho_2 \otimes \wedge^2 \rho_4\Big) \\ &\qquad = L_S\big(s, \sigma \otimes A^3(\pi), \rho_2 \otimes \wedge^2 \rho_4\big) L_S\big(s, (\sigma \otimes \omega) \otimes A^3(\pi), \rho_2 \otimes \wedge^2 \rho_4\big). \end{aligned}$$

Note that $L_S\big(s, \sigma_E \otimes (A^3(\pi))_E, \rho_2 \otimes \wedge^2 \rho_4\big)$ has a pole at $s = 1$ if and only if $L_S(s, (\sigma_E \otimes \omega_{\pi_E}) \times \Pi_E \times \Pi_E)$ has a pole at $s = 1$.

Suppose that $(A^3(\pi))_E$ is cuspidal. Then $\Pi_E \not\simeq \Pi_E \otimes \chi$ for any nontrivial character by Theorem 2.2.2. If σ is not monomial, then σ_E is cuspidal, and hence $L_S(s, \sigma_E \otimes (A^3(\pi))_E, \rho_2 \otimes \wedge^2 \rho_4)$ is holomorphic at $s = 1$. If σ is monomial, then σ_E is an automorphic representation induced from two grössencharacters. Hence again $L_S(s, (\sigma_E \otimes \omega_{\pi_E}) \times \Pi_E \times \Pi_E)$ is holomorphic at $s = 1$. Therefore $L_S(s, \sigma \otimes A^3(\pi), \rho_2 \otimes \wedge^2 \rho_4)$ is holomorphic at $s = 1$ for any σ. This is a contradiction.

Statement (2) is equivalent to statement (3). Suppose $(A^3(\pi))_E$ is not cuspidal. Since $(A^3(\pi))_E$ is equivalent to $A^3(\pi_E)$, $A^3(\pi_E)$ is not cuspidal. So $\mathrm{Ad}(\pi_E) \simeq \mathrm{Ad}(\pi_E) \otimes \chi$ for a nontrivial grössencharacter of E. Since Π_E is equivalent to $\mathrm{Ad}(\pi_E)$, we have $\Pi_E \simeq \Pi_E \otimes \chi$ for a nontrivial grössencharacter χ of E.

Statement (3) implies statement (1). Let $\sigma' = \sigma \otimes \omega_\pi = \pi(\chi)$ be the monomial representation of $\mathrm{GL}_2(\mathbb{A}_F)$ attached to χ. Let η be the quadratic character attached to E/F. Then $\sigma'_E = \chi \boxplus \chi'$. Consider the equality

$$L_S(s, \sigma'_E \times \Pi_E \times \Pi_E) = L_S(s, \sigma' \times \Pi \times \Pi) L_S\big(s, (\sigma' \otimes \eta) \times \Pi \times \Pi\big).$$

Then $L_S(s, \sigma'_E \times \Pi_E \times \Pi_E) = L_S(s, (\chi \otimes \Pi_E) \times \Pi_E) L_S(s, (\chi' \otimes \Pi_E) \times \Pi_E)$. Since $\chi \otimes \Pi_E \simeq \Pi_E$, $L_S(s, (\chi \otimes \Pi_E) \times \Pi_E)$ has a pole at $s = 1$. Hence either $L(s, \sigma' \times \Pi \times \Pi)$ or $L_S(s, (\sigma' \otimes \eta) \times \Pi \times \Pi)$ has a pole at $s = 1$. This implies that either $L_S(s, \sigma \otimes \tau, \rho_2 \otimes \wedge^2 \rho_4)$ or $L_S(s, (\sigma \otimes \eta) \otimes \tau, \rho_2 \otimes \wedge^2 \rho_4)$ has a pole at $s = 1$. Hence $A^4(\pi)$ is not cuspidal. □

By the above proposition, we see the following.

THEOREM 3.3.7
$A^4(\pi) = \mathrm{Sym}^4(\pi) \otimes \omega_\pi^{-1}$ is a cuspidal representation of $\mathrm{GL}_5(\mathbb{A})$ except in the following three cases:

(1) *π is monomial;*

(2) *π is not monomial and $A^3(\pi)$ is not cuspidal; this is the case when there exists a nontrivial grössencharacter μ such that $\mathrm{Ad}(\pi) \simeq \mathrm{Ad}(\pi) \otimes \mu$;*

(3) *$A^3(\pi)$ is cuspidal and there exists a nontrivial quadratic character η such that $A^3(\pi) \simeq A^3(\pi) \otimes \eta$, or, equivalently, there exists a nontrivial grössencharacter χ of E such that $\mathrm{Ad}(\pi_E) \simeq \mathrm{Ad}(\pi_E) \otimes \chi$, where E/F is the quadratic extension determined by η. In this case, $A^4(\pi) = \sigma_1 \boxplus \sigma_2$, where $\sigma_1 = \pi(\chi^{-1}) \otimes \omega_\pi$ and $\sigma_2 = \mathrm{Ad}(\pi) \otimes (\omega_\pi \eta)$.*

Proof
We only need to prove the last assertion. By the proof of Proposition 3.3.6, $L_S(s, \tilde{\sigma}_1 \times A^4(\pi))$ has a pole at $s = 1$. Consider $L(s, \widetilde{A^3(\pi)} \times (A^3(\pi) \otimes \eta))$. It has a pole at $s = 1$ since $A^3(\pi) \simeq A^3(\pi) \otimes \eta$. By formal calculation,

$$\begin{aligned}
&L_S\Big(s, \widetilde{A^3(\pi)} \times \big(A^3(\pi) \otimes \eta\big)\Big) \\
&\quad = L_S\big(s, \pi, \mathrm{Sym}^6 \otimes (\omega_\pi^{-3}\eta)\big) L_S\big(s, A^4(\pi) \otimes (\omega_\pi^{-1}\eta)\big) \\
&\qquad \times L_S\big(s, \mathrm{Ad}(\pi) \otimes \eta\big) L_S(s, \eta) \\
&\quad = L_S\Big(s, \big(\mathrm{Ad}(\pi) \otimes (\omega_\pi^{-1}\eta)\big) \times A^4(\pi)\Big) L_S(s, \eta).
\end{aligned}$$

The L-function $L(s, \eta)$ has no zeros at $s = 1$, and therefore

$$L_S\Big(s, \big(\mathrm{Ad}(\pi) \otimes (\omega_\pi^{-1}\eta)\big) \times A^4(\pi)\Big)$$

has a pole at $s = 1$. Hence $\sigma_2 = \mathrm{Ad}(\pi) \otimes (\omega_\pi \eta)$.

We note that since χ is a nontrivial cubic character, $\chi|_{\mathbb{A}_F^*} = 1$, and the central character of $\pi(\chi)$ is $\eta \cdot \chi|_{\mathbb{A}_F^*} = \eta$. □

COROLLARY (of the proof)
Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $A^3(\pi)$ is cuspidal. If $A^4(\pi)$

is not cuspidal, then $L(s, \pi, \mathrm{Sym}^6 \otimes (\omega_\pi^{-3}\eta))$ *has a pole at* $s = 1$*. Here* η *is as in Theorem 3.3.7(3).*

Finally, based on Langlands's calculations in [18], it is reasonable to claim the following.

CONJECTURE
Let π *be a nonmonomial cuspidal representation of* $\mathrm{GL}_2(\mathbb{A})$*. Then*
(1) $\mathrm{Sym}^3(\pi)$ *is not cuspidal if and only if* π *is of tetrahedral type;*
(2) $\mathrm{Sym}^3(\pi)$ *is cuspidal, but* $\mathrm{Sym}^4(\pi)$ *is not, if and only if* π *is of octahedral type;*
(3) $\mathrm{Sym}^4(\pi)$ *and* $\mathrm{Sym}^5(\pi)$ *are cuspidal, but* $\mathrm{Sym}^6(\pi)$ *is not, if and only if* π *is of icosahedral type.*

Remark. The final form of part (3) of the conjecture is an outcome of a number of communications with J.-P. Serre as well as calculations done in [9].

The purpose of our next proposition is to demonstrate the first two parts of the conjecture.

PROPOSITION 3.3.8
Let π *be a nonmonomial cuspidal representation of* $\mathrm{GL}_2(\mathbb{A})$*. Then*
(1) $\mathrm{Sym}^3(\pi)$ *is not cuspidal if and only if* π *is of tetrahedral type;*
(2) $\mathrm{Sym}^3(\pi)$ *is cuspidal, but* $\mathrm{Sym}^4(\pi)$ *is not, if and only if* π *is of octahedral type.*

Proof
Part (1) is [13, Lemma 6.5]. Part (2) is proved the same way. In fact, observe first that by Proposition 3.3.6(3) there exists a quadratic extension E/F such that $\Pi_E \cong \Pi_E \otimes \chi$ for a nontrivial grössencharacter χ. Notice that $\Pi_E = \mathrm{Ad}(\pi_E)$ and that therefore $\mathrm{Ad}(\pi_E) \cong \mathrm{Ad}(\pi_E) \otimes \chi$. By Theorem 2.2.2, $\mathrm{Sym}^3(\pi_E)$ is not cuspidal, and therefore, by part (1), π_E is of tetrahedral type. Consequently, there exists a two-dimensional tetrahedral representation σ_E of W_E such that $\pi_E = \pi(\sigma_E)$. Since σ_E is invariant under $\mathrm{Gal}(E/F)$, it can be extended to a two-dimensional continuous representation σ of W_F, which is now octahedral. Let $\pi' = \pi(\sigma)$, which is of octahedral type. Clearly, $\pi'_E \cong \pi_E$, and therefore $\pi' \cong \pi \otimes \eta^a$ for some $a = 0, 1$, where η is the grössencharacter attached to E/F. But σ is unique only up to twisting by a power of η, and therefore by changing the choice of η if necessary, we have $\pi \cong \pi' = \pi(\sigma)$. We are done. $\square$

4. Applications

We give several applications of cuspidality of third and fourth symmetric powers. In this section we let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ unless otherwise specified. Let π_v be an unramified local component with the Satake parameter $\mathrm{diag}(\alpha_v, \beta_v)$. Set $a_v(\pi) = \alpha_v + \beta_v$.

THEOREM 4.1

Let $S(\pi)$ be the set of places where π_v is tempered. Then

$$\underline{\delta}\big(S(\pi)\big) \geq \frac{34}{35}.$$

Proof

We follow [23]. Let ω be the central character of π. Let $a_v = a_v(\pi)$. Then by direct computation we see that $a_v(\mathrm{Sym}^2(\pi)) = a_v^2 - \omega_v$ and $a_v(A^4(\pi)) = \omega_v^{-1} a_v^4 - 3a_v^2 + \omega_v$.

For m, k, l nonnegative integers to be chosen below, let

$$\eta = m[\omega] \boxplus k\,\mathrm{Sym}^2(\pi) \boxplus l A^4(\pi).$$

Then

$$\begin{aligned} a_v(\eta) &= m\omega_v + k(a_v^2 - \omega_v) + l(\omega_v^{-1} a_v^4 - 3a_v^2 + \omega_v) \\ &= (m - k + l)\omega_v + (k - 3l)a_v^2 + l\omega_v^{-1} a_v^4. \end{aligned}$$

Let $T(\pi, 2) = \{v \mid |a_v| \geq 2\}$. Then note that for $v \in T(\pi, 2)$, $\alpha_v = \alpha q^r$, $\beta_v = \alpha q^{-r}$, for $|\alpha| = 1$ and $r \geq 0$ (see [23, Claim 4.6]). So except for finitely many places, $S(\pi) = \{v \mid |a_v| \leq 2\}$. If $v \in T(\pi, 2)$, $a_v = \alpha(q^r + q^{-r})$. So

$$\big|a_v(\eta)\big| \geq m + 3k + 5l.$$

Hence $v \in T(\eta, m + 3k + 5l)$. Thus, by [23, (4.4)],

$$\overline{\delta}\big(T(\eta, m + 3k + 5l)\big) \leq \frac{m^2 + k^2 + l^2}{(m + 3k + 5l)^2}.$$

This holds for every choice of (nonnegative) triples (m, k, l). It can be verified that the minimum of the right-hand side occurs when $k = 3m$, $l = 5m$, yielding

$$\overline{\delta}\big(T(\pi, 2)\big) \leq \frac{1}{35}. \qquad \square$$

Higher symmetric power L-functions and the Sato-Tate conjecture

In the following, let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$, and let S be a finite set of places, including the archimedean ones, such that π_v is unramified for

$v \notin S$. Let $\operatorname{diag}(\alpha_v, \beta_v)$ be the Satake parameter for π_v for $v \notin S$. Then the partial mth symmetric power L-function is defined to be

$$L(s, \pi_v, \operatorname{Sym}^m)^{-1} = \prod_{i=0}^{m} (1 - \alpha_v^{m-i} \beta_v^i q_v^{-s}),$$
$$L_S(s, \pi, \operatorname{Sym}^m) = \prod_{v \notin S} L(s, \pi_v, \operatorname{Sym}^m).$$

PROPOSITION 4.2
Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ such that $\operatorname{Sym}^3(\pi)$ is cuspidal. Then $L_S(s, \pi, \operatorname{Sym}^5)$ is invertible for $\operatorname{Re} s \geq 1$; that is, it is holomorphic and nonzero for $\operatorname{Re} s \geq 1$.

Proof
Consider

$$L_S\big(s, \operatorname{Sym}^2(\pi) \times \operatorname{Sym}^3(\pi)\big) = L_S(s, \pi, \operatorname{Sym}^5) L_S\big(s, \operatorname{Sym}^3(\pi) \otimes \omega_\pi\big) L_S(s, \pi \otimes \omega_\pi^2).$$

Note that $L_S(s, \operatorname{Sym}^3(\pi) \otimes \omega_\pi) L_S(s, \pi \otimes \omega_\pi^2)$ is invertible for $\operatorname{Re} s \geq 1$. Note also that the left-hand side is invertible for $\operatorname{Re} s \geq 1$. Hence our result follows. □

PROPOSITION 4.3
Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ such that $\operatorname{Sym}^3(\pi)$ is cuspidal. Then every partial sixth symmetric power L-function has a meromorphic continuation and satisfies a standard functional equation. Moreover, if $\omega_\pi^3 = 1$, they are all invertible for $\operatorname{Re} s \geq 1$.

Proof
By standard calculations,

$$\begin{aligned} &L_S\big(s, \operatorname{Sym}^3(\pi) \times \operatorname{Sym}^3(\pi)\big) \\ &\qquad = L_S(s, \pi, \operatorname{Sym}^6) L_S\big(s, \operatorname{Sym}^4(\pi) \otimes \omega_\pi\big) L_S\big(s, \operatorname{Sym}^2(\pi) \otimes \omega_\pi^2\big) L_S(s, \omega_\pi^3). \end{aligned}$$

Meromorphic continuation and a functional equation follow.

If $\omega_\pi^3 = 1$, then $\operatorname{Sym}^3(\pi)$ is self-contragredient. This implies that the left-hand side has a pole at $s = 1$, while in the right-hand side $L(s, \omega_\pi^3)$ has a pole at $s = 1$. By assumption, $\operatorname{Sym}^4(\pi)$ is either cuspidal or of the form $\sigma_1 \boxplus \sigma_2$, where σ_1 and σ_2 are cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ and $\mathrm{GL}_3(\mathbb{A})$, respectively. Hence $L_S(s, \operatorname{Sym}^4(\pi) \otimes \omega_\pi) L_S(s, \operatorname{Sym}^2(\pi) \otimes \omega_\pi^2)$ is invertible for $\operatorname{Re} s \geq 1$. This implies our last claim. □

PROPOSITION 4.4

If $\mathrm{Sym}^4(\pi)$ *is cuspidal, then partial sixth symmetric power* L*-functions are all invertible for* $\operatorname{Re} s \geq 1$.

Proof

Consider the equality

$$\begin{aligned} &L_S\big(s, \mathrm{Sym}^2(\pi) \times \mathrm{Sym}^4(\pi)\big) \\ &\qquad = L_S(s, \pi, \mathrm{Sym}^6) L_S\big(s, \mathrm{Sym}^4(\pi) \otimes \omega_\pi\big) L_S\big(s, \mathrm{Sym}^2(\pi) \otimes \omega_\pi^2\big). \end{aligned}$$

Since the left-hand side and $L_S(s, \mathrm{Sym}^4(\pi) \otimes \omega_\pi) L_S(s, \mathrm{Sym}^2(\pi) \otimes \omega_\pi^2)$ are invertible for $\operatorname{Re} s \geq 1$, the same holds for $L_S(s, \pi, \mathrm{Sym}^6)$. $\square$

COROLLARY

Let π *be a cuspidal representation of* GL_2 *such that* $\mathrm{Sym}^3(\pi)$ *is cuspidal. If* $\omega_\pi^3 \neq 1$ *and* $\mathrm{Sym}^3(\pi)$ *is self-contragredient, then* $L(s, \pi, \mathrm{Sym}^6)$ *has a pole at* $s = 1$.

PROPOSITION 4.5

Let π *be a cuspidal representation of* $\mathrm{GL}_2(\mathbb{A})$ *such that* $\mathrm{Sym}^3(\pi)$ *is cuspidal. Then every partial seventh symmetric power* L*-function has a meromorphic continuation and satisfies a standard functional equation. Moreover, they are all invertible for* $\operatorname{Re} s \geq 1$.

Proof

By standard calculations,

$$\begin{aligned} &L_S\big(s, \mathrm{Sym}^3(\pi) \times \mathrm{Sym}^4(\pi)\big) \\ &\quad = L_S(s, \pi, \mathrm{Sym}^7) L_S(s, \pi, \mathrm{Sym}^5 \otimes \omega_\pi) L_S\big(s, \mathrm{Sym}^3(\pi) \otimes \omega_\pi^2\big) L_S(s, \pi \otimes \omega_\pi^3). \end{aligned}$$

Meromorphic continuation and functional equations then follow. By assumption, $\mathrm{Sym}^4(\pi)$ is either cuspidal or of the form $\sigma_1 \boxplus \sigma_2$, where σ_1 and σ_2 are cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ and $\mathrm{GL}_3(\mathbb{A})$, respectively. Hence the left-hand side is invertible for $\operatorname{Re} s \geq 1$.

In the proof of Proposition 4.2, using $L_S(s, (\mathrm{Sym}^2(\pi) \otimes \omega_\pi) \times \mathrm{Sym}^3(\pi))$, one can see that $L_S(s, \pi, \mathrm{Sym}^5 \otimes \omega_\pi)$ is invertible for $\operatorname{Re} s \geq 1$. From this, we obtain our assertion. $\square$

PROPOSITION 4.6

Let π *be a cuspidal representation of* GL_2 *such that* $\mathrm{Sym}^3(\pi)$ *is cuspidal. Then every partial eighth symmetric power* L*-function has a meromorphic continuation and satisfies a standard functional equation. If* $\mathrm{Sym}^4(\pi)$ *is cuspidal and* $\omega_\pi^4 = 1$*, then they are all invertible for* $\operatorname{Re} s \geq 1$.

Proof
By standard calculations,

$$\begin{aligned} &L_S\big(s, \mathrm{Sym}^4(\pi) \times \mathrm{Sym}^4(\pi)\big) \\ &\qquad = L_S(s, \pi, \mathrm{Sym}^8) L_S(s, \pi, \mathrm{Sym}^6 \otimes \omega_\pi) L_S\big(s, \mathrm{Sym}^4(\pi) \otimes \omega_\pi^2\big) \\ &\qquad\quad \times L_S\big(s, \mathrm{Sym}^2(\pi) \otimes \omega_\pi^3\big) L_S(s, \omega_\pi^4). \end{aligned}$$

This proves the meromorphic continuation and functional equation. If $\mathrm{Sym}^4(\pi)$ is cuspidal and $\omega_\pi^4 = 1$, then $\mathrm{Sym}^4(\pi)$ is self-contragredient. Hence the left-hand side has a simple pole at $s = 1$, while in the right-hand side $L(s, \omega_\pi^4)$ has a simple pole at $s = 1$. By Proposition 4.4, and by considering $L_S(s, \mathrm{Sym}^2(\pi) \times (\mathrm{Sym}^4(\pi) \otimes \omega_\pi))$, we see that

$$L_S(s, \pi, \mathrm{Sym}^6 \otimes \omega_\pi) L_S\big(s, \mathrm{Sym}^4(\pi) \otimes \omega_\pi^2\big) L_S\big(s, \mathrm{Sym}^2(\pi) \otimes \omega_\pi^3\big)$$

is invertible for $\mathrm{Re}\, s \geq 1$. This completes our claim. □

PROPOSITION 4.7
Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ such that $\mathrm{Sym}^3(\pi)$ is cuspidal. Then every partial ninth symmetric power L-function has a meromorphic continuation and satisfies a standard functional equation. If $\mathrm{Sym}^4(\pi)$ is cuspidal, then $L_S(s, \pi, \mathrm{Sym}^9)$ has at most a simple pole or a simple zero at $s = 1$. If $\mathrm{Sym}^4(\pi)$ is not cuspidal, then $L_S(s, \pi, \mathrm{Sym}^9)$ is invertible for $\mathrm{Re}\, s \geq 1$.

Proof
Suppose first that $\mathrm{Sym}^4(\pi)$ is cuspidal. Consider the case $E_8 - 2$ of [27]. Let M be a maximal Levi subgroup, and denote by A the connected component of its center. Since E_8 is simply connected, the derived group M_D of M is simply connected as well, and hence $M_D = \mathrm{SL}_4 \times \mathrm{SL}_5$. Thus

$$M = (\mathrm{GL}_1 \times \mathrm{SL}_4 \times \mathrm{SL}_5)/(A \cap M_D),$$

where $A \cap M_D \simeq \mathbb{Z}/20\mathbb{Z}$. Let π_i, $i = 1, 2$, be cuspidal representations of $\mathrm{GL}_4(\mathbb{A})$ and $\mathrm{GL}_5(\mathbb{A})$ with central characters ω_i, $i = 1, 2$, respectively. Let π_{i0}, $i = 1, 2$, be irreducible constituents of $\pi_1|_{\mathrm{SL}_4(\mathbb{A})}$ and $\pi_2|_{\mathrm{SL}_5(\mathbb{A})}$, respectively. Then $\Sigma = \omega_1^5 \omega_2^8 \otimes \pi_{10} \otimes \pi_{20}$ can be considered a cuspidal representation of $M(\mathbb{A})$. We then get the L-function $L(s, \pi_1 \otimes \pi_2, \rho_4 \otimes \wedge^2 \rho_5)$ as our first L-function. In fact, there are five

L-functions in the constant term of the Eisenstein series, namely,

$$\begin{aligned}
L_S(s,\Sigma,r_1) &= L_S(s,\pi_1\otimes\pi_2,\rho_4\otimes\wedge^2\rho_5),\\
L_S(s,\Sigma,r_2) &= L_S\big(s,\pi_1\otimes(\tilde\pi_2\otimes\omega_2),\wedge^2\rho_4\otimes\rho_5\big),\\
L_S(s,\Sigma,r_3) &= L_S\big(s,\tilde\pi_1\times(\pi_2\otimes\omega_1\omega_2)\big),\\
L_S(s,\Sigma,r_4) &= L_S(s,\tilde\pi_2,\wedge^2\rho_5\otimes\omega_1\omega_2^2),\\
L_S(s,\Sigma,r_5) &= L_S(s,\pi_1\otimes\omega_1\omega_2^2).
\end{aligned}$$

(See [27] for the trivial central character case and [11] for the general case for detailed calculations.) Each of the L-functions, especially $L_S(s,\Sigma,r_1)$, has a meromorphic continuation and satisfies a standard functional equation (see [27]).

We apply the above to $\pi_1 = A^3(\pi)$ and $\pi_2 = \mathrm{Sym}^4(\pi)$. By standard calculations, we have

$$\begin{aligned}
&L_S(s,\pi_1\otimes\pi_2,\rho_4\otimes\wedge^2\rho_5)\\
&\quad= L_S(s,\pi,\mathrm{Sym}^9)L_S(s,\pi,\mathrm{Sym}^7\otimes\omega_\pi)L_S(s,\pi,\mathrm{Sym}^5\otimes\omega_\pi^2)^2\\
&\qquad\times L_S\big(s,\mathrm{Sym}^3(\pi)\otimes\omega_\pi^3\big)^2 L_S(s,\pi\otimes\omega_\pi^4).
\end{aligned}$$

The meromorphic continuation and functional equation of $L_S(s,\pi,\mathrm{Sym}^9)$ now follow from those of $L_S(s,\pi_1\otimes\pi_2,\rho_4\otimes\wedge^2\rho_5)$. Moreover,

$$L_S(s,\pi,\mathrm{Sym}^7\otimes\omega_\pi)L_S(s,\pi,\mathrm{Sym}^5\otimes\omega_\pi^2)^2 L_S\big(s,\mathrm{Sym}^3(\pi)\otimes\omega_\pi^3\big)^2 L_S(s,\pi\otimes\omega_\pi^4)$$

is invertible at $s=1$ by Propositions 4.2 and 4.5. So it is enough to prove that $L_S(s,\Sigma,r_1)$ has at most a simple pole or simple zero at $s=1$.

By [26], the product $\prod_{i=1}^5 L_S(1+is,\Sigma,r_i)$ does not have a zero at $s=0$. But none of the L-functions $L_S(s,\Sigma,r_i)$, $i=3,4,5$, has a pole at $s=1$. In fact, they are all entire and have no zeros for $\mathrm{Re}\, s\ge 1$. (See [7] for the case r_4 and $\omega_1\omega_2^2=1$. The general case can be seen to be the same by observing that the twisted exterior square L-function appears as the normalizing factor of a certain Eisenstein series if we consider Spin_{2n} (cf. [11]). Alternatively, by direct calculation, we have $L_S(s,\Sigma,r_4) = L_S(s,\pi,\mathrm{Sym}^6\otimes\omega_\pi^{15})L_S(s,\mathrm{Sym}^2(\pi)\otimes\omega_\pi^{17})$. The necessary properties of $L_S(s,\pi,\mathrm{Sym}^6\otimes\omega_\pi^{15})$, and in particular its invertibility at $s=1$, are now obtained by considering $L_S(s,\mathrm{Sym}^2(\pi)\times(\mathrm{Sym}^4(\pi)\otimes\omega_\pi^{15}))$ in the proof of Proposition 4.4.) The second L-function, $L_S(s,\Sigma,r_2)$, appears as the first L-function in the case D_8-3. Hence it has at most a simple pole at $s=1$. Alternatively, by direct calculation, we see that, since $\wedge^2(A^3(\pi)) = A^4(\pi)\oplus\omega_\pi$,

$$\begin{aligned}
&L_S\big(s,\pi_1\otimes(\tilde\pi_2\otimes\omega_2),\wedge^2\rho_4\otimes\rho_5\big)\\
&\quad= L_S\Big(s,\mathrm{Sym}^4(\pi)\times\big(\mathrm{Sym}^4(\pi)\otimes\omega_\pi^5\big)\Big)L_S\big(s,\mathrm{Sym}^4(\pi)\otimes\omega_\pi^7\big).
\end{aligned}$$

Hence $L_S(s, \pi_1 \otimes (\tilde{\pi}_2 \otimes \omega_2), \wedge^2 \rho_4 \otimes \rho_5)$ has at most a simple pole at $s = 1$ and is invertible for $\operatorname{Re} s > 1$. Therefore $L_S(s, \Sigma, r_1)$ has at most a simple zero at $s = 1$.

By [27], the product $\prod_{i=1}^5 L_S(is, \Sigma, r_i)$ has at most a simple pole at $s = 1$. However, $\prod_{i=2}^5 L_S(is, \Sigma, r_i)$ has no zeros at $s = 1$. Therefore $L_S(s, \Sigma, r_1)$ has at most a simple pole at $s = 1$.

Next, suppose $\mathrm{Sym}^4(\pi)$ is not cuspidal. Then $\mathrm{Sym}^4(\pi) = \sigma_1 \boxplus \sigma_2$, where σ_1 and σ_2 are cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ and $\mathrm{GL}_3(\mathbb{A})$, respectively. In this case, by standard calculations,

$$\begin{aligned} L_S(s, &\pi_1 \otimes \pi_2, \rho_4 \otimes \wedge^2 \rho_5) \\ &= L_S(s, \pi_1 \times \sigma_1 \times \sigma_2) L_S(s, \pi_1 \otimes \omega_{\sigma_1}) L_S\big(s, \pi_1 \times (\tilde{\sigma}_2 \otimes \omega_{\sigma_2})\big). \end{aligned}$$

Here $L_S(s, \pi_1 \otimes \omega_{\sigma_1}) L_S(s, \pi_1 \times (\tilde{\sigma}_2 \otimes \omega_{\sigma_2}))$ is invertible for $\operatorname{Re} s \geq 1$. By [13],

$$L_S(s, \pi_1 \times \sigma_1 \times \sigma_2) = L_S\big(s, \pi_1 \times (\sigma_1 \boxtimes \sigma_2)\big),$$

where $\sigma_1 \boxtimes \sigma_2$ is the functorial product that is an automorphic representation of $\mathrm{GL}_6(\mathbb{A})$. Since σ_1 is monomial (see Theorem 3.3.7), by the main theorem of [13], $\sigma_1 \boxtimes \sigma_2$ is either cuspidal or unitarily induced from two cuspidal representations of $\mathrm{GL}_3(\mathbb{A})$. Hence $L_S(s, \pi_1 \times \sigma_1 \times \sigma_2)$ is invertible for $\operatorname{Re} s \geq 1$, and therefore the same conclusion holds for $L_S(s, \pi, \mathrm{Sym}^9)$. □

PROPOSITION 4.8

Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ such that $\mathrm{Sym}^3(\pi)$ is cuspidal. Let $\mathrm{diag}(\alpha_v, \beta_v)$ be the Satake parameter for an unramified local component. Then $|\alpha_v|, |\beta_v| < q_v^{1/9}$. If $\mathrm{Sym}^4(\pi)$ is not cuspidal, then the full Ramanujan conjecture is valid.

Proof

If $\mathrm{Sym}^4(\pi)$ is cuspidal, use Proposition 4.7 and [27, Lemma 5.8]. If $\mathrm{Sym}^4(\pi)$ is not cuspidal, then by Proposition 3.3.8, π is of Galois type for which $|\alpha_v| = |\beta_v| = 1$. □

The following result coincides with Langlands's calculations in [18].

PROPOSITION 4.9

Let π be a nonmonomial cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ with a trivial central character. Suppose $m \leq 9$.

(1) *Suppose $\mathrm{Sym}^3(\pi)$ is not cuspidal. Then $L_S(s, \pi, \mathrm{Sym}^m)$ is invertible at $s = 1$, except for $m = 6, 8$; the L-functions $L_S(s, \pi, \mathrm{Sym}^6)$ and $L_S(s, \pi, \mathrm{Sym}^8)$ each have a simple pole at $s = 1$.*

(2) *Suppose* $\mathrm{Sym}^3(\pi)$ *is cuspidal, but* $\mathrm{Sym}^4(\pi)$ *is not. Then* $L_S(s,\pi,\mathrm{Sym}^m)$ *is invertible at* $s=1$ *for* $m=1,\ldots,7$ *and* $m=9$*; the L-function* $L_S(s,\pi,\mathrm{Sym}^8)$ *has a simple pole at* $s=1$.

Proof
(1) By Theorem 2.2.2 and Section 3.2,

$$\mathrm{Sym}^3(\pi)=(\pi\otimes\mu)\boxplus(\pi\otimes\mu^2),\qquad \mathrm{Sym}^4(\pi)=\mathrm{Sym}^2(\pi)\boxplus\mu\boxplus\mu^2,$$

where μ is a nontrivial grössencharacter such that $\mathrm{Ad}(\pi)\simeq\mathrm{Ad}(\pi)\otimes\mu$. We explicitly calculate $L_S(s,\pi,\mathrm{Sym}^m)$. Let $\Pi=\mathrm{Sym}^2(\pi)$. Then

$$L_S(s,\pi,\mathrm{Sym}^3)=L_S(s,\pi\otimes\mu)L_S(s,\pi\otimes\mu^2),$$
$$L_S(s,\pi,\mathrm{Sym}^4)=L_S(s,\Pi)L_S(s,\mu)L_S(s,\mu^2).$$

They are both invertible for $\mathrm{Re}\,s\ge 1$. From the equality in Proposition 4.2, we have $L_S(s,\pi,\mathrm{Sym}^5)=L_S(s,\pi)L_S(s,\pi\otimes\mu)L_S(s,\pi\otimes\mu^2)$, which is clearly invertible for $\mathrm{Re}\,s\ge 1$.

Using the equality in Proposition 4.3, we have $L_S(s,\pi,\mathrm{Sym}^6)=L_S(s,\Pi)^2\cdot L_S(s,1)$. Since $L(s,\Pi)$ is invertible at $s=1$, $L_S(s,\pi,\mathrm{Sym}^6)$ has a simple pole at $s=1$.

From the equality in Proposition 4.5, we have

$$L_S(s,\pi,\mathrm{Sym}^7)=L_S(s,\pi)^2L_S(s,\pi\otimes\mu)L_S(s,\pi\otimes\mu^2).$$

Hence it is invertible for $\mathrm{Re}\,s\ge 1$.

For $L_S(s,\pi,\mathrm{Sym}^8)$, consider the equality in Proposition 4.4 with $\omega_\pi=1$; we have $L_S(s,\Pi\times\Pi)=L_S(s,\pi,\mathrm{Sym}^6)L_S(s,\mu)L_S(s,\mu^2)$. Hence $L_S(s,\Pi\times\Pi)=L_S(s,\Pi)^2L_S(s,1)L_S(s,\mu)L_S(s,\mu^2)$. Then from the equality in Proposition 4.6, $L_S(s,\pi,\mathrm{Sym}^8)=L_S(s,\Pi)^2L_S(s,\mu)L_S(s,\mu^2)L_S(s,1)$, and therefore $L_S(s,\pi,\mathrm{Sym}^8)$ has a simple pole at $s=1$.

For $L_S(s,\pi,\mathrm{Sym}^9)$, consider the equality in Proposition 4.7 with $\omega_\pi=1$; by standard calculations, we see that

$$L_S(s,\pi,\mathrm{Sym}^9)=L_S(s,\pi\otimes\eta)^2L_S(s,\pi\otimes\eta^2)^2L_S(s,\pi).$$

We therefore conclude that $L_S(s,\pi,\mathrm{Sym}^9)$ is invertible for $\mathrm{Re}\,s\ge 1$.

(2) In this case, $\mathrm{Sym}^4(\pi)=\sigma_1\boxplus\sigma_2$, where σ_1 and σ_2 are cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ and $\mathrm{GL}_3(\mathbb{A})$, respectively. If $\omega_\pi=1$, then σ_1 and σ_2 are self-contragredient. We only have to discuss the case $m=8$. Consider the equality in Proposition 4.6 with $\omega_\pi=1$ in which the left-hand side has a double pole at $s=1$. But $L_S(s,\pi,\mathrm{Sym}^6)L_S(s,\pi,\mathrm{Sym}^4)L_S(s,\pi,\mathrm{Sym}^2)$ is invertible at $s=1$. Thus $L_S(s,\pi,\mathrm{Sym}^8)$ has a simple pole at $s=1$. $\square$

Remark. The above proposition is no longer true if the central character is not trivial (see Proposition 3.2.1, Corollaries to Theorem 3.3.7 and Proposition 4.4).

We now give an application of these properties of symmetric power L-functions to the Sato-Tate conjecture (see [25], [30]). This we do by following Serre's method (see [30, Appendix]). In what follows, let $\pi = \bigotimes_v \pi_v$ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ with a trivial central character such that $\mathrm{Sym}^4(\pi)$ is cuspidal. We also assume that π satisfies the Ramanujan-Petersson conjecture. Recall that we let $a_v = \alpha_v + \beta_v$, where π_v is an unramified local component with the Satake parameter $\mathrm{diag}(\alpha_v, \beta_v)$.

We use exactly the same notation as in [30, Appendix]. First let us calculate $T_n(x)$, the polynomial that gives the trace of the nth symmetric power of an element of $\mathrm{SL}_2(\mathbb{C})$ whose trace is x:

$$T_0 = 1, \qquad T_1 = x, \qquad T_2 = x^2 - 1, \qquad T_3 = x^3 - 2x,$$
$$T_4 = x^4 - 3x^2 + 1, \qquad T_5 = x^5 - 4x^3 + 3x, \qquad T_6 = x^6 - 5x^4 + 6x^2 - 1,$$
$$T_7 = x^7 - 6x^5 + 10x^3 - 4x, \qquad T_8 = x^8 - 7x^6 + 15x^4 - 10x^2 + 1,$$
$$T_9 = x^9 - 8x^7 + 21x^5 - 20x^3 + 5x.$$

Next recall the quantity

$$I(T_n) = \lim_{N\to\infty} \frac{\sum_{q_v \le N} T_n(a_v)}{\pi(N)} = k_n,$$

where the nth symmetric power L-function has an order $-k_n$ at $s = 1$ and $\pi(N)$ is the number of places such that $q_v \le N$. Hence $I(T_0) = 1$ and $I(T_n) = 0$ for $n = 1, \dots, 8$ by Propositions 4.2–4.6. Let $I(T_9) = k$. By Proposition 4.7, we know that $k \in \{-1, 0, 1\}$.

We now calculate $I(x^n)$:

$$I(x) = 0, \qquad I(x^2) = 1, \qquad I(x^3) = 0, \qquad I(x^4) = 2,$$
$$I(x^5) = 0, \qquad I(x^6) = 5, \qquad I(x^7) = 0, \qquad I(x^8) = 14, \qquad I(x^9) = k.$$

With notation as in [30, Appendix], $d = 9$ and $m = 4$. Using this, we can calculate the orthogonal polynomials, $P_0, P_1, \dots, P_4$ to get

$$P_0 = 1, \qquad P_1 = T_1 = x, \qquad P_2 = T_2 = x^2 - 1, \qquad P_3 = T_3 = x^3 - 2x,$$
$$P_4 = T_4 = x^4 - 3x^2 + 1, \qquad P_5 = T_5 - k(x^4 - 3x^2 + 1).$$

There are three possibilities:

(1) $k = 0$: $P_5 = T_5 = x^5 - 4x^3 + 3x$; the roots are $-\sqrt{3}, -1, 0, 1, \sqrt{3}$;

(2) $k = 1$: $P_5 = T_5 - T_4 = x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1$; the roots are $-2\cos(2k\pi/11)$, $k = 1, 2, 3, 4, 5$;

(3) $k = -1$: $P_5 = T_5 + T_4 = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$; the roots are $2\cos(2k\pi/11)$, $k = 1, 2, 3, 4, 5$. Note that $2\cos(2\pi/11) \cong 1.68\ldots$.

THEOREM 4.10
For every $\epsilon > 0$, there are sets T^+ and T^- of positive lower (Dirichlet) density such that $a_v > 1.68\ldots - \epsilon$ for all $v \in T^+$ and $a_v < -1.68\ldots + \epsilon$ for all $v \in T^-$.

Remark. In [30, Appendix] and [23], where the third, fourth, and fifth symmetric power L-functions were used, the result is $a_v > \sqrt{2} - \epsilon$.

In the next proposition, we do not assume the Ramanujan-Petersson conjecture for π.

PROPOSITION 4.11
Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ with a trivial central character. Then for every $\epsilon > 0$, there exists a set T of positive lower (Dirichlet) density such that $|a_v| > 1.68\ldots - \epsilon$ for all $v \in T$.

Remark. Let σ be a two-dimensional continuous representation of W_F, the Weil group of $\overline{F}/F$. Assume that there exists an automorphic cuspidal representation $\pi(\sigma)$ of $\mathrm{GL}_2(\mathbb{A})$ preserving root numbers and L-functions for pairs (cf. [16], [13]). This is possible except perhaps when σ is icosahedral of special kind (cf. [13, §10]). When σ is of icosahedral type and $\pi(\sigma)$ exists, $L(s, \mathrm{Sym}^4 \sigma)$ is a five-dimensional irreducible, and therefore entire, Artin L-function. But it is not primitive since $\mathrm{Sym}^4 \sigma$ is monomial (cf. [9]). We refer to [9] for an interesting application of this to automorphic induction.

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